Complete interpolating sequences, the discrete Muckenhoupt condition, and conformal mapping

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Definition of complete interpolating sequences

- $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is called complete interpolating sequence if

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- Paley-Wiener space PW_π^2: entire functions of exponential type $\leq \pi$ and in $L^2(\mathbb{R})$.
- f depends continuously on $\{a_n\}$:

$$c\|f\|^2_{L^2(\mathbb{R})} \leq \sum_{n\in\mathbb{Z}} |a_n|^2 \leq C\|f\|^2_{L^2(\mathbb{R})}$$

Riesz bases of exponentials

- Paley-Wiener Theorem: Fourier transform provides isometry

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Fourier transform

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$\{e^{\lambda_n it}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi,\pi)$ (or unconditional basis)

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  $\{e^{\lambda_n it}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi,\pi)$ (or unconditional basis)
- Question: Which sequences are complete interpolating sequences?

Necessary and sufficient conditions for complete interpolating sequences

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- $\{\lambda_n\}$ are the zeros of a sine-type function: sufficient but not necessary condition
- Pavlov's theorem (1979): necessary and sufficient conditions for $\{\lambda_n\}$

Theorem (Pavlov 1979)

A sequence $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ *is a complete interpolating sequence if and only if*

1. *it is separated, i.e.* $\delta := \inf_{n\neq m} |\lambda_n - \lambda_m| > 0$,
2. *the limit*

$$F(z) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{z}{\lambda_n}\right)$$

   exists uniformly on compact subsets of $\mathbb{C}$ *and defines an entire function* F *of exponential type* π*,*

3. *the function* $w(x) := |F(x+iy)|^2,\ x \in \mathbb{R}, y \neq 0$ *satisfies the Muckenhoupt condition*

$$\int_I w(x)dx \int_I \frac{1}{w(x)}dx \leq C|I|^2. \qquad (A_2)$$

Theorem (Pavlov 1979) (Lyubarskii, Seip 1997)

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3. *for a relatively dense subsequence* $\{\lambda_{n_k}\}_{k\in\mathbb{Z}}$ *and the numbers* $d_k := |F'(\lambda_{n_k})|^2$ *we have the discrete Muckenhoupt condition*

$$\sum_{n\in I} d_n \sum_{n\in I} d_n^{-1} \leq C|I|^2 \qquad (\tilde{A}_2)$$

Goals of this work

- give equivalent characterization of complete interpolating sequences
- provide another representation of the generating function

$$F(z) = \lim_{R\to\infty} \prod_{|\lambda_n|<R} \left(1 - \frac{z}{\lambda_n}\right)$$

- inspiration: extend a characterization of sine-type functions (subclass of the class of generating functions of complete interpolating sequences)

Sine-type functions

Definition

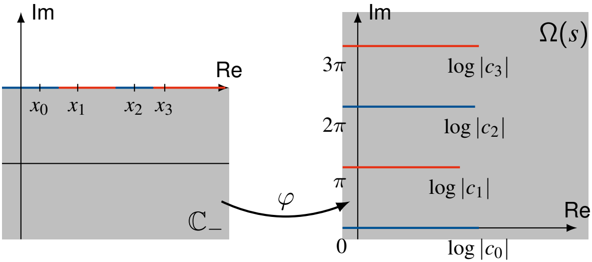
An entire function F of exponential type is called a sine-type function (of type σ) if

1. *F has exponential type σ $(0 < \sigma < \infty)$ in each of the half-planes $\mathbb{C}_+ := \{\mathrm{Im}\, z > 0\}$ and $\mathbb{C}_- := \{\mathrm{Im}\, z < 0\}$,*
2. *the zeros $\{\lambda_n\}_{n\in\mathbb{Z}}$ of F are separated and located in some strip $\{|\mathrm{Im}\, z| \leq h\}$*
3. *for some $y \neq 0,\ C, c > 0$ and all $x \in \mathbb{R}$ holds $c \leq |f(x + \mathrm{i}y)| \leq C$.*

Theorem (Levin 1961, Golovin 1964)

The zeros of a sine-type function of type π are a complete interpolating sequence.

