Besov Spaces and Frames on Stratified Lie Groups

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This is joint work with Hartmut Führ.

Outline of talk

- Historical comments
- Basic notations and definitions
- Definition of homogeneous Besov spaces
- Equivalence of norms
- Frame characterization of Besov sapces
- Synthesis Theorem
- References

Historical comments

- Frazier and Jawerth ('85) obtain two types of decompositions for distributions in the space of homogeneous Besov spaces on $\mathbb{R}^n$ for all $s \in \mathbb{R}$
- In the Euclidean case there are many equivalent characterizations of Besov spaces: a useful reference is given by the books by Triebel ('83, '92)
- Y. S.Han and E.T.Sawyer ('94) define Besov spaces on general spaces of homogeneous type, for a range of indices
- Y.Han and D.Yang ('02) give a characterization of these spaces using frames which they construct. These frames cannot be expected to be nearly tight and results are obtained only for smoothness index $s \in (0, 1)$
- Dahmen and Schneider ('99) obtained characterizations of the Besov spaces with wavelet bases on manifolds for $s > 0$
- Geller and Mayeli (preprint '07) characterize the Besov spaces on compact manifolds using nearly tight frames for $-\infty < s < \infty$

Basic notations and definitions

- $\psi_a(x) = a^Q\ \psi(ax)$ $(a > 0,\ Q$ is homogeneous dimension of $G)$
- $\psi_{a,y}(x) = a^Q\ \psi(y^{-1} \cdot ax)\ y \in G$
- $\psi_{j,y} = 2^{jQ}\ \psi(y^{-1} \cdot 2^j x)\ j \in \mathbb{Z}$

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- $\psi \in L^2(G)$ is called a wavelet if for any $f \in L^2(G)$ the following holds:

$$f = \int_{y \in G} \int_{a=0}^{\infty} < f, \psi_{a,y} > \psi_{a,y} \, \frac{dady}{a^{Q+1}}$$

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- For $a > 1$ and discrete set $\Gamma \subset G$ we say ψ generates a nearly tight frame if for any $f \in L^2(G)$

$$A_a \|f\|^2 \leq \sum_{j,\gamma} |\langle f, \psi_{a^j,\gamma} \rangle|^2 \leq B_a \|f\|^2$$

where $A_a/B_a \sim 1$ as $a \to 1$.

Definition of Besov spaces

We begin by fixing $\hat{\psi} \in \mathcal{S}(\mathbb{R}^+)$ with $\sup \hat{\psi} \subset [1/2, 2]$ for which the "dyadic resolution of unity" equation holds:

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^{-2j}\xi)|^2 = 1 \quad a.e\ \xi \in \mathbb{R}^+ \ .$$

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If ψ denotes the associated convolution kernel to the operator $\hat{\psi}(\mathcal{L})$, the associated convolution kernel to the operator $\hat{\psi}(2^{-2j}\mathcal{L})$ will be ψ_j. Therefore in the sense of $\mathcal{S}'(G)/\mathcal{P}$ we have:

$$u = \sum_{j \in \mathbb{Z}} u * \psi_j^* * \psi_j \quad \forall u \in \mathcal{S}'(G)/\mathcal{P} .$$

Definition of Besov spaces *(continued)*

Definition Let $s \in \mathbb{R}$, $1 \leq p,\ q \leq \infty$.

$$\dot{\mathcal{B}}^s_{p,q} := \left\{ u \in \mathcal{S}'(G)/\mathcal{P} :\ \|u\|^{\psi}_{\dot{\mathcal{B}}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \|u * \psi_j^*\|_p^q \right)^{1/q} < \infty \right\}.$$

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Some properties of ψ (Geller/M., JFAA, 2006)

- is schwartz and admissible (wavelet)
- of all vanishing moments: $\int_G x^\alpha \psi\ d = 0$ for all multi-indices α
- ψ generates a nearly tight wavelet frame for L^2
- $\psi_j * \psi_m = \psi_m * \psi_j$ for all $j, m \in \mathbb{Z}$
- $\psi_j * \psi_m = 0$ for $|m - j| > 1$

Definition of Besov spaces *(continued)*

Some properties of $\dot{\mathcal{B}}^s_{p,q}$

- the definition is independent of generating ψ in norm equivalence (this is a direct consequence of next slides)
- $\dot{\mathcal{B}}^s_{p,q}$ is Banach space

Definition of Besov spaces *(continued)*

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- the definition is independent of generating ψ in norm equivalence (this is a direct consequence of next slides)
- $\dot{\mathcal{B}}^s_{p,q}$ is Banach space

Proposition For any $u \in \dot{\mathcal{B}}^s_{p,q}$ the equality $u = \sum_{j\in\mathbb{Z}} u * \psi_j^* * \psi_j$ holds in $\dot{\mathcal{B}}^s_{p,q}$ norm.

