Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 11

1. Let $X := [0,1] \times \mathbb{R}$, let $N := 2$, and let $Y := \{(0,0), (0.5, 0.5), (1,0)\}$.

   (a) Choose $\lambda_1 = \lambda_2 := 0.25$. Obtain formulae for the contractive homeomorphisms u_i, the mappings p_i, and an explicit expression for the RB-operator T. Show that the function $x \mapsto 2x(1-x)$ is the unique fixed point of T.

   (b) Choose $\lambda_1 = \lambda_2 := 0.5$. Obtain formulae for the contractive homeomorphisms u_i, the mappings p_i, and an explicit expression for the RB-operator T. Show that the Tagaki function τ is the unique fixed point of T.

2. Let $X := [0,1] \times \mathbb{R}$, let $N := 4$, and let $Y := \{(0,0), (0.25, -0.5), (0.5, 0), (0.75, 0.5), (1,0)\}$. Set $\lambda_1 := -0.5$, $\lambda_j := 0.25$, for $j = 2,3,4$. Obtain formulae for the contractive homeomorphisms u_i, the mappings p_i, and an explicit expression for the RB-operator T. The unique fixed point k of T is called *Kiesswetter's fractal function*. Display the graph of k.

3. *A more general construction of fractal interpolation functions*

   Given: a nonempty interval $[a,b] \subset \mathbb{R}$, a partition $P := (a =: x_0 < x_1 < \ldots < x_N := b)$ of $[a,b]$ with $1 < N \in \mathbb{N}$, and a set of interpolation points $Y := \{(x_j, y_j^k) : j = 0,1,\ldots,N;\ k = 0,1,\ldots,K\}$, for some $K \in \mathbb{N}_0$. Write y_j for y_j^0.
   Let $L_i : [x_0, x_N] \to [x_{i-1}, x_i]$ be the unique affine function such that

   $$L_i(x_0) = x_{i-1},\ L_i(x_N) = x_i, \quad \forall i = 1,\ldots,N. \tag{1}$$

   Note that (1) in particular implies that

   $$L_j(x_N) = L_{j+1}(x_0), \quad j = 1,\ldots,N-1, \tag{2}$$

   and that $a_i := \operatorname{Lip} L_i = \frac{x_i - x_{i-1}}{x_n - x_0} < 1$. Hence, $([x_0, x_N]; \{L_i : i = 1,\ldots,N\})$ is an IFS on $[x_0, x_N]$ whose associated fractal set is $[x_0, x_N]$.

   Goal: Construct a fractal interpolation function $f : [a,b] \to \mathbb{R}$ such that

   (a) $f \in C^K[a,b]$ and

   $$f^{(k)}(x_j) = y_j^{(k)}, \quad \forall j = 0,1,\ldots,N;\ \forall k = 0,1,\ldots,K. \tag{3}$$

   (b) The graph of f over $[x_{i-1}, x_i]$ is obtained from that of f over $[x_0, x_N]$ by

      i. contracting and translating horizontally by L_i;

      ii. contracting vertically by λ_i and then translating vertically by some suitable function q_i so that (3) is satisfied, i.e., for all $i = 1,\ldots,N$,

   $$\begin{aligned} f(L_i(x)) &= \lambda_i\, f(L_i(x)) + q_i(x), \quad x \in [x_0, x_N], \text{ or equivalently} \\ f(x) &= \lambda_i (f \circ L_i^{-1})(x) + (q_i \circ L_i^{-1})(x), \quad x \in [x_{i-1}, x_i]. \end{aligned} \tag{4}$$

   Functions used in the construction:

   (a) A C^K base function $b : [x_0, x_N] \to \mathbb{R}$ satisfying

   $$b^{(k)}(x_0) = y_0^k, \quad b^{(k)}(x_N) = y_N^k, \quad \forall k = 0,1,\ldots,K. \tag{5}$$

(b) a C^K height function $h : [x_0, x_N] \to \mathbb{R}$ with the property that

$$h^{(k)}(x_j) = y_j^k, \quad \forall j = 0, 1, \dots, N;\ \forall k = 0, 1, \dots, K. \tag{6}$$

Note that, by means of an affine transformation, one may without loss of generality assume that $y_0 = y_N = 0$ and then choose $b \equiv 0$. Show that for this particular choice of y_0, y_N and b, (4) becomes

$$f(x) = \lambda_i (f \circ L_i^{-1})(x) + h(x), \quad x \in [x_{i-1}, x_i]. \tag{7}$$

