Hölder regularity estimation by Hart Smith and Curvelet transforms

Jouni Sampo

Lappeenranta University Of Technology
Department of Mathematics and Physics
Finland

18th September 2007

- This research is done in collaboration with Dr. Songkiat Sumetkijakan (Chulalongkorn University, Department of Mathematics, Bangkok, Thailand)

Outline

Uniform and Pointwice Hölder Regularity

Definition

Let $\alpha > 0$ and $\alpha \notin \mathbb{N}$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *pointwise Hölder regular with exponent* α *at* $\boldsymbol{u}$, denoted by $f \in C^\alpha(\boldsymbol{u})$, if there exists a polynomial $P_{\boldsymbol{u}}$ of degree less than α and a constant $C_{\boldsymbol{u}}$ such that for all $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{u}$

$$|f(\boldsymbol{x}) - P_{\boldsymbol{u}}(\boldsymbol{x} - \boldsymbol{u})| \leq C_{\boldsymbol{u}} \|\boldsymbol{x} - \boldsymbol{u}\|^\alpha. \quad (1)$$

Let Ω be an open subset of $\mathbb{R}^d$. If (1) holds for all $\boldsymbol{x}, \boldsymbol{u} \in \Omega$ with $C_{\boldsymbol{u}}$ being a uniform constant independent of $\boldsymbol{u}$, then we say that f is *uniformly Hölder regular with exponent* α *on* Ω or $f \in C^\alpha(\Omega)$.

Uniform and Pointwise Hölder Exponents

The *uniform* and *pointwise Hölder exponents* of f on Ω and at $\boldsymbol{u}$ are defined as

$$\alpha_l(\Omega) := \sup\{\alpha \colon f \in C^\alpha(\Omega)\}$$

and

$$\alpha_p(\boldsymbol{u}) := \sup\{\alpha \colon f \in C^\alpha(\boldsymbol{u})\}.$$

Local Hölder exponent

Definition

Let $(I_i)_{i\in\mathbb{N}}$ be a family of nested open sets in $\mathbb{R}^d$, i.e. $I_{i+1} \subset I_i$, with intersection $\cap_i I_i = \{\boldsymbol{u}\}$. *The local Hölder exponent* of a function f at $\boldsymbol{u}$, denoted by $\alpha_l(\boldsymbol{u})$, is

$$\alpha_l(\boldsymbol{u}) = \lim_{i\to\infty} \alpha_l(I_i).$$

In many situations, local and pointwise Hölder exponents coincide, e.g., if $f(x) = |x|^\gamma$ then $\alpha_p(\mathbf{0}) = \alpha_l(\mathbf{0}) = \gamma$. However, local Hölder exponents $\alpha_l(\boldsymbol{u})$ is also sensitive to oscillating behavior of f near the point $\boldsymbol{u}$. A simple example is $f(x) = |x|^\gamma \sin\left(1/|x|^\beta\right)$ for which $\alpha_p(\mathbf{0}) = \gamma$ but $\alpha_l(\mathbf{0}) = \frac{\gamma}{1+\beta}$, i.e., α_l is influenced by the wild oscillatory behavior of f near $\mathbf{0}$.

Directional Hölder regularity

Definition

Let $\boldsymbol{v} \in \mathbb{R}^d$ be a fixed unit vector and $\boldsymbol{u} \in \mathbb{R}^d$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is *pointwise Hölder regular with exponent* α *at* $\boldsymbol{u}$ *in the direction* $\boldsymbol{v}$, denoted by $f \in C^\alpha(\boldsymbol{u}; \boldsymbol{v})$, if there exist a constant $C_{\boldsymbol{u},\boldsymbol{v}}$ and a polynomial $P_{\boldsymbol{u},\boldsymbol{v}}$ of degree less than α such that

$$|f(\boldsymbol{u} + \lambda \boldsymbol{v}) - P_{\boldsymbol{u},\boldsymbol{v}}(\lambda)| \leq C_{\boldsymbol{u},\boldsymbol{v}} |\lambda|^\alpha$$

holds for all λ in a neighborhood of $0 \in \mathbb{R}$.
If one can choose $C_{\boldsymbol{u},\boldsymbol{v}}$ so that it is independent of $\boldsymbol{u}$ for all $\boldsymbol{u} \in \Omega \subseteq \mathbb{R}^d$ and the inequality holds for all $\lambda \in \mathbb{R}$ such that $\boldsymbol{u} + \lambda \boldsymbol{v} \in \Omega$, then we say that f is *uniformly Hölder regular with exponent* α *on* Ω *in direction* $\boldsymbol{v}$ or $f \in C^\alpha(\Omega; \boldsymbol{v})$.

