A survey on frame theory

Frames in general Hilbert spaces

Construction of dual Gabor frames

Construction of tight wavelet frames

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1 Goal and scope

$(\mathcal{H}, \langle \cdot, \cdot \rangle)$: Hilbert space.

Want: Expansions

$$f = \sum c_k e_k$$

of signals $f \in \mathcal{H}$ in terms of convenient building blocks c_k.

Desirable properties could be:

- Easy to calculate the coefficients c_k
- Only few large coefficients c_k for the relevant signals f.

Part I: Expansions in general Hilbert spaces

- Bases;
- Shortcomings of bases;
- Frames;
- Central themes: tight frames, pairs of dual frames.

Part II: Expansions in $L^2(\mathbb{R})$ of the

(i) Gabor type:

$$e_k \sim e^{2\pi imbx} g(x-na),\ m,n \in \mathbb{Z},$$

for some $g \in L^2(\mathbb{R}),\ a,b > 0$;

(i) Wavelet type:

$$e_k \sim 2^{j/2}\psi(2^j x - k),\ j,k \in \mathbb{Z},$$

for some $\psi \in L^2(\mathbb{R})$.

Central themes:

- Why are frames needed?
- Explicit construction of tight frames and dual frame pairs.

Operators on $L^2(\mathbb{R})$:

Translation by $a \in \mathbb{R}$:

$$T_a : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),\ (T_a f)(x) = f(x - a).$$

Modulation by $b \in \mathbb{R}$

$$E_b : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),\ (E_b f)(x) = e^{2\pi i b x} f(x).$$

Dilation by $a \neq 0$:

$$D_a : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),\ (D_a f)(x) = \frac{1}{\sqrt{|a|}} f(\frac{x}{a}).$$

With this notation, the Gabor system can be written

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x - na)\}_{m,n\in\mathbb{Z}}.$$

The wavelet system has the form ($D := D_{1/2}$)

$$\{D^j T_k \psi\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}.$$

1.1 The Fourier transform

For $f \in L^1(\mathbb{R})$, the *Fourier transform* is defined by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}\,dx, \ \gamma \in \mathbb{R}.$$

The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R})$.

Plancherel's equation:

$$\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle, \ \forall f, g \in L^2(\mathbb{R}), \ \text{and} \ ||\hat{f}|| = ||f||.$$

Important commutator relations:

$$\begin{aligned} \mathcal{F}T_a &= E_{-a}\mathcal{F}, \ \ \mathcal{F}E_a = T_a\mathcal{F}, \\ \mathcal{F}D_a &= D_{1/a}\mathcal{F}, \ \ \mathcal{F}D = D^{-1}\mathcal{F}. \end{aligned}$$

2 Bessel sequences

Let $\mathcal{H}$ be a separable Hilbert space.

Definition 2.1 *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}$ *is called a Bessel sequence if there exists a constant* $B > 0$ *such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \ ||f||^2, \ \forall f \in \mathcal{H}.$$

Theorem 2.2 *Let* $\{f_k\}_{k=1}^{\infty}$ *be a sequence in* $\mathcal{H}$*, and* $B > 0$ *be given. Then* $\{f_k\}_{k=1}^{\infty}$ *is a Bessel sequence with Bessel bound* B *if and only if*

$$T : \{c_k\}_{k=1}^{\infty} \to \sum_{k=1}^{\infty} c_k f_k$$

defines a bounded operator from $\ell^2(\mathbb{N})$ *into* $\mathcal{H}$ *and* $||T|| \leq \sqrt{B}$.

Corollary 2.3 *If* $\{f_k\}_{k=1}^{\infty}$ *is a Bessel sequence in* $\mathcal{H}$*, then* $\sum_{k=1}^{\infty} c_k f_k$ *converges unconditionally for all* $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$.

Pre-frame operator associated to a Bessel sequence:

$$T : \ell^2(\mathbb{N}) \to \mathcal{H},\ T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$$

T is also called the *synthesis operator.*

The adjoint operator - the *analysis operator:*

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}),\quad T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}.$$

The *frame operator:*

$$S : \mathcal{H} \to \mathcal{H},\quad Sf = TT^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k.$$

The series defining S converges unconditionally for all $f \in \mathcal{H}$ by Corollary 2.3.

3 Various bases

Definition 3.1 *Consider a sequence* $\{e_k\}_{k=1}^{\infty}$ *of vectors in* $\mathcal{H}$.

(i) The sequence $\{e_k\}_{k=1}^{\infty}$ *is a basis for* $\mathcal{H}$ *if there for each* $f \in \mathcal{H}$ *exist unique scalar coefficients* $\{c_k(f)\}_{k=1}^{\infty}$ *such that*

$$f = \sum_{k=1}^{\infty} c_k(f) e_k.$$

(ii) A basis $\{e_k\}_{k=1}^{\infty}$ *is an unconditional basis if the series in (i) converges unconditionally for each* $f \in \mathcal{H}$.

(iii) A basis $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis if* $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal system, i.e., if*

$$\langle e_k, e_j \rangle = \delta_{k,j}.$$

Theorem 3.2 *For an orthonormal system* $\{e_k\}_{k=1}^{\infty}$*, the following are equivalent:*

(i) $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis.*

(ii) $f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k, \ \forall f \in \mathcal{H}$.

(iii) $\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, e_k \rangle \langle e_k, g \rangle, \ \forall f, g \in \mathcal{H}$.

(iv) $\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 = ||f||^2, \ \forall f \in \mathcal{H}$.

(v) $\overline{span}\{e_k\}_{k=1}^{\infty} = \mathcal{H}$.

(vi) If $\langle f, e_k \rangle = 0, \ \forall k \in \mathbb{N}$, *then* $f = 0$.

Corollary 3.3 *If* $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis, then each* $f \in \mathcal{H}$ *has an unconditionally convergent expansion*

$$f = \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k.$$

Theorem 3.4 *Let* $\{e_k\}_{k=1}^{\infty}$ *be an orthonormal basis for* $\mathcal{H}$*. Then the orthonormal bases for* $\mathcal{H}$ *are precisely the sets* $\{Ue_k\}_{k=1}^{\infty}$*, where* $U : \mathcal{H} \to \mathcal{H}$ *is a unitary operator.*

Proposition 3.5 *Assume that* $\{e_k\}_{k=1}^{\infty}$ *is a sequence of normalized vectors in* $\mathcal{H}$ *and that*

$$\sum_{k=1}^{\infty} |\langle f, e_k \rangle|^2 = ||f||^2, \ \forall f \in \mathcal{H}.$$

Then $\{e_k\}_{k=1}^{\infty}$ *is an orthonormal basis for* $\mathcal{H}$*.*

3.2 Riesz bases

Definition 3.6 *A Riesz basis for $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.*

Theorem 3.7 *If $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$, there exists a unique sequence $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that*

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

The sequence $\{g_k\}_{k=1}^{\infty}$ is also a Riesz basis, and the series converges unconditionally for all $f \in \mathcal{H}$.

$\{g_k\}_{k=1}^{\infty}$ is called the *dual Riesz basis.*

From the proof:

$$\{g_k\}_{k=1}^{\infty} = \{(U^{-1})^* e_k\}_{k=1}^{\infty}.$$

Dual of $\{g_k\}_{k=1}^{\infty}$:

$$\left\{ \left(\left((U^{-1})^* \right)^{-1} \right)^* e_k \right\}_{k=1}^{\infty} = \{U e_k\}_{k=1}^{\infty} = \{f_k\}_{k=1}^{\infty}.$$

So $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are duals of each other: they are called a *pair of dual Riesz bases.*

Definition 3.8 *Two sequences* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *in a Hilbert space are biorthogonal if*

$$\langle f_k, g_j \rangle = \delta_{k,j}.$$

Corollary 3.9 *For a pair of dual Riesz bases* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *the following hold:*

(i) $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are biorthogonal.*

(ii) For all $f \in \mathcal{H}$,

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k.$$

Proposition 3.10 *If* $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ *is a Riesz basis for* $\mathcal{H}$, *there exist constants* $A, B > 0$ *such that*

$$A\ ||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\ ||f||^2,\ \forall f \in \mathcal{H}.$$

The largest possible value for the constant A *is* $\frac{1}{||U^{-1}||^2}$, *and the smallest possible value for* B *is* $||U||^2$.

Theorem 3.11 *For a sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{H}$, the following conditions are equivalent:*

(i) $\{f_k\}_{k=1}^\infty$ *is a Riesz basis for* $\mathcal{H}$.

(ii) $\{f_k\}_{k=1}^\infty$ *is complete in* $\mathcal{H}$, *and there exist constants* $A, B > 0$ *such that for every finite scalar sequence* $\{c_k\}$ *one has*

$$A \sum |c_k|^2 \leq \left|\left| \sum c_k f_k \right|\right|^2 \leq B \sum |c_k|^2.$$

4 The expansion property for a non-basis:

Example 4.1 The family

$$\{e_k\}_{k\in\mathbb{Z}} = \{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$$

forms an ONB for $L^2(0,1)$.

Consider an open subinterval $I \subset]0,1[$ with $|I| < 1$.

Identify $L^2(I)$ with the subspace of $L^2(0,1)$ consisting of the functions which are zero on $]0,1[\backslash I$.

For a function $f \in L^2(I)$:

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k \ \text{ in } \ L^2(0,1). \qquad (1)$$

Since

$$\left\| f - \sum_{|k|\le n} \langle f, e_k\rangle e_k \right\|^2_{L^2(I)} \to 0 \text{ as } n \to \infty,$$

we also have

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k \quad \text{in} \quad L^2(I). \tag{2}$$

That is, the (restrictions to I of the) functions $\{e_k\}_{k\in\mathbb{Z}}$ also have the expansion property in $L^2(I)$. However, they are not a basis for $L^2(I)$! To see this, define the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in I, \\ 1 & \text{if } x \notin I. \end{cases}$$

Then $\tilde{f} \in L^2(0,1)$ and we have the representation

$$\tilde{f} = \sum_{k\in\mathbb{Z}} \langle \tilde{f}, e_k\rangle e_k \quad \text{in} \quad L^2(0,1). \tag{3}$$

By restricting to I, the expansion (3) is also valid in $L^2(I)$; since $f = \tilde{f}$ on I, this shows that

$$f = \sum_{k\in\mathbb{Z}} \langle \tilde{f}, e_k\rangle e_k \quad \text{in} \quad L^2(I). \tag{4}$$

Thus, (2) and (4) are both expansions of f in $L^2(I)$, and they are non-identical; the argument is that since $f \neq \tilde{f}$ in $L^2(0,1)$, the expansions (1) and (3) show that

$$\{\langle f, e_k \rangle\}_{k \in \mathbb{Z}} \neq \{\langle \tilde{f}, e_k \rangle\}_{k \in \mathbb{Z}}.$$

Conclusion: The restriction of the functions $\{e_k\}_{k \in \mathbb{Z}}$ to I is not a basis for $L^2(I)$, but the expansion property is preserved. $\square$

It is natural to consider sequences which are not bases, but nevertheless have the expansion property.

