THE MONOGENIC CURVELET TRANSFORM

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ABSTRACT

In this article, we reconsider the continuous curvelet transform from a signal processing point of view. We show that the analyzing elements of the curvelet transform, the curvelets, can be understood as analytic signals in the sense of the partial Hilbert transform. We then replace the usual curvelets by the monogenic curvelets, which are analytic signals in the sense of the Riesz transform. They yield a new transform, called the monogenic curvelet transform, which has the interesting property that it behaves at the fine scales like the usual curvelet transform and at the coarse scales like the monogenic wavelet transform. In particular, the new transform is highly anisotropic at the fine scales and yields a well-interpretable amplitude/phase decomposition of the transform coefficients over all scales.

Index Terms— Curvelet transform, Analytic signal, Monogenic signal, Hilbert transform, Riesz transform

1. INTRODUCTION

The continuous curvelet transform is a multiscale transform which allows to resolve the singularities of an image together with their orientations, e.g., the positions and the directions of edges [1]. To this end, the curvelet transform increases the anisotropy of its analyzing elements – the curvelets – as the scale decreases, thus the curvelets have higher directional selectivity at the fine scales. However, from a signal processing point of view, the curvelets have another nice property, which has not been exposed so far. That is, the curvelets are analytic signal filters in the sense of the partial Hilbert transform.

Analytic signal filters are important in signal processing, because they yield a meaningful decomposition of the filtered signal into amplitude and phase, where the amplitude has the interpretation of the envelope of the signal. The analytic signal of one-dimensional functions is based on the Hilbert transform. The aforementioned two-dimensional generalization of the analytic signal by the partial Hilbert transform has several drawbacks, which we point out in section 2.2. We resolve these problems by switching to another generalization of the analytic signal, the monogenic signal. Thus we replace the usual curvelets by monogenic curvelets, which yield the new monogenic curvelet transform.

Several authors already use the monogenic signal in image processing. Especially the monogenic wavelet transform [2, 3, 4] opens many new applications like AM/FM analysis [3] or descreening [2]. The disadvantage of the monogenic wavelet transform is their poor directional selectivity. To gain higher anisotropy, the authors in [5] propose higher order Riesz transforms, which leads to a similar approach like the steerable pyramid. However, the degree of directional selectivity of that approach does not adapt to the scale, so the resolution of the orientations of the image singularities is still not optimal [1]. The monogenic curvelets, in contrast, adopt the scale-adaptive anisotropy of the usual curvelets, which results in a better resolution of the orientations.

The article is organized as follows. First we give an introduction to the analytic signal and its generalizations to two dimensions, namely the Hilbert-analytic signal and the monogenic signal. We then identify the usual curvelets as Hilbert-analytic signal. From there we derive the monogenic curvelets and the new monogenic curvelet transform. Finally, we show that the monogenic curvelet transform behaves like the usual curvelet transform at the fine scales and like the monogenic wavelet transform at the coarse scales.

2. ANALYTIC SIGNAL CONCEPTS

Throughout this article we use \((r, \omega)\) for polar coordinates in the frequency domain, and \(x = (x_1, x_2)\) and \(\xi = (\xi_1, \xi_2)\) for Cartesian coordinates in the spatial domain and in the frequency domain, respectively. Further, \(f\) is always a real-valued and square-integrable function.

2.1. The Analytic Signal in 1D

Let \(\mathcal{H} : L^2(\mathbb{R}, \mathbb{R}) \rightarrow L^2(\mathbb{R}, \mathbb{R})\) be the Hilbert transform defined in the Fourier domain by

\[ \widehat{\mathcal{H}f}(s) = i \text{ sign}(s) \hat{f}(s). \]