Directional Vanishing Moments

Definition

A function f of two variables is said to have an *L-order directional vanishing moments along a direction* $\mathbf{v} = (v_1, v_2)^T$ (suppose that $v_1 \neq 0$; if $v_1 = 0$ then $v_2 \neq 0$ and we can swap the two dimensions) if

$$\int_{\mathbb{R}} t^n f(t, tv_2/v_1 - c)dt = 0, \quad \forall c \in \mathbb{R}, \; 0 \leq n \leq L.$$

- Essentially, the above definition means that any 1-D slices of the function have vanishing moments of order L.

Building Function with Directional Vanishing Moments

- At spatial domain the design is challenging if vanishing moment in many directions are needed
- In frequency domain the design is relatively easy
  - If $\hat{f}^{(n)}(\omega_1, \omega_2)$ vanish at line $\omega_2 = -v_1/v_2\omega_1$ for all $n = 0, \ldots, L$ then f have L-order directional vanishing moments along a direction $\boldsymbol{v} = (v_1, v_2)^T$

Building Function with Directional Vanishing Moments

- At spatial domain the design is challenging if vanishing moment in many directions are needed
- In frequency domain the design is relatively easy
  - If $\hat{f}^{(n)}(\omega_1, \omega_2)$ vanish at line $\omega_2 = -v_1/v_2\omega_1$ for all $n = 0, \ldots, L$ then f have L-order directional vanishing moments along a direction $\boldsymbol{v} = (v_1, v_2)^T$

Kernel of Hart Smith Transform

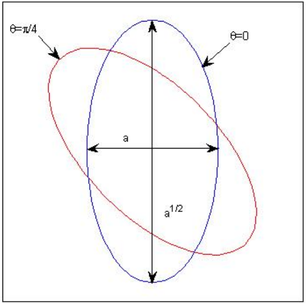
For a given $\varphi \in L^2(\mathbb{R}^2)$, we define

$$\varphi_{a\boldsymbol{b}\theta}(\boldsymbol{x}) = a^{-\frac{3}{4}}\varphi\left(\mathrm{D}_{\frac{1}{a}}\mathrm{R}_{-\theta}\left(\boldsymbol{x}-\boldsymbol{b}\right)\right),$$

for

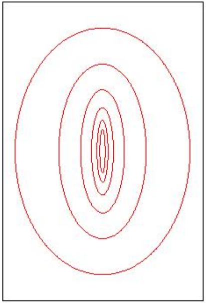
- $\theta \in [0, 2\pi)$, $\boldsymbol{b} \in \mathbb{R}^2$
- $\mathrm{R}_{-\theta}$ is the matrix affecting planar rotation of θ radians in clockwise direction.
- $0 < a < a_0$, where a_0 is a fixed coarsest scale
- $\mathrm{D}_{\frac{1}{a}} = \mathrm{diag}\left(\frac{1}{a}, \frac{1}{\sqrt{a}}\right)$

Kernel of Hart Smith Transform



- Width and length of essential support of kernel function $\varphi_{\boldsymbol{ab}\theta}(\boldsymbol{x})$ are about a and $a^{1/2}$.

Kernel of Hart Smith Transform



- Essential support of kernel function $\varphi_{\mathbf{a}\mathbf{b}\theta}(\mathbf{x})$ become like a needle when scale a become smaller.

Reconstruction Formula for Hart Smith Transform

Theorem

There exists a Fourier multiplier M of order 0 so that whenever $f \in L^2(\mathbb{R}^2)$ is a high-frequency function supported in frequency space $\|\xi\| > \frac{2}{a_0}$, then

$$f = \int_0^{a_0} \int_0^{2\pi} \int_{\mathbb{R}^2} \langle \varphi_{a\mathbf{b}\theta}, Mf \rangle \, \varphi_{a\mathbf{b}\theta} \, d\mathbf{b} \, d\theta \, \frac{da}{a^3} \quad \text{in } L^2(\mathbb{R}^2). \tag{2}$$

The function Mf is defined in the frequency domain by a multiplier formula $\widehat{Mf}(\xi) = m(\|\xi\|)\hat{f}(\xi)$, where m is a standard Fourier multiplier of order 0 (that is, for each $k \geq 0$, there is a constant C_k such that for all $t \in \mathbb{R}$, $|m^{(k)}(t)| \leq C_k \left(1 + |t|^2\right)^{-k/2}$).