5 Frames in Hilbert spaces

Definition 5.1 *A sequence* $\{f_k\}_{k=1}^{\infty}$ *of elements in* $\mathcal{H}$ *is a frame for* $\mathcal{H}$ *if there exist constants* $A, B > 0$ *such that*

$$A\ ||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\ ||f||^2, \quad \forall f \in \mathcal{H}.$$

The numbers A and B are called *frame bounds.* The *optimal upper frame bound* is the infimum over all upper frame bounds, and the *optimal lower frame bound* is the supremum over all lower frame bounds.

Lemma 5.2 *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *is a sequence of elements in* $\mathcal{H}$ *and that there exist constants* $A, B > 0$ *such that*

$$A\ ||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\ ||f||^2$$

for all f *in a dense subset* V *of* $\mathcal{H}$*. Then* $\{f_k\}_{k=1}^{\infty}$ *is a frame for* $\mathcal{H}$ *with bounds* A, B*.*

Definition 5.3 *A sequence* $\{f_k\}_{k=1}^{\infty}$ *in* $\mathcal{H}$ *is a tight frame if there exist a number* $A > 0$ *such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 = A\,||f||^2, \ \forall f \in \mathcal{H}.$$

The (exact) number A *is called the frame bound.*

Recall: the pre-frame operator is

$$T : \mathcal{H} \to \ell^2(\mathbb{N}), \ T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k.$$

Frame operator associated to general frame $\{f_k\}_{k=1}^{\infty}$:

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = TT^*f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

$\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, so the series defining S converges unconditionally for all $f \in \mathcal{H}$ by Corollary 2.3.

Some important properties of S:

Lemma 5.4 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds A, B. Then the following holds:*

(i) S is bounded, invertible, self-adjoint, and positive.

(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame operator S^{-1} and frame bounds B^{-1}, A^{-1}.

(iii) If A, B are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds B^{-1}, A^{-1} are optimal for $\{S^{-1}f_k\}_{k=1}^{\infty}$.

$\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the *canonical dual frame* of $\{f_k\}_{k=1}^{\infty}$.

Frame decomposition:

Theorem 5.5 *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame with frame operator* S*. Then*

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k, \quad \forall f \in \mathcal{H},$$

and

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \ \forall f \in \mathcal{H}. \tag{5}$$

Both series converge unconditionally for all $f \in \mathcal{H}$*.*

Proof. Let $f \in \mathcal{H}$. Via Lemma 5.4,

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1} f, f_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k.$$

Since $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence and

$$\{\langle f, S^{-1} f_k \rangle\}_{k=1}^{\infty} \in \ell^2(\mathbb{N}),$$

the series converges unconditionally by Corollary 2.3. The expansion (5) follows from

$$f = S^{-1} S f. \qquad \square$$

Corollary 5.6 *If* $\{f_k\}_{k=1}^{\infty}$ *is a tight frame with frame bound* A*, then the canonical dual frame is* $\{A^{-1}f_k\}_{k=1}^{\infty}$*, and*

$$f = \frac{1}{A}\sum_{k=1}^{\infty}\langle f, f_k\rangle f_k, \ \forall f \in \mathcal{H}. \qquad (6)$$

Proof. For a tight frame,

$$\langle Sf, f\rangle = A||f||^2 = \langle Af, f\rangle;$$

since S is self-adjoint, this implies that

$$S = AI.$$

□

By scaling of the vectors $\{f_k\}_{k=1}^{\infty}$ in a tight frame, we can always obtain that $A = 1$; in that case (6) has exactly the same form as the representation via an orthonormal basis.

Tight frames can be used without any additional computational effort compared to the use of ONB's.

Other advantages of tight frames:

- If $\{f_k\}_{k=1}^{\infty}$ consists of functions with compact support or fast decay, the same is the case for the functions in the canonical dual frame.
- If $\{f_k\}_{k=1}^{\infty}$ consists of functions with a special structure (Gabor structure or wavelet structure) the same is the case for the functions in the canonical dual frame.

The corresponding statements do not hold for a general frame $\{f_k\}_{k=1}^{\infty}$ and its canonical dual frame $\{S^{-1}f_k\}_{k=1}^{\infty}$!

To each frame $\{f_k\}_{k=1}^{\infty}$ one can associate a tight frame:

Theorem 5.7 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame operator S. Denote the positive square root of S^{-1} by $S^{-1/2}$. Then $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a tight frame with frame bound equal to 1, and*

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1/2}f_k \rangle S^{-1/2}f_k, \ \forall f \in \mathcal{H}.$$

Problems:

- Not easy to find $\{S^{-1/2}f_k\}_{k=1}^{\infty}$;
- "Nice properties" of $\{f_k\}_{k=1}^{\infty}$ not necessarily inherited by $\{S^{-1/2}f_k\}_{k=1}^{\infty}$.

5.1 Dual frames

Another way to avoid the problem of inverting the frame operator S:

A frame which is not a Riesz basis is said to be *overcomplete.*

Theorem 5.8 *Assume that $\{f_k\}_{k=1}^\infty$ is an overcomplete frame. Then there exist frames*

$$\{g_k\}_{k=1}^\infty \neq \{S^{-1}f_k\}_{k=1}^\infty$$

for which

$$f = \sum_{k=1}^\infty \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}.$$

$\{g_k\}_{k=1}^\infty$ is called a *dual frame* of $\{f_k\}_{k=1}^\infty$.

If the canonical dual frame is difficult to find, maybe there exist other duals which are easy to find???

Lemma 5.9 *Assume that* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are Bessel sequences in* $\mathcal{H}$. *Then the following are equivalent:*

(i) $f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k, \ \forall f \in \mathcal{H}.$

(ii) $f = \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k, \ \forall f \in \mathcal{H}.$

(iii) $\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, f_k \rangle \langle g_k, g \rangle, \ \forall f, g \in \mathcal{H}.$

In case the conditions are satisfied, $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are dual frames for* $\mathcal{H}$.

Characterization of all dual frames:

Theorem 5.10 *Let* $\{f_k\}_{k=1}^{\infty}$ *be a frame for* $\mathcal{H}$. *The dual frames of* $\{f_k\}_{k=1}^{\infty}$ *are precisely the families*

$$\{g_k\}_{k=1}^{\infty} = \left\{ S^{-1} f_k + h_k - \sum_{j=1}^{\infty} \langle S^{-1} f_k, f_j \rangle h_j \right\}_{k=1}^{\infty},$$

where $\{h_k\}_{k=1}^{\infty}$ *is a Bessel sequence in* $\mathcal{H}$.

Does not (immediately) help!

Theorem 5.11 *A Riesz basis* $\{f_k\}_{k=1}^\infty$ *for* $\mathcal{H}$ *is a frame for* $\mathcal{H}$*, and the Riesz basis bounds coincide with the frame bounds. The dual Riesz basis equals the canonical dual frame* $\{S^{-1}f_k\}_{k=1}^\infty$.

Theorem 5.12 *Let* $\{f_k\}_{k=1}^\infty$ *be a frame for* $\mathcal{H}$*. Then the following are equivalent.*

(i) $\{f_k\}_{k=1}^\infty$ *is a Riesz basis for* $\mathcal{H}$.

(ii) If $\sum_{k=1}^\infty c_k f_k = 0$ *for some* $\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$*, then* $c_k = 0, \ \forall k \in \mathbb{N}$.

5.3 Characterizations of frames

Theorem 5.13 *A sequence* $\{f_k\}_{k=1}^\infty$ *in* $\mathcal{H}$ *is a frame for* $\mathcal{H}$ *if and only if*

$$T : \{c_k\}_{k=1}^\infty \to \sum_{k=1}^\infty c_k f_k$$

is a well-defined mapping of $\ell^2(\mathbb{N})$ *onto* $\mathcal{H}$*.*

Compare to the characterization of Bessel sequences in terms of T (T well-defined)!

Theorem 5.14 *Let* $\{e_k\}_{k=1}^\infty$ *be an arbitrary orthonormal basis for* $\mathcal{H}$*. The frames for* $\mathcal{H}$ *are precisely the families* $\{Ue_k\}_{k=1}^\infty$*, where* $U : \mathcal{H} \to \mathcal{H}$ *is a bounded and surjective operator.*

Compare to the characterization of

- ONB's (U unitary);
- Riesz bases (U bounded and bijective).

5.4 A frame where no subsequence is a basis

Intuitively: A frame consists of a basis + some extra elements (redundance). Good as intuitive feeling - but wrong in the technical sense:

Example 5.15 Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Let

$$\{f_k\}_{k=1}^{\infty} := \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\};$$

that is, each vector $\frac{1}{\sqrt{\ell}}e_\ell, \ell \in \mathbb{N}$, is repeated ℓ times. Then $\{f_k\}_{k=1}^{\infty}$ is a tight frame for $\mathcal{H}$ with frame bound $A = 1$. No subfamily is a Riesz basis. □

More complicated: In each separable Hilbert space, there exists a frame for which no subfamily is a basis!

Part II: Constructions in $L^2(\mathbb{R})$.

Final goal:

Obtain convenient expansions in $L^2(\mathbb{R})$,

$$f = \sum_{k \in I} \langle f, g_k \rangle f_k, \ \forall f \in L^2(\mathbb{R}).$$

Convenient means:

- f_k, g_k are given explicitly;
- f_k, g_k have compact support (no truncation necessary in numerical calculations);
- f_k, g_k decay fast in the Fourier domain;
- If f_k has a certain structure (e.g., Gabor structure), g_k has the same structure.

