Directional Haar Wavelet Frames on Triangles

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Outline

Motivation

Basic problem

Construction of a simple system $\{e_i : i \in I\}$ such that every function f of a certain class can be represented by

$$f = \sum_{i \in I} \langle f, e_i \rangle e_i.$$

2D-signals (images)

Traditional approach with tensor product wavelets

$$f(x_1, x_2) = \sum_{(i,j) \in I} \langle f, \psi_i \psi_j \rangle \, \psi_i(x_1) \psi_j(x_2)$$

Important drawback: no rotation invariance $\rightarrow$ no optimal representation of images with orientated discontinous structures

Successful approaches

X-lets

- Contourlets
  - M.N. Do & M. Vetterli, *The contourlet transform: an efficient directional multiresolution image representation*, IEEE Trans. Image Process. **14**(12) (2005), 2091-2106.
  - D.D. Po & M.N. Do, *Directional multiscale modeling of images using the contourlet transform*, IEEE Trans. Image Process. **15** (2006), 1610–1620.
- Curvelets
  - E.J. Candès & D.L. Donoho, *Curvelets – a suprisingly effective nonadaptive representation for objects with edges*, in: Curves and Surfaces, C. Rabut, A. Cohen and L.L. Schumaker (Eds.), Vanderbilt University Press, Nashville, 2000, 105–120.
  - E.J. Candès & D.L. Donoho, *New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities*, Comm. Pure and Appl. Math. **56** (2004), 216–266.
- Shearlets
  - K. Guo, W.-Q. Lim, D. Labate, G. Weiss, & E. Wilson, *Wavelets with composite dilations and their MRA properties*, Appl. Comput. Harmon. Anal. **20** (2006), 231–249.
  - K. Guo & D. Labate, *Optimally sparse multidimensional representation using shearlets*, SIAM J. Math. Anal. **39** (2007), 298–318.

Drawbacks

- Compact support in Fourier domain
- No multiresolution analysis (MRA) associated

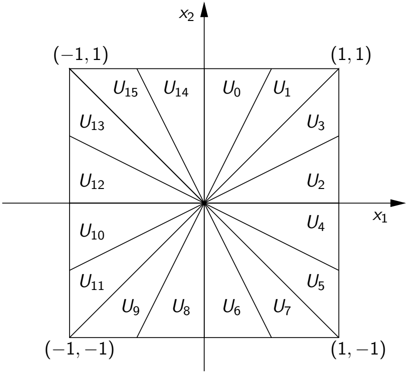
Outline

Domain division

Consider the domain $\Omega := [-1,1]^2$ and divide it into 16 triangles with the same area.

$$U_0 := \text{conv}\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1/2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$$

$$U_1 := \text{conv}\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1/2\\1\end{pmatrix}\}$$



Group of isometries of the square $[-1,1]^2$:
$\mathcal{B} := \{B_i : i = 0, ..., 7\}$ with

$$B_0 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \; B_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \; B_2 = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix},$$

$$B_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \; B_4 = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix},$$

$$B_5 = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}, \; B_6 = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \; B_7 = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}.$$

For $i = 0, ..., 7$:

$$U_{2i} = \{B_i^{-1}x : x \in U_0\} = B_i^{-1}U_0,$$
$$U_{2i+1} = \{B_i^{-1}x : x \in U_1\} = B_i^{-1}U_1.$$

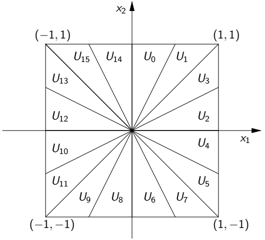
Construction of scaling functions

Define on the triangles U_i *nonseparable scaling functions of Haar type*

$$\begin{aligned}\phi_0(x) &= \phi_0(x_1,x_2) := \chi_{U_0}(x_1,x_2) = \chi_{[0,1]}(\frac{2x_1}{x_2}) \cdot \chi_{[0,1]}(x_2),\\ \phi_1(x) &= \phi_1(x_1,x_2) := \chi_{U_1}(x_1,x_2) = \chi_{[1,2]}(\frac{2x_1}{x_2}) \cdot \chi_{[0,1]}(x_2).\end{aligned}$$