Equivalence of norms

Classical definition in terms of the heat kernel associated to the sub-Laplacian

Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, $k \geq 0$ with $k > s/2$. Then we say $u \in \mathcal{S}'(G)/\mathcal{P}$ is in $\dot{\mathcal{B}}^s_{p,q}$ if

$$\left(\int_0^\infty t^{-sq/2} \|(tL)^k e^{-tL} u\|_p^q \, dt/t \right)^{1/q} < \infty$$

This definition has been given for instance by Saka ('79), Furioli ('06)

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Result Littlewood-Paley norm characterization of Besov spaces through wavelets

Equivalence of norms (*continued*)

Theorem For $s \in \mathbb{R}$ and $k > 0$ with $k > s/2,\ k > s/2(1-q)$ and for $u \in \mathcal{S}'/\mathcal{P}$ with assumptions

- $\sum_{j\in\mathbb{Z}} 2^{jsq}\|u * \psi_j^*\|_p^q < \infty$
- $(tL)^k e^{-tL} u \in L^p$

we have the norm equivalence:

$$\left(\int_0^\infty t^{-sq/2}\|(tL)^k e^{-tL}u\|_p^q \, dt/t\right)^{1/q} \asymp \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|u * \psi_j^*\|_p^q\right)^{1/q},$$

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Technical aspects of the proof

- Discrete Calderón formula: $u = \sum_j u * \psi_j^* * \psi_j$,
- $u * \psi_j^* = c_k \int_0^\infty (tL)^{2k-s/2} e^{-tL}(u * \psi_j^*) \, dt/t$
- estimation of $(tL)^k e^{-tL}\psi_j$ in L^1-norm (next lemma)

Equivalence of norms (*continued*)

Lemma For $j \in \mathbb{Z}$, $t > 0$, and any $k > 0$ is

$$\|(tL)^k e^{-tL}\psi_j\|_1 \le Ct^k 2^{2jk} \begin{cases} e^{-c2^{2j}t} & \text{for} \quad 0 < 2^j t \le 1 \\ (2^{2j}t)^Q e^{-c2^{2j}t} & \text{for} \quad \text{otherwise.} \end{cases}$$

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Proof applying the multiplier estimation in Furioli '06 and that ψ has all vanishing moments the assertions hold.

Frame characterization of Besov spaces

The aim of this section is to discretize the Besov norm by sampling the convolution products $u * \psi_j^*$.

Definition Fix a discrete set $\Gamma \subset G$. The coefficient space $\dot{b}^s_{p,q}$ associated to $\dot{\mathcal{B}}^s_{p,q}$ is defined as

$$\dot{b}^s_{p,q} := \left\{ \{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma} : \quad \|\{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma}\|_{\dot{b}^s_{p,q}} < \infty \right\}$$

where

$$\|\{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma}\|_{\dot{b}^s_{p,q}} = \left(\sum_j \left(\sum_\gamma \left(2^{j(s-Q/p)} |c_{j,\gamma}| \right)^p \right)^{q/p} \right)^{1/q}$$

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The whole idea here is to say that

$$u \in \dot{\mathcal{B}}^s_{p,q} \Longleftrightarrow \{\langle u, \psi_{j,\gamma}\rangle\}_{j,\gamma} \in \dot{b}^s_{p,q}$$

Frame characterization of Besov spaces *(continued)*

Theorem There exists a sampling set Γ such that on $\dot{B}^s_{p,q}$ one has the norm equivalence

$$\left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|u*\psi_j^*\|_p^q\right)^{1/q} \asymp \left(\sum_j \left(\sum_\gamma \left(2^{j(s-Q/p)}|\langle u,\psi_{j,\gamma}\rangle|\right)^p\right)^{q/p}\right)^{1/q}$$

Above Theorem is an immediate consequence of the following sampling theorem:

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Above Theorem is an immediate consequence of the following sampling theorem:

