Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 6

1. Let $\mathfrak{C}$ denote the middle-thirds Cantor set. Show that $\dim_B \mathfrak{C} = \log 2/\log 3$.

2. Let $M \subset \mathbb{R}^n$ be bounded. Denote by $N_\delta(M)$ the largest number of disjoint balls of radius $\delta > 0$ with centers in M. Define

$$L(M) := \lim_{\delta\to 0+} \frac{N_\delta(M)}{-\log\delta}, \quad \text{(provided this limit exists).}$$

   Show that $\dim_B M = L(M)$.

3. Let $M \subset \mathbb{R}^n$ be bounded and let

$$M_\delta := \{x \in \mathbb{R}^n \,:\, |x-y| \le \delta \text{ for some } y \in M\}$$

   denote the δ-parallel body of M. Define the *Minkowski-Bouligand dimension* of M by

$$\dim_{MB} M := n - \lim_{\delta\to 0+} \frac{\log \mathrm{vol}^n M_\delta}{\log\delta}, \quad \text{(provided this limit exists),}$$

   where $\mathrm{vol}^n E$ denotes the n-dimensional volume of a bounded set $E \subset \mathbb{R}^n$. Show that $\dim_B M = \dim_{MB} M$.

4. Using the definition of box dimension for a bounded subset M of $\mathbb{R}^n$, show that

   (a) $\lim_{\varepsilon\to 0+} \mathcal{N}_\varepsilon(M)\varepsilon^s = \infty$ if $s < \dim_B M$;

   (b) $\lim_{\varepsilon\to 0+} \mathcal{N}_\varepsilon(M)\varepsilon^s = 0$ if $s > \dim_B M$.

   Hence, the quantity

$$\mathcal{C}^s_\varepsilon(M) := \mathcal{N}_\varepsilon(M)\varepsilon^s = \inf\left\{\sum_{i=1}^\infty \varepsilon^s \,:\, \{U_i\} \text{ is a finite cover of } M \text{ by balls of radius } \varepsilon\right\}$$

   plays the role of $\mathcal{H}^s_\varepsilon(M)$. The limit superior of $\mathcal{C}^s_\varepsilon(M)$ as $\varepsilon \to 0+$ is sometimes called the fractal content $\mathcal{C}^s(M)$ of M.