# Investigations with respect to the maximal operator of Fejér means on Vilenkin systems

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# 1 Fundamental introduction

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo $m_k$. Define the group $G_m$ as the complete direct product of the groups $Z_{m_j}$, with the product of the discrete topologies of $Z_{m_j}$'s.

The direct product $\mu$ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

If the sequence $m$ is bounded, then $G_m$ is called a bounded *Vilenkin group*, else its name is an unbounded one. The elements of $G_m$ can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$ $(x_j \in Z_{m_j})$. It is easy to give a base for the neighborhoods of $G_m$ :

$$I_0(x) := G_m, \qquad I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$$

for $x \in G_m,\ n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_+$.

If we define the so-called generalized number system based on $m$ in the following way: $M_0 := 1, M_{k+1} := m_k M_k (k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum\limits_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of $n_j$'s differ from zero.

The norm (or quasinorm) of the *space* $L_p(G_m)$ is defined by $\mu$.

$$\|f\|_p := \left( \int\limits_{G_m} |f|^p \mu \right)^{1/p} \quad (0 < p < \infty).$$

Next, we introduce on $G_m$ an orthonormal system which is called the *Vilenkin system*. At first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions in this way

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \ (\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as follows.

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_1(G_m)$ [12].

Now, introduce analogues of the usual definitions of the Fourier-analysis. If $f \in L_1(G_m)$ we can establish the following definitions in the usual way:

Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k d\mu \qquad (k \in \mathbb{N}),$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k \qquad (n \in \mathbb{N}_+,\ S_0 f := 0),$$

*Fejér means*:

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_n f \qquad (n \in \mathbb{N}_+).$$

The *space weak-$L_p\left(G_m\right)$* consists of all measurable functions $f$ for which

$$\|f\|_{\text{weak}-L_p(G_m)} := \sup_{\lambda>0} \lambda\mu\left(|f| > \lambda\right)^{1/p} < \infty.$$

The $\sigma$-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $\mathcal{F}_n$. Denote by $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ a martingale with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ (for details see, e. g. [13, 18]).

The *maximal function* of a martingale $f$ is defined by

$$f^* := \sup_{n\in\mathbb{N}} \left|f^{(n)}\right|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) := \sup_{n\in\mathbb{N}} \frac{1}{\mu\left(I_n(x)\right)} \left|\int_{I_n(x)} f(u)\,\mu(u)\right|.$$

For $0 < p < \infty$ the *Hardy martingale spaces* $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G_m)$ then it is easy to show that the sequence $(S_{M_n}(f) : n \in \mathbb{N})$ is a martingale. If $f$ is a martingale, that is $f = (f^{(n)} : n \in \mathbb{N})$, then the Vilenkin-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)}(x)\,\overline{\psi}_i(x)\,\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in \mathbb{N})$ obtained from $f$.

For a martingale $f$ the *maximal operators of the Fejér means* are defined by

$$\sigma^* f(x) := \sup_{n\in\mathbb{N}} |\sigma_n f(x)|.$$

## 2 The boundedness of $\sigma^*$ from $H_p$ to $L_p$

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n f$ is due to Fine [4]. Later, Schipp [9] showed that the maximal operator $\sigma^* f$ is of weak type (1, 1), from which the a. e. convergence follows by standard argument . Schipp result implies by interpolation also the boundedness of $\sigma^* : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$ but Fujii [5] proved that $\sigma^*$ is bounded from the dyadic Hardy space $H_1$ to the space $L_1$.

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1 \quad (\lambda > 0)$$

can be found in Zygmund [21] for the trigonometric series, in Schipp [9] for Walsh series and in Pál, Simon [8] for bounded Vilenkin series. For the one-dimensional bounded Vilenkin system Simon [10] verified that $\sigma^*$ is bounded from $H_1$ to $L_1$. Weisz [14, 20] generalized this results and proved the boundedness of $\sigma^*$ from the martingale Hardy space $H_p$ to the space $L_p$ for $p > 1/2$. Simon [11] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$ Weisz [19] proved that $\sigma^*$ is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. By interpolation it follows that $\sigma^*$ is not bounded from $H_p$ to the space weak-$L_p$ for all $0 < p < 1/2$.