In concrete applications, other requirements might be relevant (differentiability, zero-means, vanishing moments, approximation order etc.)

Typical generators:

B-splines: The B-splines $N_n, \ n \in \mathbb{N}$, are given inductively by

$$N_1 = \chi_{[0,1]}, \ N_{n+1}(x) \ = \ N_n * N_1(x)$$
$$= \int_0^1 N_n(x-t)\, dt.$$

The functions N_n are called *B-splines*, and n is the *order*.

Fundamental properties:

Theorem 6.1 *Given $n \in \mathbb{N}$, the B-spline N_n has the following properties:*

(i) supp $N_n = [0, n]$ *and* $N_n > 0$ *on* $]0, n[$.

(ii) $\int_{-\infty}^{\infty} N_n(x) dx = 1.$

(iii) $\sum_{k \in \mathbb{Z}} N_n(x-k) = 1$

(iv) For any $n \in \mathbb{N}$,

$$\widehat{N_n}(\gamma) = \left(\frac{1 - e^{-2\pi i \gamma}}{2\pi i \gamma} \right)^n .$$

.

For any function $f : \mathbb{R} \to \mathbb{R}$, let

$$f(x)_+ = \max\{0, f(x)\}.$$

Theorem 6.2 *For each* $n = 2, 3, \ldots,$ *the B-spline* N_n *can be written*

$$N_n(x) = \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x - j)_+^{n-1}.$$

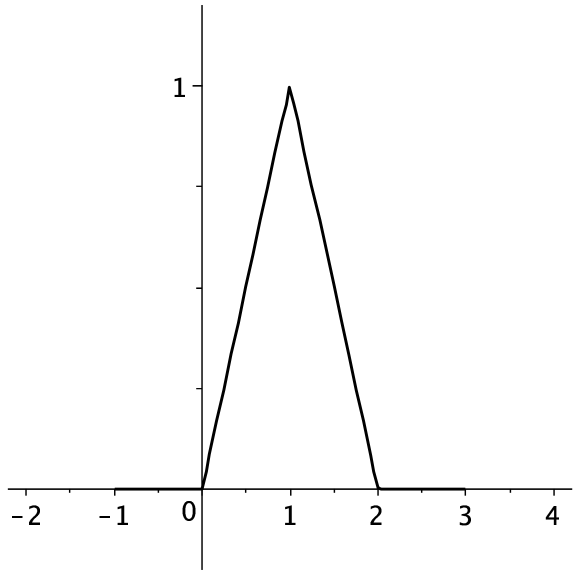


Figure 1: The B-spline N_2.

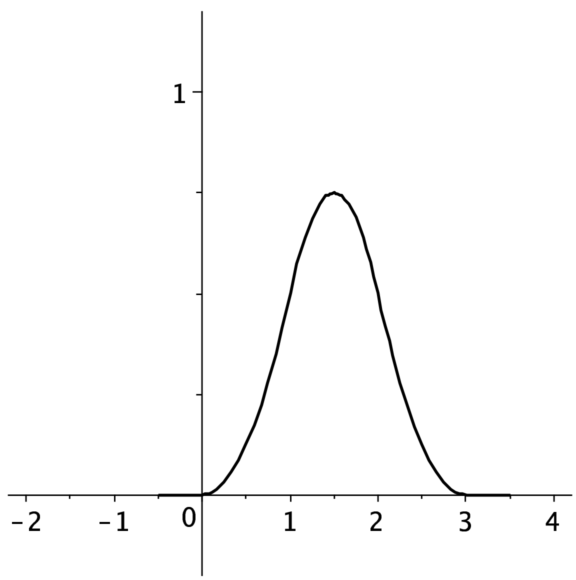


Figure 2: The B-spline N_3.

6.1 Symmetric B-splines:

For $n \in \mathbb{N}$, let

$$B_n(x) := T_{-\frac{n}{2}} N_n(x) = N_n(x + \frac{n}{2}).$$

The symmetric B-splines have similar properties:

Corollary 6.3 *For $n \in \mathbb{N}$, the B-spline B_n has the following properties:*

(i) $\sum_{k\in\mathbb{Z}} B_n(x-k) = 1.$

(ii) $\widehat{B_n}(\gamma) = \left(\frac{e^{\pi i\gamma} - e^{-\pi i\gamma}}{2\pi i\gamma}\right)^n = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^n.$

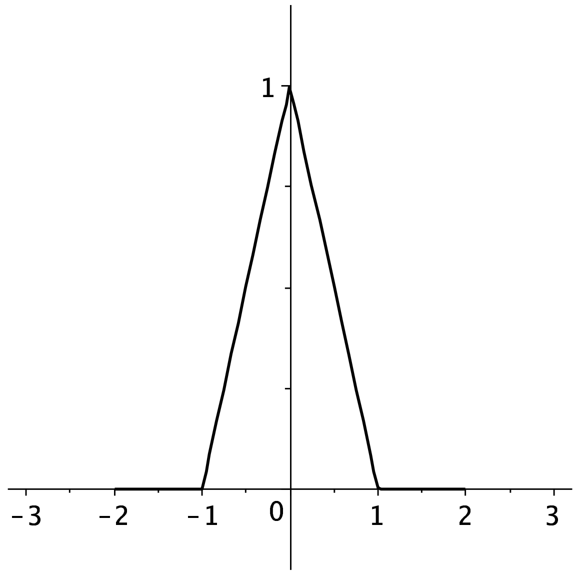


Figure 3: The B-spline B_2.

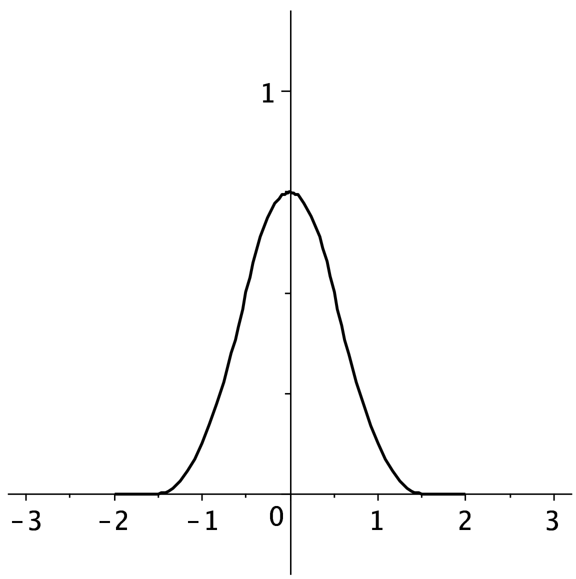


Figure 4: The B-spline B_3.

7 Gabor systems

Gabor systems, have the form

$$\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$$

for some $g \in L^2(\mathbb{R}),\ a,b>0$. Short notation:

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x-na)\}$$

Gabor bases exist:

Example 7.1 Let $\chi_{[0,1]}$ denote the characteristic function for the interval $[0,1]$. Then

$$\{e^{2\pi ikx}\chi_{[0,1]}(x)\}_{k\in\mathbb{Z}}$$

is an ONB for $L^2(0,1)$. By translation, for each $n\in\mathbb{Z}$ the space $L^2(n,n+1)$ has the ONB

$$\{e^{2\pi ik(x-n)}\chi_{[0,1]}(x-n)\}_{k\in\mathbb{Z}} = \{e^{2\pi ikx}\chi_{[0,1]}(x-n)\}_{k\in\mathbb{Z}}.$$

We conclude that $L^2(\mathbb{R})$ has the ONB

$$\left\{e^{2\pi ikx}\chi_{[0,1]}(x-n)\right\}_{k,n\in\mathbb{Z}}.$$

□

Problem: The function $\chi_{[0,1]}$ is discontinuous and has very slow decay in the Fourier domain:

$$\widehat{\chi}_{[0,1]}(\gamma) = \int_0^1 e^{-2\pi i x\gamma}\,dx = e^{-\pi i\gamma}\frac{\sin\pi\gamma}{\pi\gamma}.$$

Thus, the function is not suitable for time-frequency analysis.

Question: Can we obtain more suitable Gabor bases by replacing $\chi_{[0,1]}$ by a smoother function g?

No! A continuous function with compact support can not generate a Gabor Riesz basis.

A related short coming - the *Balian–Low Theorem:*

Theorem 7.2 *Assume that* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a Riesz basis. Then*

$$\left(\int_{\mathbb{R}} |xg(x)|^2 dx\right)\left(\int_{\mathbb{R}} |\gamma\hat{g}(\gamma)|^2 d\gamma\right) = \infty.$$

In words:

A function g generating a Gabor Riesz basis can not be well localized in both time and frequency.

For example: impossible that the estimates

$$|g(x)| \le \frac{C}{(1+x^2)^{1/2}},$$
$$|\hat{g}(\gamma)| \le \frac{C}{(1+\gamma^2)^{1/2}}$$

hold simultaneously.

This motives the construction of Gabor frames!

Example 7.3

- The Gaussian
$$g(x) = e^{-x^2}$$
generates a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for all $a, b \in]0,1[$. The Fourier transform
$$\hat{g}(x) = \sqrt{\pi}e^{-\pi^2x^2}$$
has exponential decay. Also, the dual generator $S^{-1}g$ has exponential decay in time and frequency.
- The function
$$h(x) = S^{-1/2}e^{-x^2}$$
generates a tight Gabor frame $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ for all $a, b \in]0,1[$, and h as well as $\hat{h}$ decay exponentially.

Problem: $S^{-1/2}e^{-x^2}$ and $S^{-1}e^{-x^2}$ are not given explicitly.

Solution: Don't construct a nice frame and *expect* the canonical dual to be nice.

Construct simultaneously dual pairs $\{E_{mb}T_{na}g\}$**,**$\{E_{mb}T_{na}h\}$ **such that** g **and** h **have the required properties, and**

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \forall f \in L^2(\mathbb{R}).$$

7.1 Sufficient and necessary conditions

Given a function $g \in L^2(\mathbb{R})$ and two numbers $a, b > 0$, consider the matrix-valued function

$$M(x) := (g(x - na - m/b))_{m,n\in\mathbb{Z}}\,,\ x \in \mathbb{R}.$$

Theorem 7.4 *Let $A, B > 0$ and the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ be given. Then the following holds:*

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with bound B if and only if $M(x)$ for a.e. $x \in \mathbb{R}$ defines a bounded operator on $\ell^2(\mathbb{Z})$ with norm at most $\sqrt{bB}$.