In general for $i = 0, ..., 7$:

$$\begin{aligned}\phi_{2i}(x) &:= \phi_0(B_i x) = \chi_{U_0}(B_i x) = \chi_{B_i^{-1}U_0}(x) = \chi_{U_{2i}}(x),\\ \phi_{2i+1}(x) &:= \phi_1(B_i x) = \chi_{U_1}(B_i x) = \chi_{B_i^{-1}U_1}(x) = \chi_{U_{2i+1}}(x).\end{aligned}$$



MRA properties of scaling functions

Besides *rotation/reflection* $i \in \{0, ..., 7\}$ consider *dilations* $j \in \mathbb{Z}$ and *translations* $k \in \mathbb{Z}^2$:

$$\begin{aligned} \phi_{2i,j,k}(x) &:= 2^j \phi_0(B_i(2^j x - k)), \\ \phi_{2i+1,j,k}(x) &:= 2^j \phi_1(B_i(2^j x - k)). \end{aligned}$$

The sequence $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $L^2(\mathbb{R}^2)$, given by

$$V_j := \mathrm{clos}_{L^2(\mathbb{R}^2)}\mathrm{span}\{\phi_{2i,j,k}, \phi_{2i+1,j,k} : i = 0, ..., 7; k \in \mathbb{Z}^2\},$$

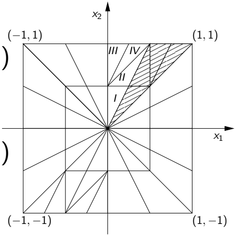
forms a generalized, stationary multiresolution analysis (MRA) of $L^2(\mathbb{R}^2)$. For the following properties are satiesfied:

1. $V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$.
2. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
3. $\mathrm{clos}_{L^2(\mathbb{R}^2)} \bigcup_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R}^2)$.
4. $\{\phi_{2i}(\cdot - k), \phi_{2i+1}(\cdot - k) : i = 0, \dots, 7; k \in \mathbb{Z}^2\}$ forms a frame of V_0.

Refinability of scaling functions

Two-scale relations for ϕ_0 and ϕ_1 ($V_0 \subset V_1$)

$$
\begin{aligned}
\phi_0 &= \phi_0(2\cdot) + \phi_0(2\cdot - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) + \phi_1(2\cdot - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) + \phi_9(2\cdot - \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)) \\
&= \tfrac{1}{2}\left(\phi_{0,1,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)} + \phi_{0,1,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} + \phi_{1,1,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} + \phi_{9,1,\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\right) \\
\phi_1 &= \phi_1(2\cdot) + \phi_1(2\cdot - \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)) + \phi_0(2\cdot - \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)) + \phi_8(2\cdot - \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)) \\
&= \tfrac{1}{2}\left(\phi_{1,1,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)} + \phi_{1,1,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} + \phi_{0,1,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} + \phi_{8,1,\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\right).
\end{aligned}
$$



Two-scale relations for $i = 0, ..., 7$

$$
\begin{aligned}
\phi_{2i} &= \phi_0(B_i\cdot) = \phi_0(2B_i\cdot) + \phi_0(2B_i\cdot - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) + \phi_1(2B_i\cdot - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) + \phi_1(B_4(2B_i\cdot - \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right))) \\
&= \tfrac{1}{2}\left(\phi_{2i,1,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)} + \phi_{2i,1,B_i^{-1}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} + \phi_{2i+1,1,B_i^{-1}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} + \phi_{(2i+9)\bmod 16,1,B_i^{-1}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\right) \\
\phi_{2i+1} &= \tfrac{1}{2}\left(\phi_{2i+1,1,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)} + \phi_{2i+1,1,B_i^{-1}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} + \phi_{2i,1,B_i^{-1}\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} + \phi_{(2i+8)\bmod 16,1,B_i^{-1}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\right).
\end{aligned}
$$

General two-scale relations for $i = 0, ..., 7, j \in \mathbb{Z}, k \in \mathbb{Z}^2$ ($V_j \subset V_{j+1}$)

$$
\begin{aligned}
\phi_{2i,j,k} = \tfrac{1}{2}\Big(&\phi_{2i,j+1,2k} + \phi_{2i,j+1,2k+B_i^{-1}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} + \phi_{2i+1,j+1,2k+B_i^{-1}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} + \\
&+ \phi_{(2i+9)\mathrm{mod}16,j+1,2k+B_i^{-1}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\Big)
\end{aligned}
$$