Reconstruction Formula for Hart Smith Transform

Because of this and the fact that $\varphi_{\boldsymbol{ab}\theta}$ and $M\varphi_{\boldsymbol{ab}\theta}$ are duals, we can write reconstruction formula also as

$$\begin{aligned} f &= \int_0^{a_0}\int_0^{2\pi}\int_{\mathbb{R}^2} \langle M\varphi_{a\boldsymbol{b}\theta}, f\rangle\, \varphi_{a\boldsymbol{b}\theta}\, d\boldsymbol{b}\, d\theta\, \frac{da}{a^3} \\ &= \int_0^{a_0}\int_0^{2\pi}\int_{\mathbb{R}^2} \langle \varphi_{a\boldsymbol{b}\theta}, f\rangle\, M\varphi_{a\boldsymbol{b}\theta}\, d\boldsymbol{b}\, d\theta\, \frac{da}{a^3}. \end{aligned}$$

- Unlike $\varphi_{a\boldsymbol{b}\theta}$, the dual $M\varphi_{a\boldsymbol{b}\theta}$ do not satisfy true parabolic dilation

Curvelets are defined in Fourier Domain

Let W be a positive real-valued function supported inside $\left(\frac{1}{2}, 2\right)$, called a *radial window*, and let V be a real-valued function supported on $[-1, 1]$, called an *angular window*, for which the following admissibility conditions hold:

$$\int_0^\infty W(r)^2 \frac{dr}{r} = 1 \quad \text{and} \quad \int_{-1}^{1} V(\omega)^2 \, d\omega = 1. \qquad (3)$$

At each scale a, $0 < a < a_0$, $\gamma_{a\mathbf{0}0}$ is defined by

$$\widehat{\gamma_{a\mathbf{0}0}}\left(r\cos(\omega), r\sin(\omega)\right) = a^{\frac{3}{4}} W(ar) V\left(\omega/\sqrt{a}\right) \quad \text{for } r \geq 0 \text{ and } \omega \in [0, 2$$

For each $0 < a < a_0$, $\boldsymbol{b} \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$, a *curvelet* $\gamma_{a\boldsymbol{b}\theta}$ is defined by

$$\gamma_{a\boldsymbol{b}\theta}(\boldsymbol{x}) = \gamma_{a\mathbf{0}0}\left(\mathrm{R}_{-\theta}\left(\boldsymbol{x} - \boldsymbol{b}\right)\right), \quad \text{for } \boldsymbol{x} \in \mathbb{R}^2. \qquad (4)$$

Reconstruction with Curvelets

Theorem

There exists a bandlimited purely radial function Φ such that for all $f \in L^2(\mathbb{R}^2)$,

$$f = \int_{\mathbb{R}^2} \langle \Phi_{\boldsymbol{b}}, f \rangle \, \Phi_{\boldsymbol{b}} \, d\boldsymbol{b} + \int_0^{a_0} \int_0^{2\pi} \int_{\mathbb{R}^2} \langle \gamma_{a\boldsymbol{b}\theta}, f \rangle \, \gamma_{a\boldsymbol{b}\theta} \, d\boldsymbol{b} \, d\theta \, \frac{da}{a^3} \quad \text{in } L^2, \tag{5}$$

where $\Phi_{\boldsymbol{b}}(\boldsymbol{x}) = \Phi(\boldsymbol{x} - \boldsymbol{b})$.

Extra conditions for kernel functions

For regularity analysis we will need that

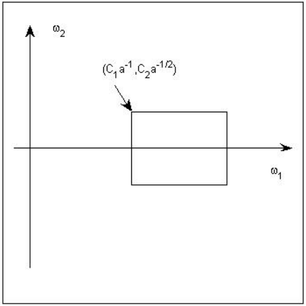
- Kernel functions have enough directional vanishing moments
- Kernel functions and their derivatives up to desired order (largest α of interest) decay fast enough.

Vanishing moments of kernel function

Lemma

There exists $C < \infty$ (independent of $a, \mathbf{b}$ and θ) such that curvelet functions $\gamma_{a\mathbf{b}\theta}$ have directional vanishing moments of any order $L < \infty$ along all directions $\mathbf{v}$ that satisfy $|\angle(\mathbf{v}_\theta, \mathbf{v})| \geq Ca^{1/2}$. Moreover if there exists finite and strictly positive constants C_1, C_1' and C_2 such that $\mathrm{supp}(\hat{\varphi}) \subset [C_1, C_1'] \times [-C_2, C_2]$, then above is true also for functions $\varphi_{a\mathbf{b}\theta}$ and $M\varphi_{a\mathbf{b}\theta}$.

