Restricted convergence of Fejér means of trigonometric Fourier series

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The system of functions

$$e^{\imath nx} \quad (n = 0, \pm 1, \pm 2, \dots)$$

$(x \in \mathbb{R}, \imath = \sqrt{-1})$ is called the trigonometric system. Let $T := [-\pi, \pi)$ and $f \in L^1(T)$. The kth Fourier coefficient of f is

$$\hat{f}(k) := \frac{1}{2\pi} \int_T f(t) e^{-\imath kt} dt,$$

where k is any integer number. The nth $(n \in \mathbb{N})$ partial sum of the Fourier series of f:

$$S_n f(y) := \sum_{k=-n}^{n} \hat{f}(k) e^{\imath ky}.$$

The nth $(n \in \mathbb{N})$ Fejér or $(C,1)$ mean of function f:

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y).$$

It is known that

$$\sigma_n f(y) = \frac{1}{\pi} \int_T f(x) K_n(y - x) dx,$$

where the function K_n is known as the nth Fejér kernel

$$K_n(u) = \frac{1}{2(n+1)} \left(\frac{\sin(\frac{u}{2}(n+1))}{\sin(\frac{u}{2})} \right)^2. \tag{1}$$

Let $f \in L^1(T^2)$. The $k = (k_1, k_2)$th Fourier coefficient of f is defined as

$$\hat{f}(k) = \hat{f}(k_1, k_2) := \frac{1}{(2\pi)^2} \int_{T \times T} f(t_1, t_2) e^{-\imath(k_1 t_1 + k_2 t_2)} d(t_1, t_2),$$

where k_1, k_2 are integers. The nth $(n \in \mathbb{N}^2)$ partial sum of the Fourier series of f is

$$S_n f(y) = S_{n_1,n_2} f(y_1, y_2) := \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \hat{f}(k_1, k_2) e^{\imath(k_1 y_1 + k_2 y_2)}.$$

The nth $(n \in \mathbb{N}^2)$ two-dimensional Fejér or $(C,1)$ mean of function f:

$$\sigma_n f(y) = \sigma_{n_1,n_2} f(y) := \frac{1}{(n_1+1)(n_2+1)} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} S_k f(y),$$

$y \in T^2$.

In 1939 Marcinkiewicz and Zygmund [7] proved their celebrated theorem on the convergence of the two-dimensional restricted $(C,1)$ means of trigonometric Fourier series. They proved for any $f \in L^1(T^2)$ the a.e. convergence

$$\sigma_{(n_1,n_2)} f \to f$$

provided $n_1/\beta \le n_2 \le \beta n_1$, where $\beta > 1$ is fixed constant. So, the set of indices (n_1, n_2) remains in some positive cone around the identical function.

In 1935 Jessen, Marcinkiewicz and Zygmund [6] proved

$$\lim_{\wedge n \to \infty} \sigma_n f = f$$

$(\wedge n = \min(n_1, n_2))$ a.e. for functions in $L^1 \log^+ L \subsetneqq L^1(T^2)$.

Problem: What kind of cone-like restriction sets can be given preserving the a.e. convergence of the two-dimensional Fejér means of integrable functions. Gát [5]

Definition 1. [5, Gát] Define the cone-like restriction sets of $\mathbb{N}^2$ as follows

$$\mathbb{N}_{\alpha,\beta,1} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha(n_1)}{\beta(n_1)} \leq n_2 \leq \alpha(n_1)\beta(n_1) \right\},$$

$$\mathbb{N}_{\alpha,\beta,2} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha^{-1}(n_2)}{\beta(n_2)} \leq n_1 \leq \alpha^{-1}(n_2)\beta(n_2) \right\}.$$

For $\alpha(x) = x, \beta(x) = \beta \in (1, +\infty)$ we have $\mathbb{N}_{\alpha,\beta,1} = \mathbb{N}_{\alpha,\beta,2} = \left\{ n \in \mathbb{N}^2 : \frac{1}{\beta} \leq \frac{n_2}{n_1} \leq \beta \right\}$ the "ordinary" restriction set used by Marcinkiewicz and Zygmund (and others). Let

