Time-Frequency Localization Operators and the Berezin Transform

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Outline

Introduction

The time-frequency localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a and windows φ_1, φ_2 is defined to be

$$A_a^{\varphi_1,\varphi_2} f(t) = \iint_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_2} f(x,\omega) M_\omega T_x \varphi_1(t)\, dx d\omega.$$

- introduced by Daubechies in 1988 as a mathematical tool to study time-frequency properties of functions
- in other contexts already used earlier (e.g. by Berezin, anti-Wick operators used for quantization)
- connection to the Berezin transform (well-known in other parts of mathematics)

Time-Frequency Analysis I

- **Time-frequency shifts:**
  for $f \in L^2(\mathbb{R}^d)$, $z = (x, \omega) \in \mathbb{R}^{2d}$:

  $$\pi(z)f(t) = M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t-x)$$

  with $T_x f(t) = f(t-x)$ translation and $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$ modulation
- **Short-time Fourier transform:**

Definition

Let $f, g \in L^2(\mathbb{R}^d)$. Then the *short-time Fourier transform* (STFT) of f with respect to the *window* g is defined as

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega \cdot t} \, dt = \langle f, \pi(z) g \rangle \, .$$

Time-Frequency Analysis II

- **Elementary properties:**
  - $V_g f$ is continuous and vanishes at infinity; it is in particular bounded with $|V_g f(x,\omega)| \leq ||f||_{L^2(\mathbb{R}^d)} \cdot ||g||_{L^2(\mathbb{R}^d)}$
  - $V_g f \in L^2(\mathbb{R}^{2d})$ and the orthogonality relation $\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \overline{\langle g_1, g_2 \rangle_{L^2(\mathbb{R}^d)}}$ holds. If $||g||_{L^2(\mathbb{R}^d)} = 1$, then $V_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is an isometry
  - for $f, g \in L^2(\mathbb{R}^d)$, $x, \omega, u, \eta \in \mathbb{R}^d$:

$$V_g(M_\eta T_u f)(x,\omega) = e^{2\pi i u\cdot\omega} V_g f(x-u, \omega-\eta),$$

    i.e. a time-frequency shift of f amounts to a translation of $V_g f$ in the time-frequency plane

Time-Frequency Analysis III

- **Inversion formula:**

Theorem

Let $g, \gamma \in L^2(\mathbb{R}^d)$ *with* $\langle g, \gamma \rangle \neq 0$. *Then for all* $f \in L^2(\mathbb{R}^d)$

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma \, dx d\omega.$$

The vector-valued integral is to be understood in the weak sense: for $F \in L^2(\mathbb{R}^{2d})$ the expression $\iint_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x \gamma \, dx d\omega$ denotes the unique $f \in L^2(\mathbb{R}^d)$ such that $\langle f, h \rangle = \iint_{\mathbb{R}^{2d}} F(x, \omega) \overline{\langle h, M_\omega T_x \gamma \rangle} \, dx d\omega$ for all $h \in L^2(\mathbb{R}^d)$.

Time-Frequency Localization Operators I

Definition

Let $a \in L^p(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. The *localization operator* with *symbol* a and *windows* φ_1, φ_2 on $L^2(\mathbb{R}^d)$ is defined by

$$A_a^{\varphi_1,\varphi_2} f = \iint_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_2} f(x,\omega) M_\omega T_x \varphi_1(t)\, dx d\omega$$

for $f \in L^2(\mathbb{R}^d)$.
We denote by $\mathcal{A}$ the mapping from symbol to operator:
$\mathcal{A} : a \mapsto \mathcal{A}a := A_a^{\varphi_1,\varphi_2}$.

- analogy with Fourier multipliers $f \mapsto \int_{\mathbb{R}^d} a(\omega) \hat{f}(\omega) e^{2\pi i \omega \cdot t}\, d\omega$

Time-Frequency Localization Operators II

- equivalent formulation:

$$A_a^{\varphi_1,\varphi_2} f = V_{\varphi_1}^*(a \cdot V_{\varphi_2} f)$$

- boundedness on $L^2(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$:
  - $p = 1$:

$$\begin{aligned} |\langle A_a^{\varphi_1,\varphi_2} f, g\rangle| &= |\left\langle V_{\varphi_1}^*(a \cdot V_{\varphi_2} f), g\right\rangle| \\ &\leq ||\varphi_1|| \cdot ||\varphi_2|| \cdot ||a||_1 \cdot ||f|| \cdot ||g||, \end{aligned}$$

    therefore $||A_a^{\varphi_1,\varphi_2} f|| \leq ||\varphi_1|| \cdot ||\varphi_2|| \cdot ||a||_1 \cdot ||f||$
  - $p = \infty$:

$$||A_a^{\varphi_1,\varphi_2} f|| = ||V_{\varphi_1}^*(a \cdot V_{\varphi_2} f)|| \leq ||\varphi_1|| \cdot ||\varphi_2|| \cdot ||a||_\infty \cdot ||f||$$

  - $1 < p < \infty$: by interpolation.