Frame condition

Definition (frame)

A countable family of elements $\{e_i : i \in I\}$ in V is a *frame* for V if there exist constants $A, B > 0$ such that

$$A \, \|f\|^2_{L^2(\mathbb{R}^2)} \leqslant \sum_{i \in I} |\langle f, e_i \rangle|^2 \leqslant B \, \|f\|^2_{L^2(\mathbb{R}^2)}.$$

A, B are called *frame bounds*. A frame is *tight* if we can choose $A = B$.

Determination of frame bounds A, B

The frame bounds A, B can be got by the eigenvalues of the frame operator S. The optimal lower frame bound A is the smallest eigenvalue for S, and the optimal upper frame bound B is the largest eigenvalue.
In finite-dimensional spaces we can consider the eigenvalues of the Gramian matrix G instead of S.

Frame condition

The family of the 16 functions $\{\phi_i(\cdot - k) : i = 0, ..., 15; k \in \mathbb{Z}^2\}$ provide the following Gram matrix:

$$G := (\langle \phi_i, \phi_{i'} \rangle)_{i,i'=0}^{15} = \frac{1}{4} \begin{pmatrix} I_4 & G_1 & G_2 & G_1^T \\ G_1^T & I_4 & G_1 & G_2 \\ G_2 & G_1^T & I_4 & G_1 \\ G_1 & G_2 & G_1^T & I_4 \end{pmatrix} \in \mathbb{R}^{16\times 16}$$

with the identity matrix I_4 of size 4×4 and with certain matrices G_1 and G_2.

Observation:

- $rank(G) = 11$.
- smallest and largest eigenvalues of G provide the frame bounds $A \approx 0.0745$ and $B = 1$.

$\Longrightarrow \{\phi_i(\cdot - k) : i = 0, ..., 15; k \in \mathbb{Z}^2\}$ is a frame of V_0, i.e. it holds the inequality

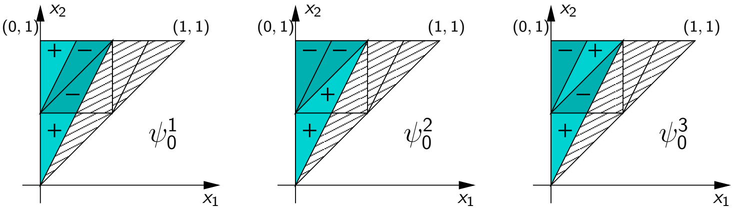
$$A\|f\|^2_{L^2(\mathbb{R}^2)} \leqslant \sum_{i=0}^{15} \sum_{k\in\mathbb{Z}^2} |\langle f, \phi_i(\cdot - k)\rangle|^2 \leqslant B\|f\|^2_{L^2(\mathbb{R}^2)} \qquad \forall f \in V_0.$$

Outline

Construction of directional wavelet functions

The locality and refinability of generating functions ϕ_0 imply to consider the following wavelet functions:

$$\begin{aligned}
\psi_0^1(x) &= \tfrac{1}{2}\left(\phi_{0,1,\binom{0}{0}}(x) + \phi_{0,1,\binom{0}{1}}(x) - \phi_{1,1,\binom{0}{1}}(x) - \phi_{9,1,\binom{1}{2}}(x)\right),\\
\psi_0^2(x) &= \tfrac{1}{2}\left(\phi_{0,1,\binom{0}{0}}(x) - \phi_{0,1,\binom{0}{1}}(x) - \phi_{1,1,\binom{0}{1}}(x) + \phi_{9,1,\binom{1}{2}}(x)\right),\\
\psi_0^3(x) &= \tfrac{1}{2}\left(\phi_{0,1,\binom{0}{0}}(x) - \phi_{0,1,\binom{0}{1}}(x) + \phi_{1,1,\binom{0}{1}}(x) - \phi_{9,1,\binom{1}{2}}(x)\right).
\end{aligned}$$



