Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 5

1. Let M, and M_j, $j \in \mathbb{N}_0$, be bounded subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $s > 0$.
   (a) Let $\lambda > 0$. Then $\mathcal{H}^s(\lambda M) = \lambda^s \mathcal{H}^s(M)$, where $\lambda M := \{\lambda m : m \in M\}$.
   (b) Show that $\mathcal{H}^0(M)$ equals the number of points in the set M.
   (c) Using the definition of $\mathcal{H}^s$, show that
      i. $\mathcal{H}^s(\emptyset) = 0$.
      ii. If $M_1 \subseteq M_2$, then $\mathcal{H}^s(M_1) \le \mathcal{H}^s(M_2)$.
      iii. For any collection of bounded sets, show that
$$\mathcal{H}^s\left(\bigcup_{j=1}^{\infty} M_j\right) \le \sum_{j=1}^{\infty} \mathcal{H}^s(M_j),$$
      where equality holds when the sets M_j are pairwise disjoint.
   (d) Show that $\mathcal{H}^s$ is invariant under translations in $\mathbb{R}^n$, i.e., $\mathcal{H}^s(M + x) = \mathcal{H}^s(M)$, where $M + x := \{x + m : m \in M\}$, $x \in \mathbb{R}^n$.

2. In the definition of $\mathcal{H}^s(M)$, replace the balls $B_r(x_0)$ by cubes $W_r(x_0)$ centered at $x_0 \in M$ with side length r, $0 < r \le \varepsilon$, where $0 < \varepsilon < 1$. Denote this new quantity by $\mathcal{W}^s(M)$.
   (a) How are $\mathcal{H}^s(M)$ and $\mathcal{W}^s(M)$ related?
   (b) Does the behavior of the function $s \mapsto \mathcal{W}^s(M)$ change?
   (c) Does one obtain the same Hausdorff-Besicovitch dimension for M?