Vanishing moments of kernel function



Frequency support of $\varphi_{ab\theta}(\boldsymbol{x})$.

Decay of kernel function

Lemma

Suppose that the windows V and W in the definition of CCT are C^∞ and have compact supports. Then for each $N = 1, 2, \dots$ there is a constant C_N such that

$$\forall \boldsymbol{x} \in \mathbb{R}^2 \quad |\partial^\nu \gamma_{\boldsymbol{ab}\theta}(\boldsymbol{x})| \leq \frac{C_N a^{-3/4-|\nu|}}{1 + \|\boldsymbol{x} - \boldsymbol{b}\|_{a,\theta}^{2N}}. \tag{6}$$

Moreover, if $\hat{\varphi} \in C^\infty$ and if there exist finite and strictly positive constants C_1, C_1', and C_2 such that $\operatorname{supp}(\hat{\varphi}) \subset [C_1, C_1'] \times [-C_2, C_2]$, then (6) also holds for functions $\varphi_{\boldsymbol{ab}\theta}$ and $M\varphi_{\boldsymbol{ab}\theta}$.

Necessary Condition for Uniform Regularity

Theorem

If a bounded function $f \in C^{\alpha}(\mathbb{R}^2)$*, then there exist a constant* C *and a fixed coarsest scale* a_0 *for which*

$$|\langle \phi_{a\mathbf{b}\theta}, f \rangle| \leq C a^{\alpha + \frac{3}{4}}$$

for all $0 < a < a_0$, $\mathbf{b} \in \mathbb{R}^2$*, and* $\theta \in [0, 2\pi)$.

Sufficient Condition for Uniform Regularity

Theorem

Let $f \in L^2(\mathbb{R}^2)$ *and* $\alpha > 0$ *a non-integer. If there is a constant* $C < \infty$ *such that*

$$|\langle \phi_{\mathbf{a}\mathbf{b}\theta}, f \rangle| \leq C a^{\alpha + \frac{5}{4}},$$

for all $0 < a < a_0$, $\mathbf{b} \in \mathbb{R}^2$, *and* $\theta \in [0, 2\pi)$, *then* $f \in C^\alpha(\mathbb{R}^2)$.

Necessary Condition for Pointwise Regularity

Theorem

If a bounded function $f \in C^{\alpha}(\mathbf{u})$ then there exists $C < \infty$ such that

$$|\langle \phi_{\mathbf{a}\mathbf{b}\theta}, f \rangle| \leq C a^{\frac{\alpha}{2}+\frac{3}{4}} \left(1 + \left\| \frac{\mathbf{b} - \mathbf{u}}{a^{1/2}} \right\|^{\alpha} \right) \quad (7)$$

for all $0 < a < a_0$, $\mathbf{b} \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$.

Sufficient Condition for Pointwise Regularity

Theorem

Let $f \in L^2(\mathbb{R}^2)$ *and* α *be a non-integer positive number. If there exist* $C < \infty$ *and* $\alpha' < 2\alpha$ *such that*

$$|\langle \phi_{\boldsymbol{a}\boldsymbol{b}\theta}, f \rangle| \leq C a^{\alpha + \frac{5}{4}} \left(1 + \left\| \frac{\boldsymbol{b} - \boldsymbol{u}}{a^{1/2}} \right\|^{\alpha'} \right), \tag{8}$$

for all $0 < a < a_0$, $\boldsymbol{b} \in \mathbb{R}^2$, *and* $\theta \in [0, 2\pi)$, *then* $f \in C^\alpha(\boldsymbol{u})$.

Necessary Conditions for direction of Singularity Line

- Now we consider situation that background is sufficiently smooth, i.e. regularity to one direction is higher.

