Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 7

1. Show that for the Sierpiński gasket $\mathfrak{S}$ one has: $\dim_H \mathfrak{S} = \dim_B \mathfrak{S} = \log 3/\log 2$.

2. Let $M \subset \mathbb{R}^n$ be bounded. Denote by $N_\delta^*(M)$ the smallest number of sets of diameter $\delta > 0$ with centers in M needed to cover M. Define

$$L^*(M) := \lim_{\delta \to 0+} \frac{N_\delta^*(M)}{-\log \delta}, \quad \text{(provided this limit exists).}$$

   Show that $\dim_B M = L^*(M)$.

3. Let $\{U_i : i \in \mathbb{N}\}$ be a collection of nonempty disjoint open subsets of $\mathbb{R}^n$ such that each U_i contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$, $a_1, a_2, r > 0$. Show that any ball B of radius r intersects at most $(1 + 2a_2)^n a_1^{-n}$ of the closures $\overline{U}_i$, $i \in \mathbb{N}$.

4. Prove that a mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ is a similitude if and only if S is of the form

$$S = H_s \circ O \circ \tau_v,$$

   where H_s, $s > 0$, is a homothety, O an orthogonal transformation, and τ_v a translation by $v \in \mathbb{R}^n \setminus \{0\}$.

5. Show that if $M \subset \mathbb{R}^n$ is nonempty and open, then $\dim_B M = n$.

6. Show that if $M \subset \mathbb{R}^n$ is bounded and $\dim_B M$ exists, then $\dim_B \overline{M} = \dim_B M$, where $\overline{M}$ denotes the closure of M. Hence conclude, using Problem 5. above, that if M is a nonempty open dense set in $\mathbb{R}^n$ then $\dim_B M = n$. Provide an example of such a set.

7. Let $M := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that $\dim_B M = \frac{1}{2}$.