(ii) Assuming that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence, it is a frame for $L^2(\mathbb{R})$ with lower frame bound A if and only if

$$bAI \leq M(x)M(x)^* \text{ a.e. } x \in \mathbb{R},$$

where I is the identity operator on $\ell^2(\mathbb{Z})$.

The following necessary condition for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$ depends on the interplay between the function g and the translation parameter a, and is expressed in terms of the function

$$G(x) := \sum_{n\in\mathbb{Z}} |g(x-na)|^2, \; x \in \mathbb{R}.$$

Proposition 7.5 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with bounds A, B. Then*

$$bA \leq \sum_{n\in\mathbb{Z}} |g(x-na)|^2 \leq bB, \; a.e. \; x \in \mathbb{R}.$$

Lemma 7.6 *Suppose that f is a bounded measurable function with compact support and that the function*

$$G(x) = \sum_{n\in\mathbb{Z}} |g(x-na)|^2, \; x \in \mathbb{R}$$

is bounded. Then

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \\
= \; & \frac{1}{b}\int_{-\infty}^{\infty} |f(x)|^2 \sum_{n\in\mathbb{Z}} |g(x-na)|^2 \, dx \\
+ \; & \frac{1}{b}\sum_{k\neq 0}\int_{-\infty}^{\infty} \overline{f(x)} f(x-k/b) \\
& \qquad \times \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \, dx.
\end{aligned}$$

Theorem 7.7 *Let $g \in L^2(\mathbb{R})$, $a, b > 0$ and suppose that*

$$B := \frac{1}{b} \sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \right| < \infty.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with bound B. If also

$$\begin{aligned} A :=& \\ & \frac{1}{b} \inf_{x\in[0,a]} \left[\sum_{n\in\mathbb{Z}} |g(x-na)|^2 \right. \\ & \qquad \left. - \sum_{k\neq 0} \left| \sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-na-k/b)} \right| \right] > 0, \end{aligned}$$

then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B.

Corollary 7.8 *Let $g \in L^2(\mathbb{R})$ be bounded and compactly supported. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for any $a, b > 0$.*

Recall:

$$G(x) = \sum_{n\in\mathbb{Z}} |g(x-na)|^2,\ x \in \mathbb{R}.$$

Corollary 7.9 *Let $a, b > 0$ be given. Suppose that $g \in L^2(\mathbb{R})$ has support in an interval of length $\frac{1}{b}$ and that the function G is bounded above and below. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B. The frame operator S and its inverse S^{-1} are given by*

$$Sf = \frac{G}{b}f, \quad S^{-1}f = \frac{b}{G}f, \quad f \in L^2(\mathbb{R}).$$

Corollary 7.10 *Suppose that $g \in L^2(\mathbb{R})$ is a continuous function with support on an interval I with length $|I|$ and that $g(x) > 0$ on the interior of I. Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for all $(a,b) \in]0,|I|[\times]0,\frac{1}{|I|}[$.*

Corollary 7.11 *For $n \in \mathbb{N}$, the B-splines B_n and N_n generate Gabor frames for all $(a,b) \in]0,n[\times]0,1/n[$.*

Question: Characterization of $(a,b) \in \mathbb{R}^2$ for which B_n generates a Gabor frame?

The exact answer is unknown, and is bound to be complicated:

1) $\{E_{mb}T_{na}B_2\}_{m,n\in\mathbb{Z}}$ can not be a frame for any $b > 0$ whenever $a \geq 2$.

1) [Gröchenig, Janssen, Kaiblinger, Pfander]: For $b = 2, 3, \ldots$, $\{E_{mb}T_{na}B_2\}_{m,n\in\mathbb{Z}}$ can not be a frame for any $a > 0$.

Example 7.12 Consider characteristic functions,

$$g := \chi_{[0,c[}, \ \ c > 0.$$

Surprisingly complicated to find the exact range of $c > 0$ and parameters $a, b > 0$ for which $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame!

A complete answer has not been obtained yet.

Via a scaling, assume that $b = 1$.

Janssen showed that

(i) $\{E_m T_{na} g\}_{m,n\in\mathbb{Z}}$ is not a frame if $c < a$ or $a > 1$.

(ii) $\{E_m T_{na} g\}_{m,n\in\mathbb{Z}}$ is a frame if $1 \geq c \geq a$.

(iii) $\{E_m T_{na} g\}_{m,n\in\mathbb{Z}}$ is not a frame if $a = 1$ and $c > 1$.

Assuming now that $a < 1, c > 1$, we further have

(iv) $\{E_m T_{na} g\}_{m,n\in\mathbb{Z}}$ is a frame if $a \notin \mathbb{Q}$ and $c \in]1, 2[$.

(v) $\{E_m T_{na} g\}_{m,n\in\mathbb{Z}}$ is not a frame if $a = p/q \in \mathbb{Q}$, $gcd(p, q) = 1$, and $2 - \frac{1}{q} < c < 2$.

(vi) $\{E_m T_{na} g\}_{m,n\in\mathbb{Z}}$ is not a frame if $a > \frac{3}{4}$ and $c = L - 1 + L(1 - a)$ with $L \in \mathbb{N}, L \geq 3$.

The graphical illustration of this result is known as *Janssen's tie.* □

Theorem 7.13 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following holds:*

(i) If $ab > 1$, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can not be a frame for $L^2(\mathbb{R})$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then

$$ab = 1 \;\Leftrightarrow\; \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \text{ is a Riesz basis.}$$

Duality principle: [Janssen], [Daubechies, Landau and Landau], [Ron and Shen].

Concerns the relationship between frame properties for a function g with respect to the lattice

$$\{(na, mb)\}_{m,n\mathbb{Z}}$$

and the so-called *dual lattice*

$$\{(n/b, m/a)\}_{m,n\mathbb{Z}}.$$

Theorem 7.14 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the Gabor system*

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if

$$\{E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$$

is a Riesz sequence with bounds abA, abB.

7.2 Time-frequency localization of Gabor expansions

No function $g \neq 0$ can have compact support simultaneously in the time-domain and the frequency-domain.

But most signals appearing in practice are *essentially localized* in the time-frequency plane:

The interesting part of the signal takes place on a finite time-interval, with frequencies belonging to a certain finite interval.

How does this affects the Gabor frame expansion?

Given a number $T > 0$, define the operator

$$Q_T : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (Q_T f)(x) = \chi_{[-T,T]}(x) f(x).$$

Will use $||(I - Q_T)f||$ as a measure for the content of the function f outside the interval $[-T, T]$: that f *essentially is localized* on the interval $[-T, T]$ means that $||(I - Q_T)f||$ is small compared to $||f||$.

Similarly, for $\Omega > 0$ define P_Ω by

$$P_\Omega : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ \widehat{P_\Omega f}(\nu) = \chi_{[-\Omega,\Omega]}(\nu) \hat{f}(\nu).$$

That $\hat{f}$ *essentially is localized* on $[-\Omega, \Omega]$ means that $||(I - P_\Omega)f||$ is small compared to $||f||$.

Now assume that f is essentially localized in both domains, i.e., on $[-T, T] \times [-\Omega, \Omega]$ for some $T, \Omega > 0$. Let

$$\begin{aligned} & B(T, \Omega) \\ = \ & \{(m, n) \in \mathbb{Z}^2 : \ mb \in [-\Omega, \Omega],\ na \in [-T, T]\}. \end{aligned}$$

Consider the frame expansion of f in terms of dual frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$,

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g.$$

Do we obtain a reasonable approximation of f if we replace the sum over $(m,n) \in \mathbb{Z}^2$ with a sum over $(m,n) \in B(T,\Omega)$?

Positive answer if we replace $B(T,\Omega)$ by a certain enlargement $B(T+\Lambda, \Omega+\Gamma)$:

Theorem 7.15 *Assume that the Gabor systems* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *form a pair of dual frames for* $L^2(\mathbb{R})$ *with upper frame bounds* B *and* D *respectively, and that for some constants* $C > 0, \alpha > 1/2$, *the decay conditions*

$$|h(x)| \leq C(1+x^2)^{-\alpha}, \quad |\hat{h}(\nu)| \leq C(1+\nu^2)^{-\alpha}$$

hold. Then, for any $\epsilon > 0$ *there exist numbers* $\Lambda, \Gamma > 0$ *such that for all* $T, \Omega > 0$,

$$\left|\left| f - \sum_{\{(m,n)\in B(T+\Lambda,\Omega+\Gamma)\}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g \right|\right|$$
$$\leq \sqrt{BD}\,(||I-Q_T)f|| + ||(I-P_\Omega)f|| + \epsilon\,||f||)$$

for all $f \in L^2(\mathbb{R})$.

7.3 Tight Gabor frames

Theorem 7.16 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the following are equivalent:*

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a tight frame for* $L^2(\mathbb{R})$ *with frame bound* $A = 1$.

(ii) For a.e. $x \in \mathbb{R}$ *the following conditions hold:*

$G(x) = \sum_{n\in\mathbb{Z}} |g(x - na)|^2 = b$;

$\sum_{n\in\mathbb{Z}} g(x - na)\overline{g(x - na - k/b)} = 0$ *for all* $k \neq 0$.

Moreover, when the equivalent conditions hold, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is an orthonormal basis for* $L^2(\mathbb{R})$ *if and only if* $\|g\| = 1$.

Corollary 7.17 *Let $a, b > 0$ be given. Assume that $\varphi \in L^2(\mathbb{R})$ is a real-valued non-negative function with support in an interval of length $1/b$, and that*

$$\sum_{n\in\mathbb{Z}} \varphi(x+na) = 1, \ \ a.e. \ x \in \mathbb{R}.$$

Then the function

$$g(x) := \sqrt{b\varphi(x)}$$

generates a tight Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with frame bound $A = 1$.

Example 7.18 For any $n \in \mathbb{N}$, the B-spline $\varphi = N_n$ satisfies the requirements in Corollary 7.17 with $a = 1$ and any $b \in]0, 1/n]$. Thus, for any $b \in]0, 1/n]$, the function

$$g(x) = \sqrt{bN_n(x)}$$

generates a tight Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ with frame bound $A = 1$. $\square$

7.4 The duals of a Gabor frame

For a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with associated frame operator S, the frame decomposition shows that

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, S^{-1}E_{mb}T_{na}g\rangle E_{mb}T_{na}g, \ \forall f \in L^2(\mathbb{R}).$$

We need to be able to calculate the canonical dual frame $\{S^{-1}E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ – difficult!