$$\mathbb{N}_{\alpha,i} := \{\mathbb{N}_{\alpha,\beta,i} : \beta > 1\}$$

for $i = 1, 2$. Let $i \in \{1, 2\}$. We say that $\mathbb{N}_{\alpha,i}$ is weaker than $\mathbb{N}_{\alpha,3-i}$, if for all $L \in \mathbb{N}_{\alpha,i}$ there exists an $\tilde{L} \in \mathbb{N}_{\alpha,3-i}$ such that $L \subset \tilde{L}$. This will be abbreviated by $\mathbb{N}_{\alpha,i} \prec \mathbb{N}_{\alpha,3-i}$. If $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$, and $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$, then we call $\mathbb{N}_{\alpha,1}$ and $\mathbb{N}_{\alpha,2}$ equivalent. We abbreviate this by

$$\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}.$$

We say that α is a cone-like restriction function (CRF), if $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$. Now let $\mathbb{N}_\alpha := \mathbb{N}_{\alpha,1} \cup \mathbb{N}_{\alpha,2}$. We say that the cone-like set $L \in \mathbb{N}_\alpha$ is based by the function α. We study the a.e. convergence of the $(C, 1)$ means $\sigma_n f$ of functions integrable that is, $f \in L^1(T^2)$, where $T := [-\pi, \pi) \times [-\pi, \pi)$. We study the convergence restricted by $n \in L, L \in \mathbb{N}_\alpha$, where α is CRF and $\wedge n \to +\infty$. It is natural to ask: How does a cone-like restriction function look like?

Proposition 2. [5, Gát] *Function α is a cone-like restriction function if and only if there exists $\zeta, \gamma_1, \gamma_2 > 1$ such that*

$$\gamma_1 \alpha(x) \leq \alpha(\zeta x) \leq \gamma_2 \alpha(x) \tag{2}$$

holds for each $x \geq 1$.

Theorem 3. *(The convergence* [5, Gát]*) Let α be CRF, $L \in \mathbb{N}_\alpha$. Then for any $f \in L^1(T^2)$ the a.e. equality*

$$\lim_{\substack{\wedge n \to \infty, \\ n \in L}} \sigma_n f = f$$

holds.

Theorem 4. *(The divergence* [5, Gát]*) Let α be CRF, $\beta : [1, +\infty) \to [1, +\infty)$ be a monotone increasing function with property $\lim_{+\infty} \beta = +\infty$, and $\delta : [1, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{+\infty} \delta = 0$. Let $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then there exists a function $f \in L^1 \log^+ L\delta(L)$ such that*

$$\limsup_{\substack{\wedge n \to \infty, \\ n \in L}} \sigma_n f = +\infty$$

a.e.

One might think that if we enlarge the cone based by α, then the convergence space from L^1 to $L^1 \log^+(L)$ (no restriction) changes somehow continuously.

No interim space between L^1 and $L^1\log^+(L)$. These Theorems immediately give

Corollary 5. [5, Gát] *Let α be CRF, $\beta : [1,+\infty) \to [1,+\infty)$ be a monotone increasing function with property $\beta(1) > 1$, and $L := \mathbb{N}_{\alpha,\beta,1}$ or $L := \mathbb{N}_{\alpha,\beta,2}$. Then*

$$\lim_{\substack{\wedge n\to\infty,\\ n\in L}} \sigma_n f = f$$

holds a.e. for all $f \in L^1(T^2)$ if and only if the function β is bounded.

This shows that the theorem of Marcinkiewicz and Zygmund on the convergence of the two-dimensional restricted $(C,1)$ means of trig. Fourier series can not be improved, i.e. the cone based by the identical function can not be enlarged infinitely preserving the a.e. convergence for each integrable functions. Moreover, it gives also a simpler proof of the theorem of Marcinkiewicz and Zygmund.

Corollary 6. [5, Gát] *Let $\beta : [1,+\infty) \to [1,+\infty)$ be a monotone increasing function with property $\beta(1) > 1$, then*

$$\lim_{\substack{\wedge n\to\infty,\\ n_1/\beta(n_1)\le n_2\le n_1\beta(n_1)}} \sigma_n f = f$$

holds a.e. for all $f \in L^1(T^2)$ if and only if the function β is bounded.

The "divergence part" of this corollary for the two-dimensional Walsh-Paley system can be read in [3, Gát] and the "convergence part" in [2, Gát].