**Theorem 1.** *(Blahota, Gát, Goginava [1]) For any bounded Vilenkin system the maximal operator $\sigma^*$ of the Fejér means is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.*

## 3 The two-dimensional case

Next, we introduce some notation with respect to the theory of *two-dimensional Vilenkin systems*. Let $\tilde{m}$ be a sequence like $m$. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence $(m_n)$ and $(M_n)$. The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. In this paper we also suppose that $m = \tilde{m}$ and $G_m^2 := G_m \times G_m$.

The rectangular partial sums of the double Vilenkin-Fourier series are defined as follows:

$$S_{M,N} f(x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j)\, \psi_i(x^1)\, \psi_j(x^2),$$

where the number

$$\widehat{f}(i,j) := \int_{G_m^2} f\left(x^1,x^2\right)\psi_i\left(x^1\right)\psi_j\left(x^2\right)d\mu\left(x^1,x^2\right).$$

is said to be the $(i,j)$-th Vilenkin-Fourier coefficient of the function $f$.

The norm (or quasinorm) of the *space* $L_p\left(G_m^2\right)$ is defined by

$$\|f\|_p := \left(\int_{G_m^2}\left|f\left(x^1,x^2\right)\right|^p d\mu\left(x^1,x^2\right)\right)^{1/p} \qquad (0<p<\infty).$$

The space weak-$L_p\left(G_m^2\right)$ consists of all measurable functions $f$ for which

$$\|f\|_{\text{weak}-L_p(G_m^2)} := \sup_{\lambda>0}\lambda\mu\left(|f|>\lambda\right)^{1/p} < \infty.$$

Let

$$I_{n,k}\left(x^1,x^2\right) := I_n\left(x^1\right)\times I_k\left(x^2\right).$$

The $\sigma$-algebra generated by the dyadic rectangles $\left\{I_{n,k}\left(x^1,x^2\right):\left(x^1,x^2\right)\in G_m^2\right\}$ will be denoted by $F_{n,k}\,(n,k\in\mathbb{N})$.

Denote by $f:=\left(f^{(n,k)},n,k\in\mathbb{N}\right)$ a martingale with respect to $\left(F_{n,k},n,k\in\mathbb{N}\right)$ (for details see, e. g. [13, 18]

The *maximal function* and the *diagonal maximal function* of a martingale $f$ is defined by

$$f^* := \sup_{n,k\in\mathbb{N}}\left|f^{(n,k)}\right|, \quad f^{\square} := \sup_{n\in\mathbb{N}}\left|f^{(n,n)}\right|,$$

respectively.

In case $f\in L_1\left(G_m^2\right)$, maximal functions can also be given by

$$f^*\left(x^1,x^2\right) = \sup_{n,k\in\mathbb{N}}\frac{1}{\mu\left(I_{n,k}(x^1,x^2)\right)}\left|\int_{I_{n,k}(x^1,x^2)} f\left(u^1,u^2\right)d\mu\left(u^1,u^2\right)\right|$$

and

$$f^{\square}\left(x^1,x^2\right) = \sup_{n\in\mathbb{N}}\frac{1}{\mu\left(I_{n,n}(x^1,x^2)\right)}\left|\int_{I_{n,n}(x^1,x^2)} f\left(u^1,u^2\right)d\mu\left(u^1,u^2\right)\right|,$$

$$\left(x^1,x^2\right)\in G_m^2,$$

respectively.