A simplification can be obtained via

Lemma 7.19 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence with frame operator S. Then the following holds:*

(i) $SE_{mb}T_{na} = E_{mb}T_{na}S$ for all $m, n \in \mathbb{Z}$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame, then also

$$S^{-1}E_{mb}T_{na} = E_{mb}T_{na}S^{-1}, \ \forall m, n \in \mathbb{Z}.$$

Theorem 7.20 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given, and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. Then the following holds:*

(i) The canonical dual frame also has the Gabor structure and is given by $\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$.

(ii) The canonical tight frame associated with $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is given by $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$.

Proposition 7.21 *Let $g \in L^2(\mathbb{R})$, and assume that g as well as $\hat{g}$ decay exponentially. Let $a, b > 0$ be given and assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame. Then $\{E_{mb}T_{na}S^{-1/2}g\}_{m,n\in\mathbb{Z}}$ is a tight frame, for which $S^{-1/2}g$ as well as $\mathcal{F}(S^{-1/2}g)$ decay exponentially.*

Theoretically perfect!

Can be applied to the Gaussian $g(x) = e^{-x^2/2}$ – but the resulting generators are not given explicitly.

7.5 Characterizations of pairs of dual Gabor frames

The dual Gabor pairs with Gabor structure are characterized in the *Wexler–Raz Theorem.*

Theorem 7.22 *Let* $g, h \in L^2(\mathbb{R})$ *and* $a, b > 0$ *be given. Then, if the two Gabor systems* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *are Bessel sequences, they are dual frames if and only if*

$$\begin{cases} \langle h, E_{m/a}T_{n/b}g\rangle = 0 \text{ for all } (m,n) \neq (0,0) \\ \langle h, g\rangle = ab. \end{cases}$$

Janssen proved the following:

Theorem 7.23 *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b-ka)}h(x-ka) = b\delta_{n,0}, \ \ a.e. \ x \in [0,a].$$

Consequence of Theorem 7.23:

Lemma 7.24 *Let g, h be real-valued, bounded, and compactly supported functions. Then for $b > 0$ sufficiently small the following two conditions are equivalent:*

(i) $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *form dual frames for* $L^2(\mathbb{R})$*;*

(ii)

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) \ = \ b, \ \ a.e. \ x \in [0,1].$$

Dual Gabor frames with $a = 1$:

Theorem 7.25 *(C.) Let* $N \in \mathbb{N}$. *Let* $g \in L^2(\mathbb{R})$ *be a real-valued bounded function with supp* $g \subseteq [0, N]$, *for which*

$$\sum_{n \in \mathbb{Z}} g(x-n) = 1.$$

Let $b \in]0, \frac{1}{2N-1}]$. *Then the function* g *and the function* h *defined by*

$$h(x) = bg(x) + 2b \sum_{n=1}^{N-1} g(x+n)$$

generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Proof. Note that

$$\text{supp } g \subseteq [0, N], \ \text{supp } h \subseteq [-N+1, N],$$

so Lemma 7.24 applies for $b \leq 1/(2N-1)$.

Also, g and h have compact support, so $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ are Bessel sequences.

We check the condition in Lemma 7.24. For $x \in [0,1]$,

$$\begin{aligned}
1 &= \left(\sum_{k=0}^{N-1} g(x+k)\right)^2 \\
&= (g(x) + g(x+1) + \cdots + g(x+N-1)) \times \\
&\qquad (g(x) + g(x+1) + \cdots + g(x+N-1)) \\
&= g(x) \times \\
&\quad [g(x) + 2g(x+1) + 2g(x+2) + \cdots + 2g(x+N-1)] \\
&\quad + g(x+1) \times \\
&\quad [g(x+1) + 2g(x+2) + 2g(x+3) + \cdots + 2g(x+N-1)] \\
&\quad + g(x+2) \times \\
&\quad [g(x+2) + 2g(x+3) + 2g(x+4) + \cdots + 2g(x+N-1)] \\
&\quad + \cdots \\
&\quad + \cdots \\
&\quad + g(x+N-2)\,[g(x+N-2) + 2g(x+N-1)] \\
&\quad + g(x+N-1)\,[g(x+N-1)] \\
&= \frac{1}{b}\sum_{k=0}^{N-1} g(x+k)h(x+k).
\end{aligned}$$

Thus the conclusion follows. □

Corollary 7.26 *(C.) For any $N \in \mathbb{N}$ and $b \in]0, \frac{1}{2N-1}]$, the functions N_N and*

$$h_N(x) := bN_N(x) + 2b\sum_{n=1}^{N-1} N_N(x+n)$$

generate a pair of dual frames $\{E_{mb}T_nN_N\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh_N\}_{m,n\in\mathbb{Z}}$.

Important features of the dual pair of frame generators (N_N, h_N) in Corollary 7.26:

- The functions N_N and h_N are splines;
- N_N and h_N have compact support, i.e., perfect time–localization;
- By choosing N sufficiently large, polynomial decay of $\widehat{N_N}$ and $\widehat{h_N}$ of any desired order can be obtained.

Example 7.27 For the B-spline

$$N_2(x) = \begin{cases} x & x \in [0,1[, \\ 2-x & x \in [1,2[, \\ 0 & x \notin [0,2[, \end{cases}$$

we can use Corollary 7.26 for $b \in]0, 1/3]$. For $b = 1/3$ we obtain the dual generator

$$\begin{aligned} h(x) &= \frac{1}{3}N_2(x) + \frac{2}{3}N_2(x+1) \\ &= \begin{cases} \frac{2}{3}(x+1) & x \in [-1,0[, \\ \frac{1}{3}(2-x) & x \in [0,2[, \\ 0 & x \notin [-1,2[. \end{cases} \end{aligned}$$

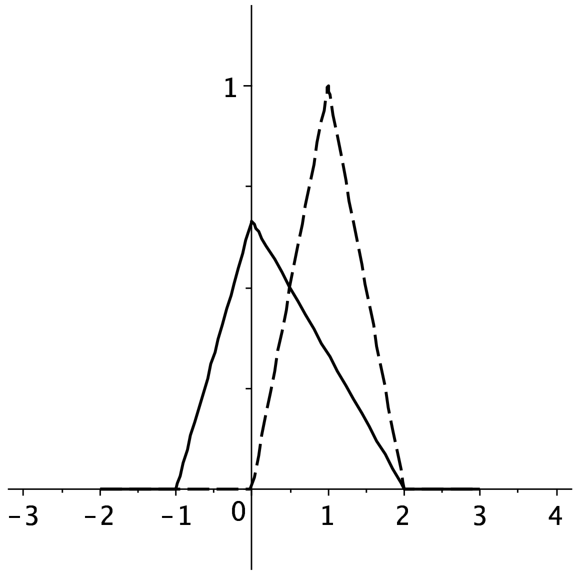


Figure 5: The B-spline N_2 and the dual generator h for $b = 1/3$.

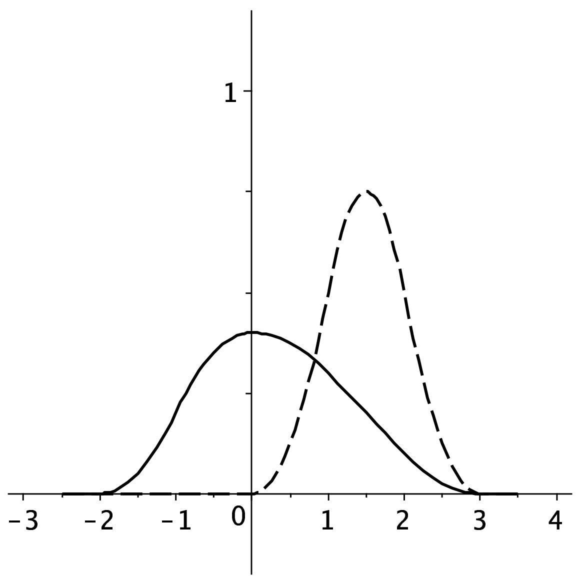


Figure 6: The B-spline N_3 and the dual generator h with $b = 1/5$.

Dual Gabor frames with general $a > 0$:

Theorem 7.28 *(C.) Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function with supp $g \subseteq [0, N]$, for which*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1.$$

Let $a, b > 0$ and choose $J \in \mathbb{N}$ such that $J \geq ab(2N-1)$. Define the function h by

$$h(x) = abg(x) + 2ab\sum_{n=1}^{N-1} g(x+n).$$

Then the functions

$$g_k = T_{\frac{a}{J}k} D_{a/J} g, \quad h_k = T_{\frac{a}{J}k} D_{a/J} h, \; k = 0, \ldots, J-1$$

generate dual multi-Gabor frames
$\{E_{mb}T_{na}g_k\}_{m,n\in\mathbb{Z},k=0,\ldots,J-1}, \{E_{mb}T_{na}h_k\}_{m,n\in\mathbb{Z},k=0,\ldots,J-1}$
for $L^2(\mathbb{R})$.

Other choices?

Yes!

Theorem 7.29 (C., Kim, 2007) *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function with supp $g \subseteq [0, N]$, for which*

$$\sum_{n \in \mathbb{Z}} g(x-n) = 1.$$

Let $b \in]0, \frac{1}{2N-1}]$. Define $h \in L^2(\mathbb{R})$ by

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

where

$$a_0 = b, \quad a_n + a_{-n} = 2b, \;\; n = 1, 2, \cdots, N-1.$$

Then g and h generate dual frames $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$.

Example 7.30

1) Take

$$\begin{cases} a_0 = b, \\ a_n = 0 \text{ for } n = -N+1, \ldots, -1 \\ a_n = 2b, n = 1, \ldots N-1 \end{cases}$$

This is Theorem 7.25. This choice gives the shortest support.