Schatten Class Properties

The following holds:

Theorem (Wong)

For $a \in L^p(\mathbb{R}^{2d})$, we have $A_a^{\varphi_1,\varphi_2} \in \mathcal{S}^p(L^2(\mathbb{R}^d))$, the Schatten p-class, and

$$||A_a^{\varphi_1,\varphi_2}||_{\mathcal{S}^p(L^2)} \leq ||\varphi_1|| \cdot ||\varphi_2|| \cdot ||a||_p,$$

so $\mathcal{A} : L^p(\mathbb{R}^{2d}) \to \mathcal{S}^p(L^2(\mathbb{R}^d))$ bounded with $||\mathcal{A}||_{L^p \to \mathcal{S}^p(L^2)} \leq ||\varphi_1|| \cdot ||\varphi_2||$.

Further Symbol Classes

Localization operators can be defined for many more classes of symbols and windows, for example:

symbol a	windows φ_1, φ_2	operator $A_a^{\varphi_1,\varphi_2}$
$L^\infty(\mathbb{R}^{2d})$	$L^2(\mathbb{R}^d)$	$B(L^2(\mathbb{R}^d))$
$L^p(\mathbb{R}^{2d})$, $1 \leq p < \infty$	$L^2(\mathbb{R}^d)$	$\mathcal{S}^p(L^2(\mathbb{R}^d))$
$\mathcal{S}'(\mathbb{R}^{2d})$	$\mathcal{S}(\mathbb{R}^d)$	$\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$
$M^{p,\infty}(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$	$M^1(\mathbb{R}^d)$	$B(M^{p,q}(\mathbb{R}^d))$, $1 \leq q \leq \infty$

The Berezin Transform I

Definition

Let $T \in B(L^2(\mathbb{R}^d))$. The *Berezin transform* $\mathcal{B}$ maps T to the function on $\mathbb{R}^{2d}$

$$\mathcal{B}T(z) := \langle T\pi(z)\varphi_2, \pi(z)\varphi_1\rangle\,, \qquad z \in \mathbb{R}^{2d}.$$

The function $\mathcal{B}T(z)$ is continuous since for arbitrary $\varphi \in L^2(\mathbb{R}^d)$ the mapping $z \mapsto \pi(z)\varphi$ is continuous from $\mathbb{R}^{2d}$ to $L^2(\mathbb{R}^d)$.

The Berezin Transform II

Lemma

$\mathcal{B}$ is a bounded operator from $\mathcal{S}^p(L^2(\mathbb{R}^d))$ to $L^p(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$.

- $p = \infty$:

$$
\begin{aligned}
|\mathcal{B}T(z)| &= |\langle T\pi(z)\varphi_2, \pi(z)\varphi_1\rangle| \\
&\leq ||T||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \cdot ||\pi(z)\varphi_2|| \cdot ||\pi(z)\varphi_1|| \\
&= ||T||_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \cdot ||\varphi_2|| \cdot ||\varphi_1||
\end{aligned}
$$

  for all $z \in \mathbb{R}^{2d}$
- $p = 1$: using the spectral representation of trace class operators
- $\infty > p > 1$: by interpolation

The Berezin Transform III

The following holds:

Theorem

The operator $\mathcal{A} : L^\infty(\mathbb{R}^{2d}) = (L^1(\mathbb{R}^{2d}))^* \to B(L^2(\mathbb{R}^d)) = (\mathcal{S}^1(L^2(\mathbb{R}^d)))^*$ *is the (Banach space) adjoint of the operator* $\mathcal{B} : \mathcal{S}^1(L^2(\mathbb{R}^d)) \to L^1(\mathbb{R}^{2d})$*, i.e.* $\mathcal{B}^* = \mathcal{A}$*.*

Facts from Functional Analysis

Let X, Y be Banach spaces and $T : X \to Y$ be a bounded operator. Its (Banach space) adjoint operator be denoted by $T^* : Y^* \to X^*$.

- T^* is injective if and only if the range of T is dense in Y with respect to the norm topology on Y.
- T is injective if and only if the range of T^* is dense in X^* with respect to the weak* topology on X^*.