Geometric construction of f via a certain IFS

(a) For $i = 1, \dots, N$, define $F_i : [x_0, x_N] \times \mathbb{R} \to \mathbb{R}$ by

$$F_i(x, y) := \lambda_i \left(y - b(x) \right) + h(L_i(x)). \tag{8}$$

(b) For $i = 1, \dots, N$, define $w_i : [x_0, x_N] \times \mathbb{R} \to [x_0, x_N] \times \mathbb{R}$ by

$$w_i(x, y) := (L_i(x), F_i(x, y)), \tag{9}$$

and $W : \mathcal{H}([x_0, x_N] \times \mathbb{R}) \to \mathcal{H}([x_0, x_N] \times \mathbb{R})$ by

$$W(A) := \bigcup_{i=1}^{N} w_i(A).$$

(c) Show that for $k = 0$ and all $i = 1, \dots, N$,

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i, \tag{10}$$

$$w_i(x_0, y_0) = (x_{i-1}, y_{i-1}), \quad w_i(x_N, y_N) = (x_i, y_i). \tag{11}$$

Proposition 1. The IFS $([x_0, x_N] \times \mathbb{R}, \mathcal{W})$, where $\mathcal{W} := \{w_i : i = 1, \dots, N\}$, generates a unique compact fractal set G. Moreover, G is the graph of a continuous function $f : [a, b] \to \mathbb{R}$ with the property that $f(x_j) = y_j$, $j = 0, 1, \dots, N$.

Outline of Proof: For θ to be chosen, let

$$d_\theta((x, y), (x', y')) := |x - x'| + \theta |y - y'|, \quad x, x' \in [a, b];\ y, y' \in \mathbb{R}, \tag{12}$$

be a metric on $[a, b] \times \mathbb{R}$. Then, setting $q_i := h \circ L_i - \lambda_i b$ and $M_i := \operatorname{Lip} q_i$, and choosing $\theta < \max\{(1 - \lambda_i)/M_i : i = 1, \dots, N\}$, the mappings w_i are contractive. Hence, $([x_0, x_N] \times \mathbb{R}, \mathcal{W})$ generates a unique compact fractal set G. Using (11) one shows that G is the graph of a *function* f and that $f(x_j) = y_j$, $j = 0, 1, \dots, N$. The continuity of f will follow from Proposition 2 below. $\square$

Remark: It is sometimes customary to write

$$F_i(x, y) = \lambda_i y + q_i(x), \quad i = 1, \dots, N, \tag{13}$$

where the functions $q_i : [a, b] \to \mathbb{R}$ are chosen so that the compatibility condition (10) is fulfilled, i.e., so that

$$\lambda_i y_0 + q_i(x_0) = y_{i-1}, \quad \lambda_i y_N + q_i(x_N) = y_i, \quad \forall i = 1, \dots, N. \tag{14}$$

If b and h satisfy (5) and (6) with $k = 0$, and one defines

$$q_i := h \circ L_i - \lambda_i b, \tag{15}$$

then (14) follows. Conversely, if (14) is true for q_i, $i = 1, \dots, N$, then choose any b so that (5) is satisfied for $k = 0$ and use (15) to define h. Then, h is well-defined at each x_j, $j = 0, 1, \dots, N$, and satisfies (6) with $k = 0$.

Construction via RB operators

Let

$$C^*[a,b] := \{g \in C[a,b] \,:\, g(a) = y_0 \wedge g(b) = y_N\}$$
$$C^{**}[a,b] := \{g \in C[a,b] \,:\, g(x_j) = y_j,\ \forall j = 0,1,\ldots,N\}.$$

Define an RB-operator $T : C^*[a,b] \to C^{**}[a,b]$ by

$$Tg := h(x) + \sum_{i=1}^{N} \lambda_i \left(g \circ L_i^{-1} - b \circ L_i^{-1}\right) \chi_{[x_{i-1},x_i]}. \tag{16}$$

Show that, for $k = 0$,

(a) Tg is well-defined, continuous and belongs to $C^{**}[a,b]$;

(b) T is a contraction in the $\| \bullet \|_\infty$ metric with $\operatorname{Lip} T = \max\{|\lambda_i \,:\, i = 1,\ldots,N|\}$;

(c) The operator T acts on $g \in C^*[a,b]$ in essentially the same way that W acts on the graph of g, i.e.,

$$\operatorname{graph}(Tg) = W(\operatorname{graph} g).$$

Proof the following proposition.

Proposition 2. The RB-operator T defined by (16) has a unique fixed point $f \in C^*[a,b]$ and $f \in C^{**}[a,b]$. Moreover, $f(x_j) = y_j$, $j = 0,1,\ldots N$, and $\operatorname{graph} f = G$, the fractal set generated by the IFS $([x_0, x_N] \times \mathbb{R}, \mathcal{W})$. In addition, $T^n g \to f$ in the $\| \bullet \|_\infty$-metric.