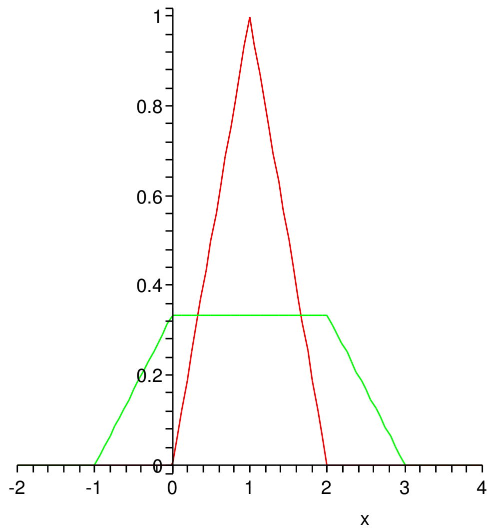
2) Take

$$a_{-N+1} = a_{-N+2} = \cdots = a_{N-1} = b:$$

if g is symmetric, this leads to a symmetric dual generator

$$h(x) = b \sum_{n=-N+1}^{N-1} g(x+n).$$

Note: $h(x) = 1$ on supp g.



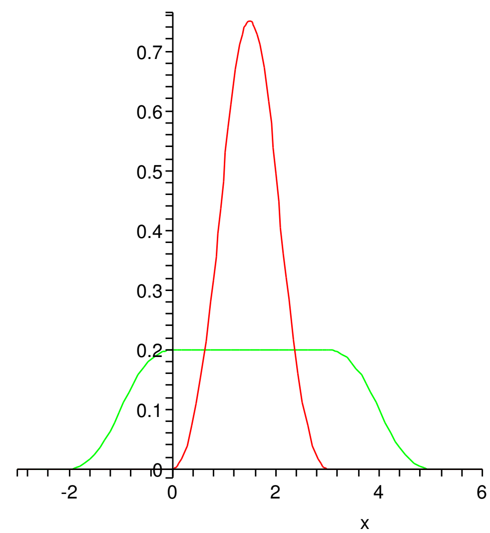


Figure 7: The generators N_2 and N_3 and their dual generators via 2) in Example 7.30.

8 Polynomial generators and duals [C., Kim]

The restriction of any polynomial to a sufficiently large interval will generate a Gabor frame for small modulation parameters:

Proposition 8.1 *Let* $N \in \mathbb{N}$*, and consider any bounded interval* $I \subset \mathbb{R}$ *with* $|I| \geq N$*. Then any (nontrivial) polynomial*

$$g(x) = \left(\sum_{k=0}^{N-1} c_k x^k \right) \chi_I(x)$$

generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *for* $b \in]0, 1/|I|]$.

The condition on the relationship between the support size and the degree of the polynomial is necessary in Proposition 8.1:

Example 8.2 Let

$$\begin{aligned} g(x) &= (x-1/2)(x-3/2)\chi_{[0,2]}(x) \\ &= (x^2-2x+3/4)\chi_{[0,2]}(x). \end{aligned}$$

Then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is not a Gabor frame for any $b>0$. In fact, for $x\in[0,1]$,

$$\sum_{k\in\mathbb{Z}} |g(x-k)|^2 = |g(x)|^2 + |g(x+1)|^2,$$

so

$$\sum_{k\in\mathbb{Z}} |g(1/2-k)|^2 = 0.$$

□

In applications of frames: crucial that the frame generator as well as the dual frame generator have a convenient form. The polynomials are very convenient - but no pair of dual generators of this form exists:

Theorem 8.3 *Two compactly supported polynomials*

$$g(x) = \left(\sum_{k=0}^{N_1} a_k x^k \right) \chi_{I_1}(x)$$

and

$$h(x) = \left(\sum_{k=0}^{N_2} b_k x^k \right) \chi_{I_2}(x)$$

of degree ≥ 1 *can not generate dual Gabor frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *for any* $b > 0$*, no matter how the intervals* I_1 *and* I_2 *are chosen.*

Remark: The proof shows that even finite linear combinations of the type

$$\tilde{g}(x) = \sum c_k T_k g, \quad \tilde{h}(x) = \sum d_k T_k h$$

with g and h as in Theorem 8.3 can't generate dual Gabor frames $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n\tilde{h}\}_{m,n\in\mathbb{Z}}$

Conclusion: for polynomial generators we must search for other simple generators for the dual. Natural choices: B-splines!

Theorem 8.4 *Let $N \in \mathbb{N}$ and $g(x) := \sum_{k=0}^{N-1} c_k x^k$ be a polynomial. Given an interval $I \subset \mathbb{R}$, let*

$$\tilde{g}(x) := g(x)\chi_I(x).$$

Let $h \in L^2(\mathbb{R})$ be such that supp $h \subset I$. Then the following are equivalent:

(i) *There exists $\beta \in \mathbb{R}$ such that $\tilde{g}(x)$ and $\tilde{h}(x) := \beta h(x)$ generate dual frames $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n\tilde{h}\}_{m,n\in\mathbb{Z}}$ for $b > 0$ sufficiently small;*

(ii)

$$\begin{cases} \sum_{k=0}^{N-1} c_k \dfrac{D^k\hat{h}(n)}{(-2\pi i)^k} = 0, \quad n \in \mathbb{Z} \setminus \{0\}; \\ \sum_{k=0}^{N-1} c_k \dfrac{D^k\hat{h}(0)}{(-2\pi i)^k} \neq 0. \end{cases}$$

Theorem 8.5 *Let $N \in \mathbb{N}$ and $g(x) := \sum_{k=0}^{N_0} c_k x^k$ be a polynomial of degree $N_0 < N$. Given an interval $I \subset \mathbb{R}$, let*

$$\tilde{g}(x) := g(x)\chi_I(x).$$

Let $h \in L^2(\mathbb{R})$ be such that supp h is an interval with supp $h \subsetneq I$. If h satisfy the Strang-Fix conditions, that is,

$$\hat{h}(0) \neq 0;$$
$$D^k \hat{h}(n) = 0,\ n \in \mathbb{N} \setminus \{0\},\ k = 0, 1, \cdots, N-1,$$

then there always exist $\alpha, \beta \in \mathbb{R}$ such that $\tilde{g}(x)$ and $\tilde{h}(x) := \beta h(x - \alpha)$ generate dual frames $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n\tilde{h}\}_{m,n\in\mathbb{Z}}$ for $b > 0$ sufficiently small.

Corollary 8.6 *Let $N \in \mathbb{N}$ and $g(x) := \sum_{k=0}^{N-1} c_k x^k$ be a polynomial. Given an interval I containing $[0, N]$, let*

$$\tilde{g}(x) := g(x)\chi_I(x).$$

Then the following hold:

(a) *If $I = [0, N]$ and*

$$\kappa := \sum_{k=0}^{N-1} c_k \frac{D^k \widehat{N_N}(0)}{(-2\pi i)^k} \neq 0,$$

then $\tilde{g}(x)$ and $h(x) := \frac{b}{\kappa} N_N(x)$ generate dual frames $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ for $b > 0$ sufficiently small.

(b) *If $I \supsetneq [0, N]$, then there exists $\alpha, \beta \in \mathbb{R}$ such that $\tilde{g}$ and*

$$h(x) := \beta N_N(x - \alpha)$$

generate dual frames $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ for $b > 0$ sufficiently small.

Conclusion: polynomial generators on sufficiently large intervals almost always generate Gabor frames with B-spline duals for sufficiently small modulation parameters.

Example 8.7 Let $g(x) = (x-1)\chi_I(x)$. First, let $I = [0, 2]$. A direct calculation shows that

$$\widehat{N_2}(0) = 1, \quad D\widehat{N_2}(0) = -2\pi i.$$

Thus

$$\kappa = \sum_{k=0}^{1} c_k \frac{D^k \widehat{N_2}(0)}{(-2\pi i)^k} = 0.$$

One can prove that

$$\sum_{k \in \mathbb{Z}} g(x-k) N_2(x-k) = 0.$$

Hence N_2 is not a dual of g for any $b > 0$ by Lemma 7.24.

Now, let $I = [-1, 3]$. Then $\beta N_2(x - \alpha_0)$ is dual generator of g for $b > 0$ sufficiently small for some constant β (which can be explicitly calculated).

Example 8.8 Let

$$\begin{aligned} g(x) &= x^2(5-x)^2\chi_{[0,5]}(x) \\ &= \left(25x^2 - 10x^3 + x^4\right)\chi_{[0,5]}(x) \\ &=: \left(\sum_{k=0}^{4} c_k x^k\right)\chi_{[0,5]}(x). \end{aligned}$$

For the B-spline N_5,

$$\frac{D^k\widehat{N_5}(0)}{(-2\pi i)^k} = \begin{cases} 20/3, & k=2; \\ 75/4, & k=3; \\ 331/6, & k=4. \end{cases}$$

Thus

$$\kappa := \sum_{k=0}^{4} c_k \frac{D^k\widehat{N_5}(0)}{(-2\pi i)^k} = 103/3 \neq 0.$$

Hence $h(x) := \frac{b}{\kappa}N_5$ is a dual of $g(x)$ for $b > 0$ sufficiently small by Corollary 8.6 (a). See Figure 8.

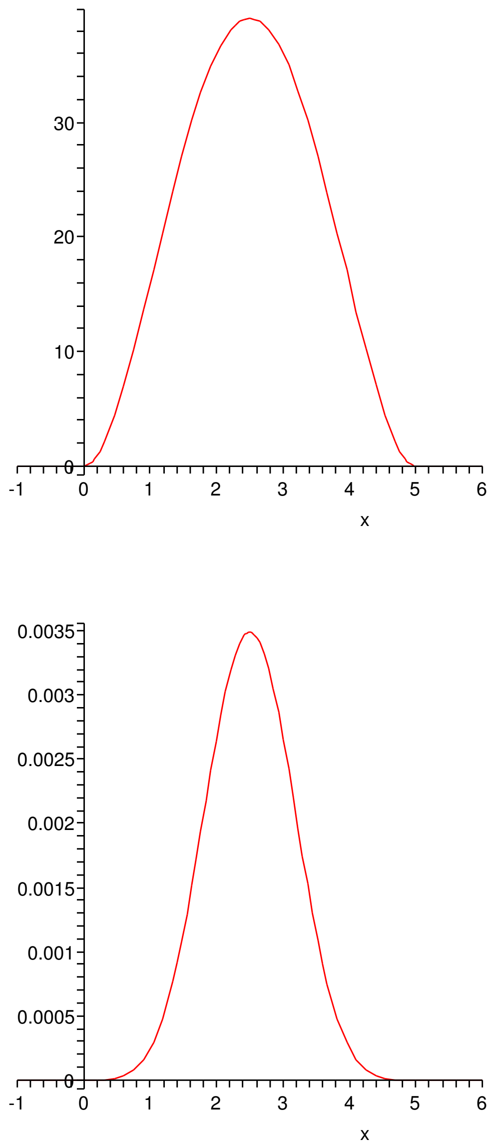


Figure 8: The generators g and h for $b = 1/5$ in Example 8.8.