For $0<p<\infty$ the *Hardy martingale spaces* $H_p(G_m^2)$ and $H_p^{\Box}(G_m^2)$ consists all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty \quad \text{and} \quad \|f\|_{H_p^{\Box}} := \left\|f^{\Box}\right\|_p < \infty.$$

If $f \in L_1\left(G_m^2\right)$ then it is easy to show that the sequence $\left(S_{M_n,M_k}\left(f\right): n,k \in \mathbb{N}\right)$ is a martingale. If $f$ is a martingale, that is $f = (f^{(n,k)} : n,k \in \mathbb{N}.)$ then the Vilenkin-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}\left(i,j\right) := \lim_{\min(k,l)\to\infty} \int_{G_m^2} f^{(k,l)}\left(x^1,x^2\right)\psi_i\left(x^1\right)\psi_j\left(x^2\right)d\mu\left(x^1,x^2\right).$$

The Vilenkin-Fourier coefficients of $f \in L_1\left(G_m^2\right)$ are the same as the ones of the martingale $\left(S_{M_n,M_k}\left(f\right): n,k \in \mathbb{N}\right)$ obtained from $f$.

For $n,k \in \mathbb{P}$ and a martingale $f$ the *Fejér mean* of order $(n,k)$ of the double Vilenkin-Fourier series of the martingale $f$ is given by

$$\sigma_{n,k}f := \frac{1}{nk}\sum_{i=0}^{n-1}\sum_{j=0}^{k-1} S_{i,j}f.$$

For the martingale $f$ the *restricted*, the *unrestricted* and the *diagonal maximal operators* of the Fejér means are defined by the

$$\sigma_\lambda^* f := \sup_{1/M_\lambda \le n/k \le M_\lambda} |\sigma_{n,k}f|, \qquad \sigma^* f := \sup_{n,k}|\sigma_{n,k}f|,$$

$$\sigma_0^* f := \sup_n |\sigma_{n,n}f|.$$

For the two-dimensional Walsh-Fourier series Weisz [16, 17] proved that the following are true

**Theorem A.** *(Weisz [16]) Let $p>1/2$. Then the maximal operator $\sigma_\lambda^*$ is bounded from the Hardy space $H_p^{\Box}$ to the space $L_p$.*

**Theorem B.** *(Weisz [17]) Let $p>1/2$. Then the maximal operator $\sigma^*$ is bounded from the Hardy space $H_p$ to the space $L_p$.*

In Theorems A. and B. the assumption $p>1/2$ is essential even in the case of bounded Vilenkin system. Moreover, we proved that the following is true.

**Theorem 2.** *(Blahota, Gát, Goginava [2]) For any bounded Vilenkin systems the maximal operator $\sigma_0^*$ is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.*

Thus, in question of boundedness of $\sigma_\lambda^*$ and $\sigma^*$, the case of double Vilenkin-Fourier series differs from that one-dimensional Vilenkin-Fourier series. By Theorem 2. and interpolation it follows that $\sigma_0^*$ is not bounded from $H_p$ to weak-$L_p$ for all $0<p<1/2$. In particular, from Theorem 2. we have that in Theorems A. and B. the assumption p ¿ 1/2 is essential. On the other hand, it would be interesting to find a decent space to replace weak-$L_{1/2}$ in order to have the relevant boundedness. However, this question does not seem to be an easy one.

# 4 Marcinkiewicz-Fejér means

The *Marcinkiewicz-Fejér means* with respect to the two-dimensional Vilenkin (or Walsh) system and the corresponding *maximal operator* are defined as follows:

$$\sigma_n^{\square} f := \frac{1}{n} \sum_{j=1}^{n} S_{j,j} f, \qquad \sigma^{\square *} f = \sup_n \left| \sigma_n^{\square} f \right|.$$

In the Walsh-Fourier case this operator is bounded from the two-dimensional dyadic martingale Hardy-Lorentz space $H_p$ to the Lorentz space $L_p$ for $p > 2/3$ (see Weisz [15]). In 2004 Gát proved [6] that for all $f \in L\left(G_m^2\right)$ the Marcinkiewicz-Fejér means of double Vilenkin-Fourier series converge a. e. to $f$. Goginava generalized the theorems of Weisz (see [15], Theorem 3.) and Gát (see [6], Theorem 1.), namely, the following is proved

**Theorem C.** *(Goginava [7]) Let $p > 2/3$. Then the maximal operator $\sigma^{\square *}$ of the Marcinkiewicz-Fejér means of double Vilenkin-Fourier series is bounded from the space $H_p\left(G_m^2\right)$ to the space $L_p\left(G_m^2\right)$.*