Theorem

Let f be bounded with local Hölder exponent $\alpha \in (0,1]$ *at point* $\mathbf{u}$ *and* $f \in C^{2\alpha+1+\varepsilon}(\mathbb{R}^2, \mathbf{v}_{\theta_0})$ *for some* $\theta_0 \in [0, 2\pi)$ *with any fixed* $\varepsilon > 0$. *Then there exist* $\alpha' \in [\alpha - \varepsilon, \alpha]$ *and* $C < \infty$ *such that for* $a > 0$ *and* $\mathbf{b} \in \mathbb{R}^2$,

$$|\langle \phi_{a\mathbf{b}\theta}, f\rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } \theta \notin \theta_0 + Ca^{1/2}[-1,1] \\ Ca^{\alpha'+\frac{3}{4}}\left(1 + \left\|\frac{\mathbf{b}-\mathbf{u}}{a}\right\|^{\alpha'}\right), & \text{if } \theta \in \theta_0 + Ca^{1/2}[-1,1] \end{cases}$$

Sufficient Conditions for direction of Singularity Line

Theorem

Let $f \in L^2(\mathbb{R}^2)$, $\boldsymbol{u} \in \mathbb{R}^2$, *and assume that* $\alpha > 0$ *is not an integer. If there exist* $\alpha' < 2\alpha$, $\theta_0 \in [0, 2\pi]$, *and* $C < \infty$ *such that*

$$|\langle \phi_{a\boldsymbol{b}\theta}, f\rangle| \leq \begin{cases} Ca^{\alpha+\frac{5}{4}}\left(1+\left\|\frac{\boldsymbol{b}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } \theta \notin \theta_0 + Ca^{1/2}[-1,1] \\ Ca^{\alpha+\frac{3}{4}}\left(1+\left\|\frac{\boldsymbol{b}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } \theta \in \theta_0 + Ca^{1/2}[-1,1] \end{cases}$$

for all $0 < a < a_0$, $\boldsymbol{b} \in \mathbb{R}^2$, *and* $\theta \in [0, 2\pi)$, *then* $f \in C^\alpha(\boldsymbol{u})$.

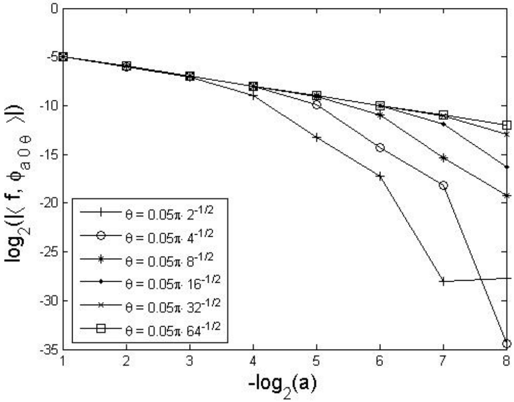


Figure: Decay behavior of $|\langle \phi_{\mathbf{a0}\theta}, f \rangle|$ across scales a at various angles θ for the function $f(\boldsymbol{x}) = e^{-\|\boldsymbol{x}\|}|x_1|^{0.25}$

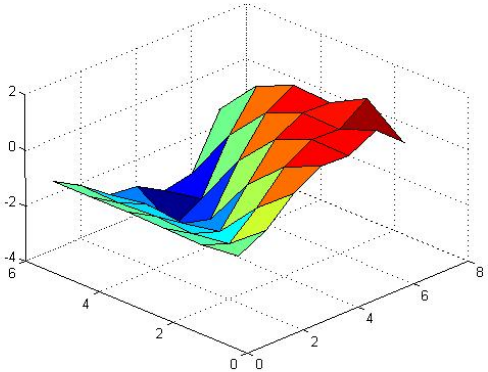


Figure: Estimation errors of the Hölder exponents by $\alpha_e(s,\theta)$ at scales 2^{-s} and angles θ for the function $f(\boldsymbol{x}) = e^{-\|\boldsymbol{x}\|}|x_1|^{0.25}$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
    - Real valued kernel functions if support include reflection respect origin
    - Compact support on Fourier domain may not be necessary
    - Theorems could work for contourlets also
- With some extra assumptions of background regularity
    - Necessary and sufficient conditions would be the same up to ε.
    - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
    - Real valued kernel functions if support include reflection respect origin
    - Compact support on Fourier domain may not be necessary
    - Theorems could work for contourlets also
- With some extra assumptions of background regularity
    - Necessary and sufficient conditions would be the same up to ε.
    - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$

Generalizations

- Assumption $\alpha < 1$ can be removed from all theorems
- Everything holds also for discrete curvelet transform
- Assumptions can be relaxed to hold only at ball of radius ε.
- Assumptions about Fourier properties of kernel functions can be relaxed
  - Real valued kernel functions if support include reflection respect origin
  - Compact support on Fourier domain may not be necessary
  - Theorems could work for contourlets also
- With some extra assumptions of background regularity
  - Necessary and sufficient conditions would be the same up to ε.
  - Similar estimates for C^2 curve (now only for line singularity)
- Generalization from $\mathbb{R}^2$ to $\mathbb{R}^d$