Fractal Geometry
Winter Semester 2010/2011

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Exercise Set 9

1. Prove the Lyapunov Dimension Theorem for IFSs with similitudes satisfying the OCS.

2. Suppose that (J, d_J) and (K, d_K) are compact metric spaces. For a $\theta > 0$ define a metric d on $J \times K$ by
$$d((t,x),(t',x')) := d_J(t,t') + \theta d_K(x,x'). \tag{1}$$
Then $(J \times K, d)$ is a compact metric space. Let $l : J \to J$ and $k : J \times K \to K$ be such that
$$d_J(l(t), l(t')) \le s_1 d_J(t,t'), \quad \forall t, t' \in J, \tag{2}$$
for some $0 \le s_1 < 1$, and
$$d(k(t,x), k(t',x)) \le c d_J(t,t') \quad \forall t,t' \in J,\ \forall x \in K, \tag{3}$$
$$d(k(t,x), k(t,x')) \le s_2 d_K(x,x'), \quad \forall x,x' \in K,\ \forall t \in J, \tag{4}$$
for some $c > 0$ and $0 \le s_2 < 1$. Show that if θ in (1) is chosen equal to $(1-s_1)/2c$, then $W : J \times K \to J \times K$ defined by $W(t,x) := (l(t), k(t,x))$ is a contraction on $J \times K$.

3. With the notation of Exercise 2 above, let $\{W_i : J \times K \to J \times K : i = 1, \ldots, N\}$, $1 < N \in \mathbb{N}$, be a finite set of maps of the form
$$W_i(t,x) = (l_i(t), k_i(t,x)), \tag{5}$$
where l_i and k_i satisfy (2), (3), and (4). Let θ be given as above. Then $(J \times K, \{W_i : i = 1, \ldots, N\})$ is an IFS with unique fractal set G. Moreover, $(J, \{l_i : i = 1, \ldots, N\})$ is an IFS with fractal set I. If $\pi_t : J \times K \to J$ denotes the projection operator defined by $\pi_t(t,x) := t$, for all $(t, x \in J \times K)$, then $I = \pi_i G$.

   (a) Suppose that
   $$\{(t_j, x_j) : j = 0, 1, \ldots, N\} \subset J \times K \tag{6}$$
   and
   $$W_i(t_0, x_0) = (t_{i-1}, x_{i-1}) \quad \text{and} \quad W_i(t_N, x_N) = (t_i, x_i), \quad i = 1, \ldots, N. \tag{7}$$
   Show that $\{(t_j, x_j) : j = 0, 1, \ldots, N\} \subset G$.

   (b) Assume that the maps $l_i : I \to I$ are invertible over their ranges $l_i(I)$ and that
   $$l_i(I) \cap l_j(I) = \emptyset \quad \text{when } |i-j| \notin \{0,1\}, \tag{8}$$
   $$l_i(I) \cap l_{i+1}(I) = t_i, \quad \text{for } i = 1, \ldots, N. \tag{9}$$

   (c) Let $\mathscr{F} := \{f \in C(I,K) : f(t_0) = x_0, f(t_N) = x_N\}$ and let $d_\infty : \mathscr{F} \times \mathscr{F} \to \mathbb{R}$ be given by $d_\infty(f,g) := \max\{d(f(t), g(t)) : t \in I\}$. Show that $(\mathscr{F}, d_\infty)$ is a complete metric space.

   (d) Define $T : \mathscr{F} \to \mathscr{F}$ by
   $$(Tf)(t) := k_i(l_i^{-1}(t), f(l_i^{-1}(t))), \quad t \in l_i(I).$$
   Show that if l_i satisfies conditions (8) and (9), then T is well-defined and continuous at each of the points $\{t_0, t_1, \ldots, t_N\}$. Moreover, $Tf \in \mathscr{F}$ whenever $f \in \mathscr{F}$.

   (e) Show that, if conditions (5) – (9) are satisfied, the operator T is a contraction on the complete metric space $(\mathscr{F}, d_\infty)$ whose unique fixed point is a function $F \in \mathscr{F}$ with graph$(F) = G$ and
   $$F(t_j) = x_j, \quad \forall j = 0, 1, \ldots, N.$$