Example 8.9 Let

$$\begin{aligned} g(x) &= x^3(6-x)^2\chi_{[0,6]}(x) \\ &= \left(36x^3 - 12x^4 + x^5\right)\chi_{[0,6]}(x) \\ &= \left(\sum_{k=0}^{5} c_k x^k\right)\chi_{[0,6]}(x). \end{aligned}$$

For N_6, we have

$$\frac{D^k\widehat{N_6}(0)}{(-2\pi i)^k} = \begin{cases} 63/2, & k=3; \\ 1087/10, & k=4; \\ 777/2, & k=5. \end{cases}$$

Thus

$$\kappa := \sum_{k=0}^{5} c_k \frac{D^k\widehat{N_6}(0)}{(-2\pi i)^k} = 2181/10 \neq 0.$$

Hence $h(x) := \frac{b}{\kappa}N_6$ is a dual of $g(x)$ for $b > 0$ sufficiently small by Corollary 8.6 (a). See Figure 9.

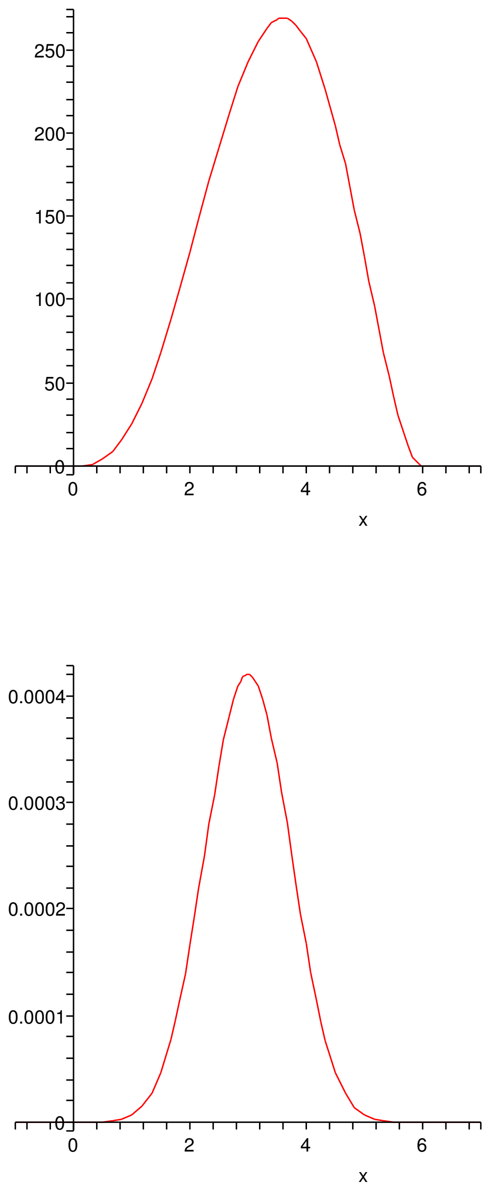


Figure 9: The generators g and h for $b = 1/6$ in Example 8.9.

Extensions:
1) Higher dimensions: (C., R. Y. Kim)

For $f \in L^2(\mathbb{R}^d)$ and $d \times d$ matrices B, C, and $m, n \in \mathbb{Z}^d$, let

$$E_{Bm}f(x) = e^{2\pi i Bm\cdot x}f(x),\ x \in \mathbb{R}^d$$

$$T_{Cn}f(x) = f(Cn - x),\ x \in \mathbb{R}^d.$$

Theorem 8.10 *Let $N \in \mathbb{N}$ and assume that $g \in L^2(\mathbb{R}^d)$ has supp $g \subseteq [0,N]^d$, and that*

$$\sum_{n \in \mathbb{Z}^d} g(x-n) = 1.$$

Assume that the $d \times d$ matrix B is invertible and $||B|| \leq \frac{1}{\sqrt{d}(2N-1)}$. For $i = 1, \ldots, d$, let F_i be the set of lattice points $\{k_j\}_{j=1}^d \in \mathbb{Z}^d$ for which the coordinates $k_j, j = 1, \ldots, d$, satisfy

$$\begin{cases} \text{if } j = 1, \ldots, i-1, \ \text{then } |k_j| \leq N-1; \\ \text{if } j = i, \quad \text{then } 1 \leq k_j \leq N-1; \\ \text{if } j = i+1, \ldots, d, \ \text{then } k_j = 0. \end{cases}$$

Define $h \in L^2(\mathbb{R}^d)$ by

$$h(x) := |\det B| \left[g(x) + 2 \sum_{i=1}^{d} \sum_{k \in F_i} g(x+k) \right].$$

Then $\{E_{Bm}T_n g\}_{m,n \in \mathbb{Z}^d}$ and $\{E_{Bm}T_n h\}_{m,n \in \mathbb{Z}^d}$ are dual frames for $L^2(\mathbb{R}^d)$.

2) Tight frames (C., R.Y. Kim):

Theorem 8.11 *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a non-negative function with supp $g \subseteq [0, N]^d$, for which*

$$\sum_{n \in \mathbb{Z}^d} g(x-n) > 0, \ a.e.\ x \in \mathbb{R}^d.$$

Assume that the $d \times d$ matrix B is invertible and $||B|| \leq \frac{1}{\sqrt{d}\ N}$. Define $h \in L^2(\mathbb{R}^d)$ by

$$h(x) := \sqrt{|\det B| \frac{g(x)}{\sum_{n \in \mathbb{Z}^d} g(x-n)}}.$$

Then the function h generates a Parseval frame $\{E_{Bm}T_n h\}_{m,n \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.

The generator in Theorem 8.11 has fast decay in the Fourier domain if it is smooth:

Lemma 8.12 *Let $k \in \mathbb{N}$ and let $f \in C^{dk}(\mathbb{R}^d)$ be compactly supported. Then*

$$|\hat{f}(\gamma)| \leq A(1 + |\gamma|^2)^{-k/2}.$$

3) Irregular B-splines (C., Wenchang Sun):

Theorem 8.13 *Let* $\{x_n : n \in \mathbb{Z}\} \subset \mathbb{R}$ *be a sequence such that*

$$\lim_{n \to \pm\infty} x_n = \pm\infty, \quad x_{n-1} \le x_n,$$

and

$$x_{n+2N-1} - x_n \le M, \qquad n \in \mathbb{Z},$$

for some constants $N \in \mathbb{N}$ *and* $M > 0$. *Let* g_n *be the* N*-th order normalized B-spline with knots* $(x_n, x_{n+1}, \cdots, x_{n+N})$ *and*

$$h_n(x) = b g_n(x) + 2b \sum_{k=1}^{N-1} g_{n-k}(x).$$

Then $\{E_{mb} g_n\}_{m,n \in \mathbb{Z}}$ *and* $\{E_{mb} h_n\}_{m,n \in \mathbb{Z}}$ *are a pair of dual frames for* $L^2(\mathbb{R})$ *provided that* $0 < b \le 1/M$.

9 Wavelet frames

9.1 Wavelet bases

Given a function $\psi \in L^2(\mathbb{R})$ and $j, k \in \mathbb{Z}$, let

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \; x \in \mathbb{R}.$$

In terms of T_k and $Df(x) = 2^{1/2}f(2x)$,

$$\psi_{j,k} = D^j T_k \psi, \; j, k \in \mathbb{Z}.$$

If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, the function ψ is called a *wavelet.*

Definition 9.1 *A multiresolution analysis for $L^2(\mathbb{R})$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$, such that the following conditions hold:*

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$.

(ii) $\overline{\cup_j V_j} = L^2(\mathbb{R})$ *and* $\cap_j V_j = \{0\}$.

(iii) $f \in V_j \Leftrightarrow [x \to f(2x)] \in V_{j+1}$.

(iv) $f \in V_0 \Rightarrow T_k f \in V_0, \; \forall k \in \mathbb{Z}$.

(v) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0.

An MRA can be used to construct an ONB for $L^2(\mathbb{R})$:

For $j \in \mathbb{Z}$, let W_j denote the orthogonal complement of V_j in V_{j+1}. Then

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

The spaces W_j satisfy the same dilation relationship as V_j, i.e.,

$$\psi \in W_0 \Leftrightarrow [x \to \psi(2^j x)] \in W_j.$$

In order to obtain an orthonormal basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ it is now enough to find $\psi \in W_0$ such that $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_0.