Two main tools in the proofs

A decomposition lemma

The dyadic subintervals of T are defined in the following way.

$$\mathcal{I}_0 := \{T\}, \quad \mathcal{I}_1 := \{[-\pi,0),[0,\pi)\},$$
$$\mathcal{I}_2 := \{[-\pi,-\pi/2),[-\pi/2,0),[0,\pi/2),[\pi/2,\pi)\},\ldots$$
$$\mathcal{I} := \bigcup_{n=0}^{\infty} \mathcal{I}_n.$$

The elements of $\mathcal{I}$ are called dyadic intervals. Let functions $\phi_j : [1,+\infty) \to [1,+\infty)$ be monotone increasing and continuous, $\lim_{+\infty}\phi_j = +\infty$ $(j=1,2)$. Set $\psi_j = \lfloor \phi_j \rfloor \quad (j=1,2)$.

Lemma 7. [5, Gát] *$f \in L^1(T^2)$, $\lambda > \|f\|_1/(2\pi)^2$. Then $\exists$ a seq. of integr. func. (f_i) such that*

$$f = \sum_{i=0}^{\infty} f_i, \|f_0\|_\infty \le C\lambda, \quad \|f_0\|_1 \le C\|f\|_1, \text{ and}$$
$$\operatorname{supp} f_i \subset I^{i,1}\times I^{i,2}, \quad \text{where } I^{i,j}\in\mathcal{I} \text{ intervals},$$
$$\operatorname{mes}\left(I^{i,j}\right) = \frac{2\pi}{2^{\psi_j(s_i)}}, \quad \text{for some } s_i \ge 1\,(j=1,2,\ i\in\mathbb{N}\setminus\{0\}).$$

Moreover, $\int_{T^2} f_i(x)dx = 0\,(i\ge 1)$, the dyadic rectangles $I^{i,1}\times I^{i,2}$ disjoint ($i\in\mathbb{N}\setminus\{0\}$), and for

$$F := \bigcup_{i=1}^{\infty}\left(I^{i,1}\times I^{i,2}\right) \quad \text{we have} \quad \operatorname{mes}(F) \le C\|f\|_1/\lambda.$$

Another basic tool: quasi locality

σ_L^* ($L \in \mathbb{N}_\alpha$, α is CRF) is a quasi-local-like

Lemma 8. [5, Gát] *Let α be CRF, $L \in \mathbb{N}_\alpha$, $f \in L^1(T^2)$, and* $\operatorname{supp} f \subset J_1 \times J_2 \in \mathfrak{I} \times \mathfrak{I}$, *with* $\operatorname{mes}(J_j) = \frac{2\pi}{2^{\psi_j(s)}}$ *for some $s \geq 1$ $(j = 1, 2)$. Suppose that*

$$\int_{T^2} f(x_1, x_2)\, d(x_1, x_2) = 0.$$

Then there follows

$$\int_{T^2 \setminus (2J_1 \times 2J_2)} \sigma_L^* f(y_1, y_2)\, d(y_1, y_2) \leq C\|f\|_1.$$

Theorem 9. [5, Gát] *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of weak type (L^1, L^1).*

Corollary 10. [5, Gát] *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of type (L^p, L^p) for all $1 < p \leq \infty$.*

The Walsh system

Dyadic expansion of $n \in \mathbf{N}$ and $x \in I := [0, 1)$

$$n = \sum_{k=0}^{\infty} n_k 2^k$$

$$x = \sum_{k=0}^{\infty} x_k 2^{-k-1}.$$

Let $(\omega_n, n \in \mathbf{N})$ represent the Walsh-Paley system. That is,

$$\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k}.$$

the n-th Walsh-Fourier coefficient of the function $f \in L^1(I)$

$$\hat{f}(n) := \int_I f(x)\omega_n(x)dx.$$

The n-th partial sum of the Walsh-Fourier series of the integrable function $f \in L^1(I)$ is

$$S_n f(y) = \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y).$$

The nth Fejér or $(C, 1)$ mean:

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f.$$

In 1955 Fine [1] proved the Fejér-Lebesgue theorem:

$$\sigma_n f \to f.$$

Set $A_n^\alpha := \frac{(1+\alpha)\ldots(n+\alpha)}{n!}$ for any $n \in \mathbf{N}, \alpha \in \mathbf{R}$, n is not a negative integer.

