Fractal Geometry
Winter Semester 2010/2011

Dr. Peter Massopust
November 17, 2010

Exercise Set 4

1. Let $(X;\mathcal{F})$ and $(Y;\mathcal{G})$ be IFSs with the same code space Σ_N and let $\mathfrak{A}_\mathcal{F}$ and $\mathfrak{A}_\mathcal{G}$ be the fractals generated by these IFSs. The graph of the fractal transformation $\mathfrak{T}_{\mathcal{F}\mathcal{G}} : \mathfrak{A}_\mathcal{F} \to \mathfrak{A}_\mathcal{G}$ can be described as follows. Consider the IFS

$$(X \times Y \times \Sigma_N; \mathcal{K}),$$

where $\mathcal{K} := \{k_i : i = 1, \ldots, N\}$ with $k_i(x, y, \sigma) := (f_i(x), g_i(y), c(i, \sigma))$. Here, $f_i \in \mathcal{F}$, $g_i \in \mathcal{G}$, and $c(i, \sigma) := i\,\sigma$ denotes the concatenation of the symbol i with the code σ. Show that:

   (a) The set $X \times Y \times \Sigma_N$ endowed with the metric

   $$d((x, y, \sigma), (x', y', \sigma')) := \max\{d_X(x, x'), d_Y(y, y'), d_F(\sigma, \sigma')\},$$

   generates the compact metric space $(X \times Y \times \Sigma_N, d)$;

   (b) The mappings k_i, $i = 1, \ldots, N$, are contractive in the metric d with maximum contractivity constant $\lambda_\mathcal{K} := \max\{\lambda_\mathcal{F}, \lambda_\mathcal{G}, \frac{1}{N+1}\}$, where $\lambda_\mathcal{F}$ and $\lambda_\mathcal{G}$ denotes the maximum contractivity constant of the IFS $(X;\mathcal{F})$ and $(Y;\mathcal{G})$, respectively.

   (c) The unique attractor of the IFS $(X \times Y \times \Sigma_N; \mathcal{K})$ is the fractal set

   $$\mathfrak{A}_\mathcal{K} = \{(x, y, \sigma) : \gamma_\mathcal{F}(\sigma) = x,\ \gamma_\mathcal{G}(y) = y,\ \sigma \in \Sigma_N\}.$$

   (d) The graph of the fractal transformation $\mathfrak{T}_{\mathcal{F}\mathcal{G}}$ is identical to the set

   $$\{(x, y) : (x, y, \sigma) \in \mathfrak{A}_\mathcal{K},\ \tau \prec \sigma \text{ for all } (x, y', \tau) \in \mathfrak{A}_\mathcal{K}\}.$$

2. **Definition.** (Homeomorphic Address Structures) *Assume that $(X;\mathcal{F})$ and $(Y;\mathcal{G})$ are two IFSs with the same code space Σ_N. The address structures $\mathcal{C}_\mathcal{F}$ and $\mathcal{C}_\mathcal{G}$ are called* homeomorphic, *in symbol $\mathcal{C}_\mathcal{F} \cong \mathcal{C}_\mathcal{G}$, iff there exists a homeomorphism $\eta : \overline{\Omega}_\mathcal{F} \to \overline{\Omega}_\mathcal{G}$ that respects the address structures, i.e., η maps each $A \in \mathcal{C}_\mathcal{F}$ onto a $B \in \mathcal{C}_\mathcal{G}$.*

   Prove the following weaker version of the theorem presented in the lecture.

   Theorem. Suppose that the address structures $\mathcal{C}_\mathcal{F}$ and $\mathcal{C}_\mathcal{G}$ of the IFSs $(X;\mathcal{F})$ and $(Y;\mathcal{G})$ are homeomorphic. Then the attractors $\mathfrak{A}_\mathcal{F}$ and $\mathfrak{A}_\mathcal{G}$ generated by these two IFSs are also homeomorphic.

   Hint: $\gamma_\mathcal{G} \circ \eta \circ \tau_\mathcal{F} : \mathfrak{A}_\mathcal{F} \to \mathfrak{A}_\mathcal{G}$ is the desired homeomorphism.

3. In reference to Problem 1 on Exercise Set 2, define a fourth map $(A, B, C) \overset{f_4}{\longmapsto} (A', B', C')$, and consider the IFS $(\mathbb{R}^2, \mathcal{F}_{r,s,t})$, where $\mathcal{F}_{r,s,t} := \{f_1, f_2, f_3, f_4\}$, whose attractor is the (filled-in) triangle $\Delta(A, B, C)$. Show that for $r, s, t, r', s', t' \in (0, 1)$, $\mathcal{C}_{r,s,t} = \mathcal{C}_{r',s',t'}$.