Note that \(\mathcal{H}f\) is also real-valued. A complex-valued function \(g : \mathbb{R} \rightarrow \mathbb{C}\) whose imaginary part is the Hilbert transform of its real part, i.e., \(\text{Im } g = -\mathcal{H}(\text{Re } g)\), is called analytic signal. In 1D signal processing the analytic signal is used to decompose a signal into amplitude and phase, so loosely spoken into a signal intensity and a signal structure.
For every different $\nu$, $\gamma_{a \theta \nu}(x) = \gamma_{a \theta 0}(\rho \nu(x - b))$

where $\theta$ is a planar rotation by the angle $\theta$ and $\gamma_{a \theta 0}$ is defined by its Fourier transform

$$\tilde{\gamma}_{a \theta 0}(r, \omega) = a^{3/4} W(a r) V\left(\frac{\omega}{\sqrt{a}}\right).$$ (1)
The angular windowing is well-defined only for scales \( a < a_0 \) smaller than a fixed scale \( a_0 \) [1]. Thus for the coarser scales \( a \geq a_0 \) the transform is continued by a purely radial window

\[
\hat{\gamma}_{a00}(r, \omega) = a W(\omega a r) \tag{2}
\]

Note that \( \gamma_{ab\theta} \) is complex-valued for \( a < a_0 \) and real-valued for \( a \geq a_0 \). The (continuous) curvelet transform \( \Gamma_f \) of a function \( f \) is defined by

\[
\Gamma_f : \begin{cases} 
\mathbb{R}^+ \times \mathbb{R}^2 \times [0, 2\pi) & \to \mathbb{C} \\
(a, b, \theta) & \mapsto \langle \gamma_{ab\theta}, f \rangle.
\end{cases} \tag{3}
\]

### 3.2. Interpretation of Curvelets as Hilbert-Analytic Signals

We show that \( \gamma_{ab\theta} \) is a Hilbert-analytic signal for \( a < a_0 \). To this end, we define the real-valued curvelet \( \beta_{a00} \) by symmetrizing \( \hat{\gamma}_{a00} \) with respect to the origin in the Fourier domain

\[
\hat{\beta}_{a00}(\xi_1, \xi_2) := \frac{1}{2}(\hat{\gamma}_{a00}(\xi_1, \xi_2) + \hat{\gamma}_{a00}(-\xi_1, -\xi_2)).
\]

Now a simple calculation (omitting the subscripts) yields \( \gamma_{a00} = \beta_{a00} - i \mathcal{H}_{a00} \). By the translation invariance of \( \mathcal{H}_{a00} \), i.e., \( (\mathcal{H}_{a00} \beta_{a00})(x) = \mathcal{H}_{a00}(\beta_{a00})(x-b) \), and the rotation covariance of \( \mathcal{H}_{a00} \), i.e., \( (\mathcal{H}_{a00} \beta_{a00})\rho x = (\mathcal{H}_{a00} \beta_{a00})(x) \), we get that \( \gamma_{ab\theta} \) is a Hilbert-analytic signal, so

\[
\gamma_{ab\theta} = \text{Re} \gamma_{ab\theta} + i \text{Im} \gamma_{ab\theta} = \beta_{ab\theta} - i \mathcal{H}_{ab\theta} \beta_{ab\theta}. \tag{4}
\]

We will refer to the usual curvelet transform also as Hilbert-analytic curvelet transform.

At this point we bring the monogenic signal into play. In section 2.3 we stated that the proper generalization of the analytic signal is not the Hilbert-analytic signal but the monogenic signal. This motivates to replace the Hilbert-analytic curvelets \( \gamma_{ab\theta} \) by new monogenic curvelets \( \mathcal{M}_{ab\theta} \). This yields a new quaternion-valued transform, which we call monogenic curvelet transform.