Theorem *(Führ, Gröchenig '07)* There exists Γ and constants $0 < c_p \le C_p < \infty$ such that for all $u \in \mathcal{S}'$ satisfying $u * \psi_j^* \in L^p$

$$c_p\, 2^{jQ/p}\|u * \psi_j^*\|_p \le \left(\sum_{\gamma\in\Gamma} |\langle u, \psi_{j,\gamma}\rangle|^p\right)^{1/p} \le C_p 2^{jQ/p}\|u * \psi_j^*\|_p \ .$$

Synthesis Theorem

Theorem Assume that $f = \sum_{j,\gamma} 2^{-jQ} c_{j,\gamma} \psi_{j,\gamma}$ converges in $\mathcal{S}'(G)/\mathcal{P}$. Then the convergence holds unconditionally in the Besov space norm for $\{c_{j,\gamma}\} \in \dot{b}^s_{p,q}$ and

$$\|f\|^{\psi}_{\dot{B}^s_{p,q}} \leq c \left(\sum_j \left(\sum_\gamma \left(2^{j(s-Q/p)} |c_{j,\gamma}| \right)^p \right)^{q/p} \right)^{1/q},$$

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or, in other words, by the definition of Besov norm

$$\|f\|^{\psi}_{\dot{B}^s_{p,q}} = \left\| 2^{ls} \| \sum_{j,\gamma} 2^{-jQ} c_{j,\gamma}\, \psi_{j,\gamma} * \psi^*_l \|_p \right\|_{l^q(\mathbb{Z})} \leq c \sum_j \left(\sum_\gamma \left(2^{j(s-Q/p)} |c_{j,\gamma}|\right)^p \right)^{q/p}.$$

Proof it is immediate by next two technical lemmas, analogues of Lemmas 3.3 and 3.4 of Frazier and Jawerth '85.

Synthesis Theorem *(continued)*

Lemma 1 Let $N, \theta \in \mathbb{N}$. Then for any $j, l \in \mathbb{Z}$, $\gamma \in \Gamma$, and $x \in G$ is:

$$\left|\psi^{j,\gamma} * \psi_l^*(x)\right| \le \begin{cases} c_1\, 2^{-(j-l)N} \; \left(1 + 2^l|2^{-j}\gamma^{-1}\cdot x|\right)^{-(Q+1)} & \text{for} \quad l \le j \\ c_2\, 2^{-(l-j)\theta} \quad \left(1 + 2^j|2^{-j}\gamma^{-1}\cdot x|\right)^{-(Q+1)} & \text{for} \quad l \ge j \end{cases}$$

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Furthermore, suppose that $G = \dot{\cup}_\gamma R_{j,\gamma}$ for any $j \in \mathbb{Z}$ where the sets $R_{j,\gamma}$ are measurable and bounded. Then

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Furthermore, suppose that $G = \dot{\cup}_\gamma R_{j,\gamma}$ for any $j \in \mathbb{Z}$ where the sets $R_{j,\gamma}$ are measurable and bounded. Then

Lemma 2 Let $\eta, j \in \mathbb{Z}$ with $\eta \leq j$ be fixed. Suppose $x_{j,\gamma} \in R_{j,\gamma}$ and $F(x) = \sum_{\gamma \in \Gamma} c_{j,\gamma} f_{j,\gamma}(x)$, where

$$|f_{j,\gamma}(x)| \leq \left(1 + 2^\eta |x_{j,\gamma}^{-1} \cdot x|\right)^{-(Q+1)}.$$

Then for $\{c_{j,\gamma}\} \in l^p(\Gamma)$ is $F \in L^p(G)$ and

$$\|F\|_{L^p} \leq C 2^{(j-\eta)Q} 2^{-jQ/p} \|c_{j,\gamma}\|_{l^p} .$$

Synthesis Theorem *(continued)*

Corollary Presuming that for $u \in \dot{B}^s_{p,q}$ the series $\sum_{j,\gamma} \langle u, \psi_{j,\gamma} \rangle \, \psi^{j,\gamma}$ converges in Besov norm, we get

$$\| \sum_{j,\gamma} \langle u, \psi_{j,\gamma} \rangle \, \psi^{j,\gamma} \|^{\psi}_{\dot{B}^s_{p,q}} \leq c \, \|u\|^{\psi}_{\dot{B}^s_{p,q}}$$

Sketch of Proof

For $u \in \dot{B}^s_{p,q}$ the associate frame coefficients are in $\dot{b}^s_{p,q}$, hence $\sum_{j,\gamma} \langle u, \psi_{j,\gamma} \rangle \, \psi^{j,\gamma}$ converges unconditionally in $\dot{B}^s_{p,q}$ norm.
Now, applying the previous theorem, the assertion holds.

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