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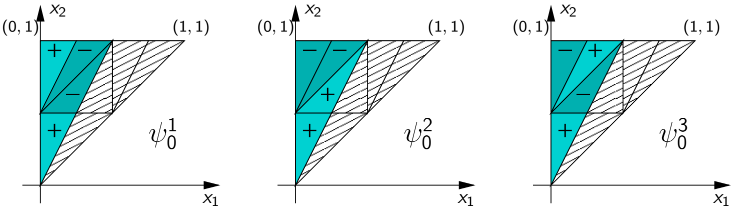
\end{aligned}$$



In general, regarding rotations $i = 0, ..., 7$, dilations $j \in \mathbb{Z}$, and translations $k \in \mathbb{Z}^2$, we get for $r = 1, 2, 3$

$$\psi^r_{2i,j,k} := 2^j \psi^r_0(B_i(2^j \cdot -k)) \quad \text{and} \quad \psi^r_{2i+1,j,k} := 2^j \psi^r_1(B_i(2^j \cdot -k)).$$

Note: These functions can be understood as wavelets with composite dilations ($A = 2I$, $B \in \mathcal{B}$).

Wavelet spaces

Wavelet spaces: $W_j := \mathrm{clos}_{L^2(\mathbb{R}^2)} \operatorname{span} \{\psi^r_{i,j,k} : i = 0, ..., 15;\ r = 1, 2, 3;\ k \in \mathbb{Z}^2\}$

$$\Longrightarrow V_{j+1} = V_j \oplus W_j \quad \forall j \in \mathbb{Z}$$

1. $W_j \subset V_{j+1} \quad \forall j \in \mathbb{Z}$, for general refinement equations hold:

$$\psi^1_{2i,j,k} = \frac{1}{2} \Big(\phi_{2i,j+1,2k} + \phi_{2i,j+1,2k+B_i^{-1}\binom{0}{1}} - \phi_{2i+1,j+1,2k+B_i^{-1}\binom{0}{1}} - \phi_{(2i+9) \bmod 16,j+1,2k+B_i^{-1}\binom{1}{2}} \Big)$$

   and analogous relations for $\psi^2_{i,j,k}$ and $\psi^3_{i,j,k}$.

2. $V_{j+1} = V_j + W_j \quad \forall j \in \mathbb{Z}$, for reconstruction formulas hold.

   E.g.

$$\begin{aligned} \phi_{0,j+1,2k} &= \tfrac{1}{2} \left(\phi_{0,j,k} + \psi^1_{0,j,k} + \psi^2_{0,j,k} + \psi^3_{0,j,k} \right), \\ \phi_{0,j+1,2k+\binom{0}{1}} &= \tfrac{1}{2} \left(\phi_{0,j,k} + \psi^1_{0,j,k} - \psi^2_{0,j,k} - \psi^3_{0,j,k} \right). \\ &\;\;\cdots \end{aligned}$$

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Goal: The set of directional Haar wavelets

$$\Psi_{DH} := \{\psi^r_{i,j,k} : i = 0, \ldots, 15;\ r = 1, 2, 3;\ j \in \mathbb{Z};\ k \in \mathbb{Z}^2\}$$

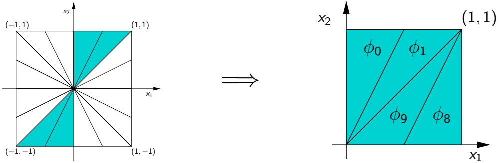
forms a Parseval frame of $L^2(\mathbb{R}^2)$.

Tight wavelet frame

Consider the subspaces V_j^0, V_j^1, V_j^2, and V_j^3 of V_j given by

$$V_j^\nu := \mathrm{clos}_{L^2(\mathbb{R}^2)} \mathrm{span}\, \{\phi_{2\nu,j,k}, \phi_{2\nu+1,j,k}, \phi_{2\nu+8,j,k}, \phi_{2\nu+9,j,k} : k \in \mathbb{Z}^2\}, \quad \nu = 0,1,2,3.$$

- The generating functions form an orthogonal basis of V_j^ν, where $\|\phi_{i,j,k}\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{4}$, $i = 0, \ldots, 15; j \in \mathbb{Z}; k \in \mathbb{Z}^2$.
- $\{V_j^\nu\}_{j\in\mathbb{Z}}$ themselves already form a multiresolution of $L^2(\mathbb{R}^2)$.