### 3.3. Definition and Properties of the Monogenic Curvelet Transform

We define the monogenic curvelet transform \( M_f \) by

\[
M_f : \begin{cases} 
\mathbb{R}^+ \times \mathbb{R}^2 \times [0, 2\pi) & \to \mathbb{H} \\
(a, b, \theta) & \mapsto \langle \mathcal{M}_{ab\theta}, f \rangle,
\end{cases} \tag{5}
\]

where

\[
\langle \mathcal{M}_{ab\theta}, f \rangle = \langle \beta_{ab\theta}, f \rangle + i \langle \mathcal{R}_1(\beta_{ab\theta}), f \rangle + j \langle \mathcal{R}_2(\beta_{ab\theta}), f \rangle.
\]

The monogenic curvelet transform has a Calderón-like reproducing formula

\[
\mathcal{M}_f(x) = \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^2} \langle \mathcal{M}_{ab\theta}, f \rangle \mathcal{M}_{ab\theta}(x) \, db \, d\theta \, \frac{da}{a^3}
\]

and a Parseval formula

\[
\|f\|_2^2 = \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^2} |M_f(a, b, \theta)|^2 \, db \, d\theta \, \frac{da}{a^3}.
\]
3.4. Comparison Between the Hilbert-Analytic Curvelet Transform and the Monogenic Curvelet Transform

A comparison of the complex Hilbert-analytic curvelet coefficients with the quaternionic monogenic curvelet coefficients requires to represent the complex numbers $\Gamma_f(a, b, \theta)$ as quaternions. To this end we embed the curvelet coefficients into the quaternions by an isometric and injective mapping $\iota$ defined by

$$\iota : \Gamma_f(a, b, \theta) \mapsto \Re(\Gamma_f(a, b, \theta)) + \i\cos(\theta) \Im(\Gamma_f(a, b, \theta)) - \j\sin(\theta) \Im(\Gamma_f(a, b, \theta)).$$

With the help of $\iota$ we get for all $a$ and $b$ the inequality

$$|\iota(\Gamma_f(a, b, \theta)) - M_f(a, b, \theta)| \leq a^2 C_{W,V} \|f\|_2$$

with a constant $C_{W,V}$ depending only on the window functions $W$ and $V$. Thus as $a$ tends to 0, the Hilbert-analytic curvelet coefficients and the monogenic curvelet coefficients converge to each other uniformly in $b$ and $\theta$.

At the coarse scales, in contrast, the transforms differ strongly. The concept of the Hilbert-analytic signal is not applicable to the isotropic scales, thus, $\gamma_{a\theta b\theta}$ remains a purely real-valued function for $a \geq a_0$. Hence the amplitude $|\gamma_{a\theta b\theta}|$ boils down to the absolute value of the real numbers. This results in an oscillatory behaviour of $|\Gamma_f(a, \cdot, \theta)|$ (Fig. 3 (a)). The concept of the monogenic signal on the other hand can be applied to all scales, so $M\beta_{a\theta b\theta}$ is an analytic signal also at the coarse scales. Thus the amplitude $|M_f(a, b, \theta)|$ has the interpretation of an envelope of $f$ and consequently does not oscillate (Fig. 3 (b)).

The similarity to the monogenic wavelet transform is more obvious. $M\beta_{a\theta b\theta}$ is a monogenic wavelet for every $a \geq a_0$. Thus for the choice $a_0 = 0$, $M_f$ simplifies to a monogenic wavelet transform as in [2] or [3]. (See Table 1).

Fig. 2. Filters of usual curvelets and monogenic curvelets for some isotropic scale $a \geq a_0$ in the time domain (a) and in the frequency domain (b).

Fig. 3. Amplitude responses of the Hilbert-analytic curvelet transform (a) and of the monogenic curvelets transform (b) for $a \geq a_0$. $\delta$ denotes the Dirac distribution. The Hilbert-analytic amplitude oscillates in radial direction whereas the monogenic amplitude decays monotonously.

4. CONCLUSION

We introduced a new transform which unifies the main advantages of the monogenic wavelet transform and of the curvelet transform. In particular, the monogenic curvelet coefficients split into meaningful amplitude and phase components over all scales. Furthermore, the anisotropy of the analyzing elements increases at the fine scales, which results in excellent directional selectivity.

5. REFERENCES