A Negative Result for the Norm Topology

Theorem

The Fourier transform $\mathcal{F} \in \mathcal{B}(L^2(\mathbb{R}^d))$ is not contained in the norm-closure of the range of $\mathcal{A}$. In particular the set of all localization operators with symbols in $L^\infty(\mathbb{R}^{2d})$ is not dense in $\mathcal{B}(L^2(\mathbb{R}^d))$ with respect to the operator norm.

The Idea of the Proof

The intuition (and idea of proof) behind this statement is that whereas the Fourier transform rotates the time-frequency content of a function by 90° in phase space, a localization operator with symbol a (L^∞-multiplyer for the short-time Fourier transform) leaves it essentially unchanged. For suitable $f \in L^2(\mathbb{R}^d)$ (depending on the symbol) the essential time-frequency contents of $\mathcal{F}f$ and $A_a^{\varphi_1,\varphi_2}f$ will be 'almost' disjoint, $\mathcal{F}f$ and $A_a^{\varphi_1,\varphi_2}f$ 'almost' orthogonal, and therefore $||\mathcal{F}f - A_a^{\varphi_1,\varphi_2}f|| \geq c||f||$ for some constant $c > 0$.

Density in the Weak*-Topology

Theorem

Consider localization operators with symbols in $L^\infty(\mathbb{R}^{2d})$. The following conditions are equivalent:

1. *The range $\mathrm{ran}(\mathcal{A})$ is weak* dense in $B(L^2(\mathbb{R}^d))$.*
2. *The Berezin transform $\mathcal{B} : \mathcal{S}^1(L^2(\mathbb{R}^d)) \to L^1(\mathbb{R}^{2d})$ is one-to-one.*
3. *The short-time Fourier transform of the windows φ_1, φ_2 is nonzero almost everywhere, i.e. $V(\varphi_1, \varphi_2)(x, \omega) \neq 0$ for almost all $(x, \omega) \in \mathbb{R}^{2d}$.*

A Very Short Sketch of Proof I

The equivalence of (1) and (2) is clear from the preceding discussion. For the equivalence of (2) and (3), we need some tools:

Definition

Let $f, g \in L^2(\mathbb{R}^d)$. The *Wigner distribution* of f and g is defined by

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i\omega\cdot t}\,dt, \qquad x,\omega \in \mathbb{R}^d.$$

A Very Short Sketch of Proof II

Lemma (Kernel Theorem)

Let $T \in \mathcal{S}^1(L^2(\mathbb{R}^d)) \subset \mathcal{S}^2(L^2(\mathbb{R}^d))$. Then there exists a unique kernel function $\sigma \in L^2(\mathbb{R}^{2d})$ such that

$$\langle Tf, g \rangle = \langle \sigma, W(g, f) \rangle$$

for all $f, g \in L^2(\mathbb{R}^d)$.

Using this, we have
$\mathcal{B}T(z) = \langle \sigma, W(\pi(z)\varphi_1, \pi(z)\varphi_2) \rangle = \langle \sigma, T_z W(\varphi_1, \varphi_2) \rangle$. Injectivity of $\mathcal{B}$ is equivalent to the statement that the subspace spanned by the translates of $W(\varphi_1, \varphi_2)$ is dense in $L^2(\mathbb{R}^d)$, which is in turn equivalent to $\widehat{W}(\varphi_1, \varphi_2)(x, \omega) \neq 0$ for almost all $(x, \omega) \in \mathbb{R}^{2d}$. Since $\widehat{W}(\varphi_1, \varphi_2)(x, \omega) = e^{\pi i x \cdot \omega} V_{\varphi_2}\varphi_1(x, \omega)$, the statement follows.

Further Results

Very much in the same way (though technically harder) there is a family of analogous results for symbols from other classes, e.g.

Theorem

Consider localization operators with symbols in $L^2(\mathbb{R}^{2d})$. The following conditions are equivalent:

1. *The range ran$(\mathcal{A})$ is weak* dense in $\mathcal{S}^2(L^2(\mathbb{R}^d))$, the Hilbert-Schmidt class.*
2. *The Berezin transform $\mathcal{B} : \mathcal{S}^2(L^2(\mathbb{R}^d)) \to L^2(\mathbb{R}^{2d})$ is one-to-one.*
3. *The short-time Fourier transform of the windows φ_1, φ_2 is nonzero almost everywhere, i.e. $V(\varphi_1, \varphi_2)(x, \omega) \neq 0$ for almost all $(x, \omega) \in \mathbb{R}^{2d}$.*

Future Perspectives

- analogous results for other classes of symbols, especially for modulation spaces
- density (or non-density) results for other topologies
- further exploitation of the relation with the Berezin transform

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