For the boundedness of the maximal operator $\sigma^{\square *}$ from the Hardy space $H_p\left(G_m^2\right)$ to the space $L_p\left(G_m^2\right)$ the assumption $p > 2/3$ is essential. The following is true

**Theorem 3.** *(Blahota, Goginava [3]) The maximal operator $\sigma^{\square *}$ of the Marcinkiewicz-Fejér means of the 2-dimensional Vilenkin-Fourier series is not bounded from the Hardy space $H_{2/3}\left(G^2\right)$ to the space $L_{2/3}\left(G^2\right)$.*

Theorems C. and 3. imply

**Corollary 1.** *(Blahota, Goginava [3]) The following conditions are equalent:*

$$\left\| \sigma^{\square *} \right\|_{H_p(G^2) \to L_p(G^2)} < \infty \quad \text{and} \quad p > 2/3.$$

# References


[1] I. Blahota, U. Goginava, G. Gát, *Maximal operators of Fejér means of Vilenkin Fourier series*, (to appear).

[2] ______, *Maximal operators of Fejér means of double Vilenkin-Fourier series, Colloquium Mathematicum*, VOL. 107, 2007 NO 2. 287-296

[3] Blahota, I., Goginava, U., *The Maximal Operator of the Marcinkiewicz-Fejér Means of the 2-Dimensional Vilenkin-Fourier Series*, (to appear).

[4] J. Fine, *Cesàro summability of Walsh-Fourier series*, Proc. Nat. Acad. Sci. USA 41(1955), 558-591.

[5] N. J. Fujii, *Cesàro summability of Walsh-Fourier series*, Proc. Amer. Math. Soc. 77(1979), 111-116.

[6] G. Gát, *Convergence of Marcinkiewicz means of integrable functions with respect to two-dimensional Vilenkin system*, Georg. Math. J. 10(3)(2004), 467-478.

[7] U. Goginava, *Marcinkiewicz-Fejér Means of double Vilenkin-Fourier Series*, Stud. Sci. Math. Hung. (to appear).

[8] J. Pál and P. Simon, *On a generalization of the concept of derivate*, Acta Math. Acad. Sci. Hungar. 29(1977), 155-164.

[9] F. Schipp, *Certain rearrangements of series in the Walsh series*, Mat. Zametki 18(1975), 193-201.

[10] P. Simon, *Investigations with respect to the Vilenkin system*, Annales Univ. Sci. Budapest Eötv., Sect. Math. 28(1985), 87-101.

[11] ———, *Cesaro summability with respect to two-parameter Walsh system*, Monatsh. Math. 131(2000), 321-334.

[12] N. Ya. Vilenkin, *A class of complete orthonormal systems*, Izv. Akad. Nauk. U.S.S.R., Ser. Mat. 11 (1947), 363–400

[13] F. Weisz, *Martingale Hardy Spaces and their Applications in Fourier Analysis*, Springer, Berlin - Heidelberg - New York, 1994.

[14] ———, *Cesàro summability of one and two-dimensional Walsh-Fourier series*, Anal. Math. 22(1996), 229-242.

[15] ———, *Convergence of double Walsh-Fourier series and Hardy spaces* Approx. Theory & its. Appl. 17, 2(2001), 32-44.

[16] ———, *Cesàro summability of two-dimensional Walsh-Fourier series*, Trans. Amer. Math. Soc. 348(1996), 2169-2181.

[17] ———, *The maximal $(C, \alpha, \beta)$ operator of two-paremeter Walsh-Fourier series*, J. Fourier Anal. Appl. 6(2000), 389-401.

[18] ———, *Summability of multi-dimensional Fourier series and Hardy space*, Kluwer Academic, Dordrecht, 2002.

[19] ———, *$\vartheta$-summability of Fourier series*, Acta Math. Hungar. 103(1-2)(2004),139-176.

[20] ———, *Bounded operators on Weak Hardy spaces and applications*, Acta Math. Hungar, 80(1998), 249-264.

[21] A. Zygmund, *Trigonometric series*, vol. 1, Cambridge Univ. Press, 1959.