Conformal mappings onto slit domains



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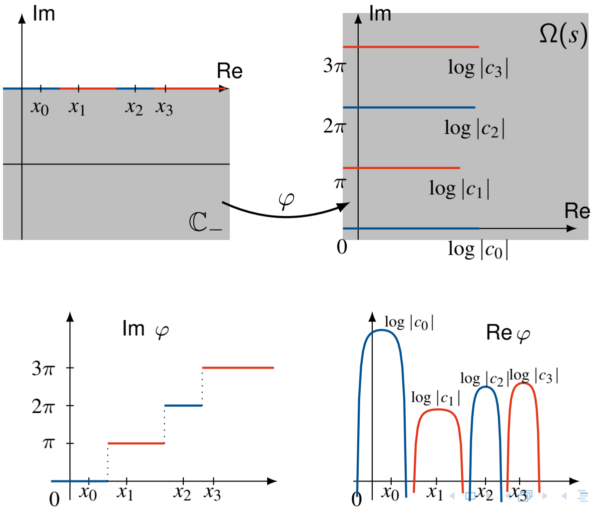
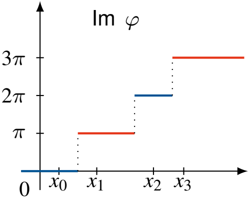
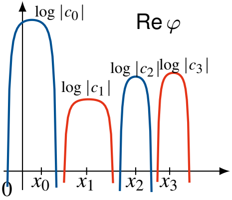
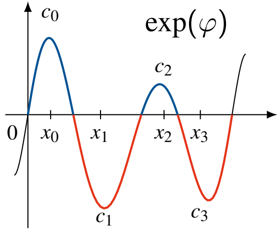


Exponentiating the conformal map







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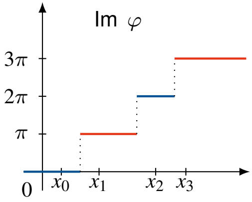
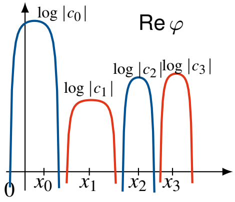
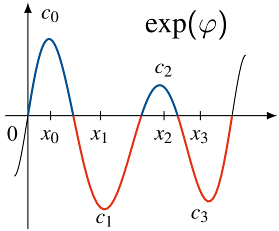
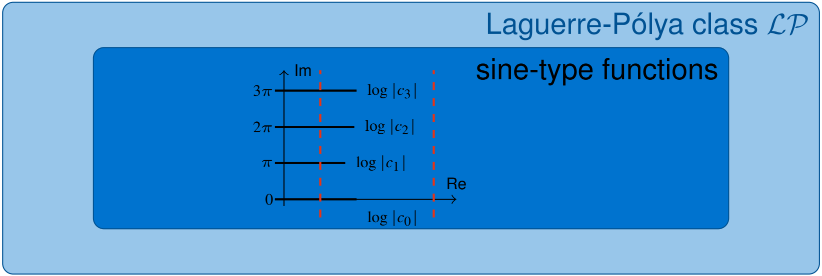






class of functions representable in the form $\exp(\varphi)$
= Laguerre-Pólya class $\mathcal{LP}$
=real entire functions approximable by real polynomials with real zeros uniformly on every compact subset of $\mathbb{C}$

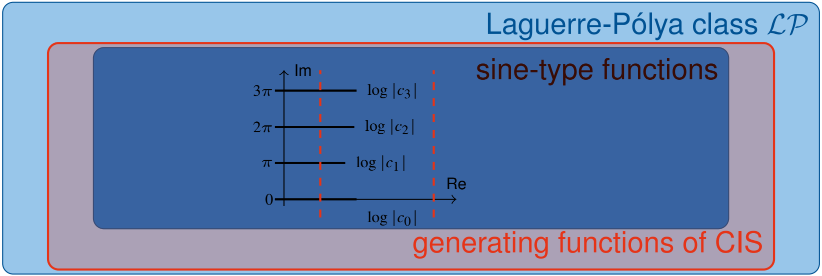
Subclasses of $\mathcal{LP}$



Theorem (Eremenko, Sodin 1992)

A real entire function F is of sine-type if and only if it is of the form $\exp(\varphi)$ with a conformal mapping $\varphi : \mathbb{C}_- \to \Omega(s)$ satisfying $\lim_{y\to-\infty} \operatorname{Re}\varphi(\mathrm{i}y) = \infty$ and $0 < c \le |c_n| \le C$.