The (C,α) means of $f \in L^1(I)$:

$$\sigma_{n+1}^{\alpha} f = \frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f.$$

In 1975 Schipp [9] proved $\sigma_n^{\alpha} f \to f$ a.e. for $f \in L^1$ and $\alpha > 0$.
The two-dimensional Walsh-Paley functions:

$$\omega_n(x) := \omega_{n_1}(x^1)\omega_{n_2}(x^2),$$

where $n = (n_1, n_2) \in \mathbf{N}^2$, $x = (x^1, x^2) \in I^2$.
The Fourier coefficients, the (rectangual) partial sums of the Fourier series:

$$\hat{f}(n) := \int_{I^2} f(x)\omega_n(x)dx,$$

$$S_{n_1,n_2} f := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\omega_{k_1,k_2}.$$

The two-dimensional Fejér or $(C,1)$ means of $f \in L(I^2)$

$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} S_{k_1,k_2} f$$

$(n \in \mathbb{P}^2)$.
1992 Móricz, Schipp and Wade: [8] $f \in L\log^+ L(I^2)$ implies

$$\sigma_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$.
Since $L\log^+ L(I^2) \subsetneq L(I^2)$, then it would be useful enlarge the convergence space if possible.
Not possible!
In 2000 Gát proved [3] that for all measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\delta(\infty) = 0$ $\exists f$

$$f \in L\log^+ L\delta(L) \quad \text{and} \quad \sigma_{n_1,n_2} f \not\to f$$

a.e. (in the Pringsheim sense).
What "positive" can be said concerning functions in $L(I^2)$?
Restricted convergence:
1992 Móricz, Schipp and Wade for the two-dimensional Walsh-Paley system:

$$\sigma_{2^{n_1},2^{n_2}} f \to f$$

a.e. for all $f \in L(I^2)$ as $\min\{n_1, n_2\} \to \infty$, provided

$$|n_1 - n_2| \le \alpha$$

for some $\alpha \ge 0$.
For the full indices, i.e. the Marcinkiewicz-Zygmund theorem was proved by Gát and Weisz in 1996 [2, 12]

What about the (C,α) summation of 2-dim. Walsh-Fourier series? Recall: $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$ $(\alpha > 0)$

$$\sigma^\alpha_{n_1+1,n_2+1}f = \frac{1}{A^\alpha_{n_1}A^\alpha_{n_2}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} A^{\alpha-1}_{n_1-k_1}A^{\alpha-1}_{n_2-k_2}S_{k_1,k_2}f.$$

1999 Weisz proved: [13]

$$\sigma^\alpha_{n_1,n_2}f \to f$$

a.e. as $\min\{n_1,n_2\} \to \infty$ for all functions $f \in L\log^+ L(I^2)$ and $\alpha > 0$.

Open problems:

(1) Cone-like restricted Fejér summability of two-dimensional Walsh-Fourier series.
(2) Multidimensional cone-like $(C,1)$ summability of trigonometric Fourier series ($\alpha_{j,k}, j < k, j,k \in \{1,...,d\}$, CRF functions, d is the dimension).
(3) Is it possible to give a larger convergence space for summability method (C,α), where $\alpha > 0$? If $\alpha \leq 1$, then not. Because for the method $(C,1)$ is not possible. What about $\alpha > 1$? Trigonometric, Walsh case.
(4) Problem (3) in the multi-dimensional case is completely open. Trigonometric, Walsh case. (Convergence results with respect to the ordinary restriction are solved for both systems).
(5) Cesàro summability with $\alpha = (\alpha_1,...,\alpha_d)$. No divergence result yet.
(6) What about other summability methods, e.g. logarithmic means:

$$t_nf := \frac{1}{\log^2 n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{S_{j,k}f}{jk}.$$

It is trivial that $f \in L\log^+ L \Rightarrow t_nf \to f$ a.e. Can be given a larger convergence space?
(7) What about the Walsh-Kaczmarz system?

For $n > 0$ denote $|n| := \max(j \in \mathbb{N} : n_j \neq 0)$.

The n-th Walsh-Kaczmarz function

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

for $n > 0$, $\kappa_0(x) := 1, x \in I$.

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}.$$

In 1998 Gát proved [4] the Fejér-Lebesgue theorem for the Walsh-Kaczmarz system. That is, $\sigma_nf \to f$ a.e. for $f \in L(I)$.

In 2004 Simon generalized [11] this with respect to (C,α) summability.

What about the 2-dimensional Walsh-Kaczmarz systerm with respect to Cesàro summability? 2001 Simon: [10]

$$\sigma_{n_1,n_2}f \to f$$

a.e. as $\min\{n_1,n_2\} \to \infty$ (Pringsheim sense) for $f \in L\log^+ L(I^2)$, and restricted convergence for functions in $L(I^2)$.

No divergence result at all, completely open.

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