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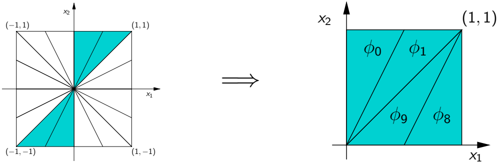


Consider the subspaces W_j^0, W_j^1, W_j^2, and W_j^3 of W_j in the same manner, i.e.,

$$W_j^\nu := \operatorname{clos}_{L^2(\mathbb{R}^2)} \operatorname{span}\{\psi^r_{2\nu,j,k}, \psi^r_{2\nu+1,j,k}, \psi^r_{2\nu+8,j,k}, \psi^r_{2\nu+9,j,k} : r = 1,2,3;\ k \in \mathbb{Z}^2\}.$$

- The generating functions form an orthogonal basis of W_j^ν, $\nu = 0,1,2,3$, where $\|\psi^r_{i,j,k}\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{4}$, $i = 0,\dots,15; j \in \mathbb{Z}; k \in \mathbb{Z}^2, r = 1,2,3$.
- $W_j^\nu \perp V_j^\nu$

Tight wavelet frame

Each generating function set

$$\Psi^{\nu} := \{\psi^r_{2\nu,j,k}, \psi^r_{2\nu+1,j,k}, \psi^r_{2\nu+8,j,k}, \psi^r_{2\nu+9,j,k} : r = 1,2,3;\ j \in \mathbb{Z};\ k \in \mathbb{Z}^2\}, \quad \nu = 0,1,2,3,$$

forms an orthogonal basis of $L^2(\mathbb{R}^2)$ with

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \sum_{\psi \in \Psi^{\nu}} \frac{|\langle f, \psi\rangle|^2}{\langle \psi, \psi\rangle} = 4 \sum_{\psi \in \Psi^{\nu}} |\langle f, \psi\rangle|^2 \qquad \forall f \in L^2(\mathbb{R}^2), \nu = 0,1,2,3.$$

The set of directional Haar wavelets

$$\Psi_{DH} := \{\psi^r_{i,j,k} : i = 0,\dots,15;\ r = 1,2,3;\ j \in \mathbb{Z};\ k \in \mathbb{Z}^2\} = \bigcup_{\nu=0}^{3} \Psi^{\nu}$$

forms a Parseval frame of $L^2(\mathbb{R}^2)$ with

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \sum_{\psi \in \Psi_{DH}} |\langle f, \psi\rangle|^2 \qquad \forall f \in L^2(\mathbb{R}^2).$$

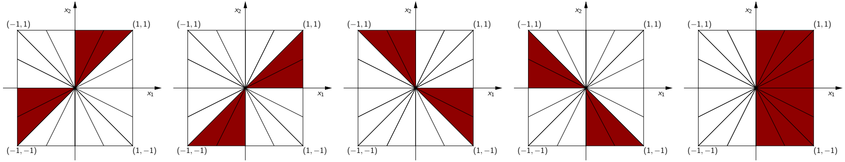
Outline

1. Introduction
2. Construction of scaling functions
3. Construction of directional wavelets
4. Applications to image processing

Redundance relations

The family of functions $\{\phi_i(\cdot - k) : i = 0, ..., 15; k \in \mathbb{Z}^2\}$ generates a frame of V_0.
We have the following dependencies:

$$
\begin{aligned}
\phi_0 + \phi_1 &= \phi_{10}(\cdot - \tbinom{1}{1}) + \phi_{11}(\cdot - \tbinom{1}{1}),\\
\phi_2 + \phi_3 &= \phi_8(\cdot - \tbinom{1}{1}) + \phi_9(\cdot - \tbinom{1}{1}),\\
\phi_4 + \phi_5 &= \phi_{14}(\cdot - \begin{pmatrix} 1 \\ -1 \end{pmatrix}) + \phi_{15}(\cdot - \begin{pmatrix} 1 \\ -1 \end{pmatrix}),\\
\phi_6 + \phi_7 &= \phi_{12}(\cdot - \begin{pmatrix} 1 \\ -1 \end{pmatrix}) + \phi_{13}(\cdot - \begin{pmatrix} 1 \\ -1 \end{pmatrix}),\\
\phi_0 + \phi_1 + \phi_2 + \phi_3 &= \phi_4(\cdot - \tbinom{0}{1}) + \phi_5(\cdot - \tbinom{0}{1}) + \phi_6(\cdot - \tbinom{0}{1}) + \phi_7(\cdot - \tbinom{0}{1}).
\end{aligned}
$$



$\Longrightarrow$ In every 2×2-block of a digital image we have 20 redundance equations.
$\Longrightarrow$ Collect these 20 equations into a matrix $U \in \mathbb{R}^{20 \times 64}$ such that $U\Phi_0 = 0$.

Directional filter bank algorithm

- Input: Initial image in V_0:

$$f = \sum_{k \in J} f_{0,k}^T \, \Phi_0(\cdot - k), \quad f_{0,k} = \langle f, \tilde{\Phi}_0(\cdot - k) \rangle$$

- Add redundancies in V_0:

$$f = \sum_{k \in J} (f_{0,k}^T + d_k^T U) \, \Phi_0(\cdot - k)$$

- Decomposition into $V_{-1} \oplus W_{-1}$ by $A \, \Phi_0 = \Psi_{-1}$

$$f = \sum_{k \in J} (f_{0,k}^T + d_k^T U) A^{-1} \, \Psi_{-1}(\cdot - k)$$

- For each $k \in J$ compute $d_k \in \mathbb{R}^{20}$ such that the l^0-norm

$$\left\| (f_{0,k}^T + d_k^T U) A^{-1} \right\|_0 = \left\| f_{-1,k}^T + d_k^T U A^{-1} \right\|_0$$

  becomes minimal.

- Let $\tilde{f}_{-1,k}^T$ be this minimized coefficient vector. Compute the sparse representation

$$f = \sum_{k \in J} \tilde{f}_{-1,k}^T \, \Psi_{-1}(\cdot - k)$$

Minimization of l^0-norm

Goal:

$$\min_{d_k \in \mathbb{R}^{20}} \left\{ \left\| f_{-1,k}^T + d_k^T U A^{-1} \right\|_0 \right\}$$

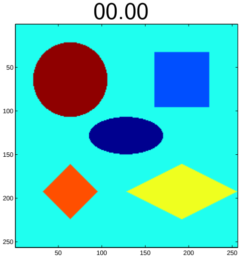
Idea: Choose $d_k \in \mathbb{R}^{20}$ such that $f_{-1,k}^T + d_k^T U A^{-1}$ contains at least 20 zeros.

- Compute the orthogonal projections of f into the subspaces $V_{-1}^{\nu} \oplus W_{-1}^{\nu}$, $\nu = 0, 1, 2, 3$.
- Consider the corresponding 64 coefficients and select the smallest coefficients first.
- Find d_k such that these coefficients will be eliminated.

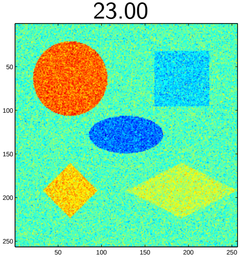
We can prove:

- Our method is optimal for piecewise constant images.
- The resulting sparse representation does not depend on the initial representation of the signal.

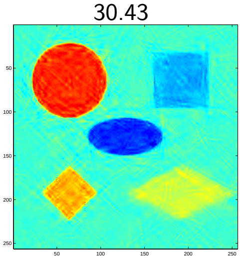
Denoising of piecewise constant images



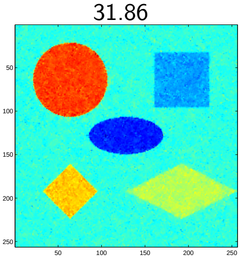
Input image.



Noisy image ($\sigma = 20$).



Denoised by contourlets.



Denoised by our method.

Future prospects

- Smoother scaling/wavelet functions
- More directions: doubling of directions in each level.
- ...

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Thanks for your attention!