Subclasses of $\mathcal{LP}$



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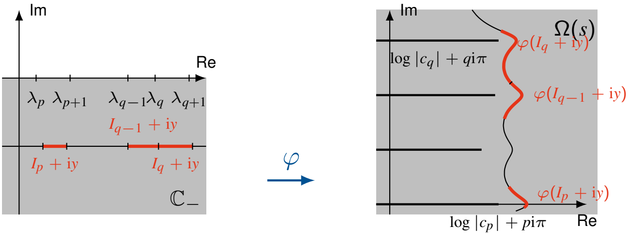
Getting an idea

Question: Which property of the slits characterizes complete interpolating sequences?



$$\varphi(x + \mathrm{i}y) \approx \log|c_n| + \mathrm{i}n\pi, \qquad x \in I_n, n = p, \dots, q$$

$$\int_I |F(x+\mathrm{i}y)|^2 dx \approx \sum_{n=p}^{q} |c_n|^2 |I_n|, \qquad \int_I |F(x+\mathrm{i}y)|^{-2} dx \approx \sum_{n=p}^{q} |c_n|^{-2} |I_n|$$

Characterization of complete interpolating sequences

Theorem

Let $s = \{c_n\}_{n\in\mathbb{Z}}$ *be a sequence with* $(-1)^n c_n \geq 0$*, and let* $\{d_n\}_{n\in\mathbb{Z}} = \{c_n^2\}_{n\in\mathbb{Z}}$ *satisfy the discrete Muckenhoupt condition*

$$\sum_{n\in I} d_n \sum_{n\in I} d_n^{-1} \leq C|I|^2.$$

Then for every conformal map $\varphi : \mathbb{C}_- \to \Omega(s)$ *with* $\lim_{y\to-\infty} \operatorname{Re} \varphi(\mathrm{i}y) = \infty$ *the function* $F = \exp(\varphi)$ *is an entire function of exponential type.* φ *can be normalized so that the exponential type of* F *is* π*, and in this case the zero set* $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ *of* F *is a complete interpolating sequence. Conversely, every complete interpolating sequence is obtained in this way.*

About the proof

Simply use

- distortion theorems for conformal mappings,
- Teichmüller theorems on conformal modules/extremal length,
- Grötzsch principle,
- some elementary geometry,
- Gehring-Hayman theorem,
- Lindelöf principle,
- facts about the Cartwright class,
- basic properties of BMO spaces,
- Poisson representation for the half-plane,
- Rodin-Warschawski theory of parallel strip domains

Uniqueness of the representation

- conformal mapping $\varphi : \mathbb{C}_- \to \Omega(s)$ with

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is unique up to conformal automorphisms of $\mathbb{C}_-$ that fix ∞

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- $\varphi(az + b), a > 0, b \in \mathbb{R}$ describes set of all such conformal maps – for which one has $F = \exp(\varphi)$ exponential type π?

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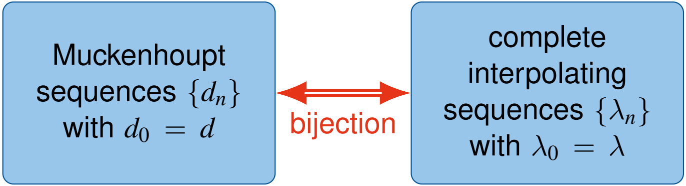
- b has no influence on the type – complete interpolating sequences can be shifted $\lambda_n \mapsto \lambda_n + b$
- correct value of a can be identified by checking Landau's necessary density conditions $D^+ = 1$ or $D^- = 1$ where
$$D^+ := \lim_{r \to \infty} \max_{x \in \mathbb{R}} \frac{|\{\lambda_n\} \cap [x - r, x + r)|}{2r}, \quad D^- := \lim_{r \to \infty} \min_{x \in \mathbb{R}} \dots$$

Discussion of our characterization

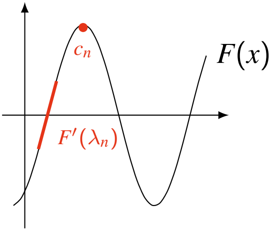
- Parameterization of complete interpolating sequences



- exponential type of generating function follows automatically
- works only for real sequences

Connection with condition of Lyubarskii and Seip

- recall: they proved characterization of complete interpolating sequences involving the discrete Muckenhoupt condition for $d_n = |F'(\lambda_n)|^2$
- our characterization involves $d_n = c_n^2$



- for complete interpolating sequences we have

$$|F'(\lambda_n)| \asymp |c_n|$$

Thank you.