Fractal Geometry
Winter Semester 2010/2011

Dr. Peter Massopust
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Exercise Set 10

1. (Problem 3, Exercise Set 9, continued) Let $J := [0,1] \subset \mathbb{R}$ and let $0 =: t_0 < t_1 < t_2 < t_3 := 1$. Choose $l_i : J \to J$ as $l_i(t) := (t_i - t_{i-1})t + t_{i-1}$, $i = 1, 2, 3$.

   (a) Show that the fractal set generated by the IFS $(J, \{l_i : i = 1, 2, 3\})$ is the unit interval $I = [0, 1]$.

   (b) Let $K := [0,1] \times [0,1] \subset \mathbb{R}^2$ and let the maps k_i, $i = 1, 2, 3$, be independent of t and each be of the form
   $$k(x,y) := \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix},$$
   where the constants a, b, c, d, e, and f are determined by requiring that
   $$\begin{array}{lll} k_1(0,0) = (0,0), & k_1(1.0,0) = (0.25,0.5), & k_3(0.5,1) = (0.5,0), \\ k_2(0,0) = (0.25,0.5), & k_2(0.5,1) = (0.5,1), & k_2(1.0,0) = (0.75,0.5), \\ k_3(0,0) = (0.75,0.5), & k_3(0.5,1) = (0.5,0), & k_3(1.0,0) = (1,0). \end{array}$$
   Determine the mappings k_i, $i = 1, 2, 3$.

   (c) Show that the fractal set generated by the IFS $(J \times K, \{k_i : i = 1, 2, 3\})$ is the graph of a function $F : [0,1] \to [0,1] \times [0,1]$ such that
   $$F(t_0) = (0,0), \quad F(t_1) = (0.25, 0.5), \quad F(t_2) = (0.75, 0.5), \quad F(t_3) = (1, 0).$$
   Writing $F := (f, h)$, where $f, g : [0,1] \to [0,1]$, show that the projection of graph F onto the xy-plane is the well-known Sierpiński triangle.

   (d) Display the graphs of f and g. The functions f and g are called *hidden variable fractal interpolation functions*. Explain why the term "hidden" is used.

2. Let $J := [0,1] \subset \mathbb{R}$ and let $t_0 := 0$, $t_1 := 0.5$, and $t_2 := 1$. Choose $l_i : J \to J$ as $l_i(t) := (t_i - t_{i-1})t + t_{i-1}$, $i = 1, 2$. Let $K := [0,1] \times [0,1] \subset \mathbb{R}^2$ and let the maps k_i, $i = 1, 2$, be independent of t and of the form
   $$k(x,y) := \begin{pmatrix} 0.5 & 0.5(-1)^{i-1} \\ 0.5(-1)^{i-1} & -0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (i-1)0.5 \\ (i-1)0.5 \end{pmatrix}.$$

   (a) Show that the fractal set generated by the IFS $(K, \{k_i : i = 1, 2\})$ is a *space-filling* or *Peano curve*.

   (b) Obtain different views of G, the fractal set generated by the IFS $(J \times K, \{w_i : i = 1, 2, 3\})$, with $w_i := (l_i, k_i)$, $i = 1, 2$.