Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 8

1. A *Jordan curve* Γ is the image of a continuous injection $f : [a, b] \to \mathbb{R}^n$. The length of Γ, $L(\Gamma)$, is defined as
$$L(\Gamma) := \sup \sum_{j=1}^{m} |f(t_i) - f(t_{i-1})|,$$
where the supremum is taken over all partitions $a = t_0 < t_1 < \ldots < t_m = b$. Γ is called *rectifiable* if $L(\Gamma) < \infty$, i.e., when f is of bounded variation. Show that for any rectifiable Jordan curve Γ: $\mathcal{H}^1(\Gamma) = L(\Gamma)$.

2. Prove that the graph G of a continuously differentiable function on $[a, b]$ satisfies $0 < \mathcal{H}^1(G) < \infty$.

3. Let G denote the graph of $f : [0, 1] \to \mathbb{R}$. Show that if $\sum_{j=1}^{m} |f(x_i) - f(x_{i-1})|^s \leq c$ for all partitions $0 = x_0 < x_1 \ldots x_m = 1$, then $\mathcal{H}^s(G) < \infty$.

4. Let $f : [a, b] \to \mathbb{R}$ be such that
$$|f(x + h) - f(x)| \leq c\, h^{2-s},$$
for all x and h with $0 < h \leq h_0$, where c and h_0 are positive constants. If G denotes the graph of f, show that $\mathcal{H}^s(G) < \infty$.

5. Let $([a, b], \mathcal{F})$ be an IFS where $\mathcal{F}$ satisfies the OSC with G an open interval. Suppose that each $f_i \in \mathcal{F}$, $i = 1, \ldots, N$, satisfies
$$q_i|x - x'| \leq |f_i(x) - f_i(x')| \leq r_i|x - x'|, \quad \forall x, x' \in \overline{G}.$$
If $\mathfrak{A}$ denotes the fractal set generated by $([a, b], \mathcal{F})$, show that $s \leq \dim_H \mathfrak{A} \leq t$, where s and t are defined by
$$\sum_{i=1}^{N} q_i^s = 1 = \sum_{i=1}^{N} r_i^s.$$
Hint: You may want to prove and then use the following lemma:

Lemma: Suppose that $\{U_j : j \in \mathbb{N}\}$ is a collection of disjoint intervals, each of length at least $2\, c\, \rho$, where $c, \rho > 0$. Then any interval B of length 2ρ intersects at most $2 + c^{-1}$ of the $\overline{U}_j$.