Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 12

Continuation of Exercise Set 11

1. Suppose that $K \in \mathbb{N}$ and that b and h are functions in $C^K[a,b]$ satisfying the conditions stated on Exercise Set 11. For $k = 1, \ldots, K$, define

$$C^{(k)*}[a,b] := \{g \in C[a,b] \,:\, g(x_0) = y_0^k \wedge g(x_N) = y_N^k\},$$
$$C^{(k)**}[a,b] := \{g \in C[a,b] \,:\, g(x_j) = y_j^k,\ j = 0,1,\ldots,N\}.$$

Define the k-th derived RB-operator

$$T^{(k)} : C^{(k)*}[a,b] \to C^{(k)**}[a,b]$$

by

$$(T^{(k)}g)(x) := h^{(k)}(x) + \frac{\lambda_i}{a_i^k}\left(((g - b^{(k)}) \circ u_i^{-1})(x)\right), \quad i = 1,\ldots,N;\ x \in [x_{i-1}, x_i]. \tag{1}$$

Show that $T^{(k)}$ is well-defined and has a unique fixed point $f \in C^{(k)**}[a,b]$.

2. Define

$$C^{K,*}[a,b] := \{g \in C^K[a,b] \,:\, g^{(k)}(x_0) = y_0^k \wedge g^{(k)}(x_N) = y_N^k,\ k = 0,1,\ldots,K\},$$
$$C^{K,**}[a,b] := \{g \in C^K[a,b] \,:\, g^{(k)}(x_j) = y_j^k,\ j = 0,1,\ldots,N,\ k = 0,1,\ldots,K\}.$$

Suppose that b and h satisfy conditions (5) and (6) on Exercise Set 11 and that $\max\{|\lambda_i|/|a_i|^k : i = 1,\ldots,N\} < 1$. Show that the function f given in Propositions 1 and 2 on Exercise Set 11 is in $C^K[a,b]$ and satisfies the following Hermite Interpolation Conditions:

$$f^{(k)}(x_j) = y_j^k, \quad k = 0,1,\ldots,K;\ j = 0,1,\ldots,N.$$

Moreover, show that the sequence $\{T^n g : n \in \mathbb{N}_0\}$ converges uniformly to f in the C^K-norm for $g \in C^{K,*}[a,b]$. More generally, the sequence $\{(T^{(k)})^n g^{(k)} : n \in \mathbb{N}_0\}$ converges uniformly to f in the C^{K-k}-norm to $f^{(k)}$ for $k = 0,1,\ldots,K$.

3. Prove the following lemma:

Lemma: Suppose that f is an affine fractal function. Let $\mathcal{N}(r)$ be the cardinality of a minimal cover of graph$f$ of the form $\mathcal{C}(r)$, $r \in \mathbb{N}_0$. If $\sum_{i=1}^N |\lambda_i| > 1$ and the set of interpolation points Y is not collinear, then

$$\lim_{r\to\infty} \frac{N^r}{\mathcal{N}(r)} = 0.$$

4. Suppose $f : [a,b] \to \mathbb{R}$ is an affine fractal function. Use the fixed point equation for f to show that

$$\int_a^b f(x)dx = \frac{\int_a^b p(x)dx}{1 - \sum_{i=1}^N a_i\lambda_i},$$

where $p := \sum_{i=1}^N p_i \chi_{[x_{i-1},x_i]}$.