We will now explain how to find a suitable function ψ. The condition $\phi \in V_0 \subset V_1$ implies by (iii) that

$$\frac{1}{\sqrt{2}} D^{-1}\phi \in V_0.$$

Since $\{T_{-k}\phi\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0, there exist coefficients $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\frac{1}{\sqrt{2}} D^{-1}\phi = \sum_{k\in\mathbb{Z}} c_k T_{-k}\phi. \qquad (7)$$

We will now rewrite (7) in terms of the Fourier transform of ϕ. Let

$$E_k(x) = e^{2\pi i k x},\ x \in \mathbb{R}.$$

Now, applying the Fourier transform on both sides of (7),

$$\frac{1}{\sqrt{2}} D\hat{\phi} = \sum_{k\in\mathbb{Z}} c_k E_k \hat{\phi}.$$

Defining the 1-periodic function $H_0 := \sum_{k\in\mathbb{Z}} c_k E_k$, this can be written as

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma), \quad a.e.\ \gamma \in \mathbb{R}. \qquad (8)$$

The equation (8) is called a *scaling equation* or *refinement equation.* Now, a certain choice of a 1-periodic function H_1, implies that the function ψ defined via

$$\hat{\psi}(2\gamma) = H_1(\gamma)\hat{\phi}(\gamma) \tag{9}$$

generates a wavelet orthonormal basis $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$. One choice of H_1 is to take

$$H_1(\gamma) = \overline{H_0(\gamma + \frac{1}{2})}e^{-2\pi i\gamma}.$$

Note that (9) leads to an explicit expression of the function ψ in terms of the given function ϕ:

Lemma 9.2 *Assume that*

$$\hat{\psi}(2\gamma) = H_1(\gamma)\hat{\phi}(\gamma)$$

holds for a 1-periodic and bounded function H_1 with Fourier expansion $H_1 = \sum_{k\in\mathbb{Z}} c_k E_k$. Then

$$\begin{aligned}\psi(x) &= \sqrt{2}\sum_{k\in\mathbb{Z}} c_k DT_{-k}\phi(x) \\ &= 2\sum_{k\in\mathbb{Z}} c_k \phi(2x+k), \ \textit{a.e. } x \in \mathbb{R}.\end{aligned}$$

In particular, if H_1 is a trigonometric polynomial, $H_1(x) = \sum_{k=N_1}^{N_2} c_k e^{2\pi ikx}$, then

$$\begin{aligned}\psi(x) &= \sqrt{2}\sum_{k=N_1}^{N_2} c_k DT_{-k}\phi(x) \\ &= 2\sum_{k=N_1}^{N_2} c_k \phi(2x+k), \ \forall x \in \mathbb{R}.\end{aligned}$$

The Haar basis can be constructed via the multiresolution analysis defined by $\phi = \chi_{[0,1[}$ and

$V_j = \{f : f \text{ is const. on } [2^{-j}k, 2^{-j}(k+1)[, \ \forall k \in \mathbb{Z}\}$.

In terms of the function ϕ, the Haar function is

$$\psi = \frac{1}{\sqrt{2}}\phi_{1,0} - \frac{1}{\sqrt{2}}\phi_{1,1}.$$

The Haar function is a special case of a *spline wavelet.*

One can consider higher order splines N_n and define associated multiresolution analyses, which leads to wavelets of the type

$$\psi(x) = \sum_{k\in\mathbb{Z}} c_k N_n(2x - k).$$

These wavelets are called *Battle–Lemarié wavelets.* Except for the case $n = 1$, all coefficients c_k are non-zero, which implies that the wavelet ψ has support equal to $\mathbb{R}$. However, the wavelets have exponential decay.

For $\phi \in L^2(\mathbb{R})$, let

$$V_j = \overline{\text{span}}\{D^j T_k \phi\}_{k\in\mathbb{Z}}.$$

Then the MRA-conditions (iii) and (iv) are satisfied.

When does V_j generate an MRA?

Lemma 9.3 *Let $\phi \in L^2(\mathbb{R})$. Then the following holds:*

(i) $\cap_j V_j = \{0\}$.

(ii) Assume that the spaces V_j are nested. If $|\hat{\phi}| > 0$ on a neighborhood of 0, then $\cup_j V_j$ is dense in $L^2(\mathbb{R})$.

Thus, if the spaces V_j are nested and the condition in (ii) is satisfied, then the MRA-conditions (i)–(iv) are satisfied. We thus need a condition ensuring that V_j are nested:

Lemma 9.4 *Assume that* $\phi \in L^2(\mathbb{R})$ *and that* $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a Bessel sequence. Define the spaces* V_j *by*

$$V_j = \overline{span}\{D^jT_k\phi\}_{k\in\mathbb{Z}}.$$

Then the following holds:

(i) If $\psi \in L^2(\mathbb{R})$ *and there exists a bounded 1-periodic function* H_1 *such that*

$$\hat{\psi}(2\gamma) = H_1(\gamma)\hat{\phi}(\gamma),$$

then $\psi \in V_1$.

(ii) If there exists a bounded 1-periodic function H_0 *such that*

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma),$$

then $V_j \subseteq V_{j+1}$ *for all* $j \in \mathbb{Z}$.

Via Lemma 9.3 and Lemma 9.4 we obtain the following:

Theorem 9.5 *Let $\phi \in L^2(\mathbb{R})$, and assume that $|\hat{\phi}| > 0$ on a neighborhood of 0. Assume further that*

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma),$$

is satisfied for a bounded 1-periodic function H_0. Define the spaces V_j by

$$V_j = \overline{span}\{D^j T_k \phi\}_{k\in\mathbb{Z}}.$$

Then the following holds:

(i) If $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system, then ϕ and the spaces V_j form a multiresolution analysis.

(i) If $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a Bessel sequence, then the spaces V_j satisfy the conditions (i)–(iv) in Definition 9.1.

Important properties for wavelet bases $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$:

- that ψ has a computationally convenient form, for example that ψ is a piecewise polynomial (a spline);
- regularity of ψ;
- symmetry (or anti-symmetry) of ψ, i.e., that

  $$\psi(x) = \psi(-x) \text{ or } \psi(x) = -\psi(-x),\ x \in \mathbb{R};$$

- compact support of ψ, or at least fast decay;
- that ψ has *vanishing moments*, i.e., that for a certain $m \in \mathbb{N}$,

  $$\int_{-\infty}^{\infty} x^\ell \psi(x)\, dx = 0 \text{ for } \ell = 0, 1, \ldots, m.$$

Short description of the role of these conditions:

- Vanishing moments: needed if we want smooth wavelets. If ψ is an m times differentiable function with bounded derivatives with reasonable decay and $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal system, then ψ has $m-1$ vanishing moments.

- Vanishing moments are essential in the context of *compression.* Assuming that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$, every $f \in L^2(\mathbb{R})$ has the representation

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}. \tag{10}$$

  All information about f is stored in the coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j,k\in\mathbb{Z}}$, and (10) tells us how to reconstruct f based on the coefficients. In practice one can not store an infinite sequence of non-zero numbers, so one has to select a finite number of the coefficients to keep. Done by

Thresholding: Chooses a certain $\epsilon > 0$ and keep only the coefficients $\langle f, \psi_{j,k} \rangle$ for which

$$|\langle f, \psi_{j,k} \rangle| \geq \epsilon.$$

If ψ has a large number of vanishing moments, then only relatively few coefficients $\langle f, \psi_{j,k} \rangle$ will be large:

Theorem 9.6 *Assume that the function* $\psi \in L^2(\mathbb{R})$ *is compactly supported and has* $N-1$ *vanishing moments. Then, for any* N *times differentiable function* $f \in L^2(\mathbb{R})$ *for which* $f^{(N)}$ *is bounded, there exists a constant* $C > 0$ *such that*

$$|\langle f, \psi_{j,k} \rangle| \leq C\, 2^{-jN} 2^{-j/2}, \ \forall j, k \in \mathbb{Z}.$$

Further relevant properties:

- Compact support (or at least fast decay) of ψ is essential for the use of computer-based methods, where a function with unbounded support always has to be truncated. For the same reason we often want the support to be small.
- The condition of ψ being symmetric is helpful in image processing, where a non-symmetric wavelet will generate non-symmetric errors, which are more disturbing to the human eye than symmetric errors.

One can not combine the classical multiresolution analysis with the desire of having a symmetric wavelet ψ:

Proposition 9.7 *Assume that* $\phi \in L^2(\mathbb{R})$ *is real-valued and compactly supported, and let*

$$V_j = \overline{span}\{D^j T_k \phi\}_{k\in\mathbb{Z}}, \; j \in \mathbb{Z}.$$

Assume that $(\phi, \{V_j\})$ *constitute a multiresolution analysis. Then, if the associated wavelet* ψ *is real-valued and compactly supported and has either a symmetry axis or an antisymmetry axis, then* ψ *is necessarily the Haar wavelet.*

Thus, under the above assumptions we are back at the function we want to avoid!

Definition 9.8 *Let $\psi \in L^2(\mathbb{R})$. A frame for $L^2(\mathbb{R})$ of the form $\{D^j T_k \psi\}_{j,k\in\mathbb{Z}}$ is called a dyadic wavelet frame.*

The associated frame operator:

$$S : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ Sf = \sum_{j,k\in\mathbb{Z}} \langle f, D^j T_k \psi\rangle D^j T_k \psi.$$

The frame decomposition:

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, S^{-1} D^j T_k \psi\rangle D^j T_k \psi,\ f \in L^2(\mathbb{R}).$$

Inconvenient - one needs to calculate

$$\langle f, S^{-1} D^j T_k \psi\rangle,\ \forall j,k \in \mathbb{Z}.$$

Improvement: can show that

$$S^{-1}D^jT_k\psi = D^jS^{-1}T_k\psi.$$

Unfortunately, in general

$$D^jS^{-1}T_k\psi \neq D^jT_kS^{-1}\psi.$$

We can not expect the canonical dual frame of a wavelet frame to have wavelet structure.

Bownik and Weber: example of a wavelet system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for which

- The canonical dual does not have the wavelet structure;
- There exist infinitely many functions $\tilde{\psi}$ for which $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ is a dual frame.

Example 9.9 Let $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ be a wavelet ONB for $L^2(\mathbb{R})$. Given $\epsilon \in]0,1[$, let

$$\theta = \psi + \epsilon D\psi.$$

Then $\{\theta_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis and the canonical dual frame is given by

$$S^{-1}\theta_{j,2k+1} = \psi_{j,2k+1} \text{ for all } j,k \in \mathbb{Z}.$$

Also, for any $k \neq 0$,

$$S^{-1}\theta_{j,2k} \;=\; \psi_{j,2k} - \epsilon\psi_{j-1,k} + \cdots + 0,\; j \in \mathbb{Z}, k \neq 0.$$

For $k = 0$,

$$S^{-1}\theta_{j,0} \sum_{n=0}^{\infty} (-\epsilon)^n \psi_{j-n,0},\; j \in \mathbb{Z}.$$

In particular, the canonical dual frame of $\{\theta_{j,k}\}_{j,k\in\mathbb{Z}}$ does *not* have the wavelet structure.

For a Riesz basis, the dual is unique: so no dual with wavelet structure exists!

Other properties which are not inherited by the canonical dual frame:

If ψ has compact support, then θ also has compact support, and all the functions $\{\theta_{j,k}\}_{j,k\in\mathbb{Z}}$ have compact support.

For the canonical dual frame $\{S^{-1}\theta_{j,k}\}_{j,k\in\mathbb{Z}}$, the functions have compact support when $k \neq 0$. However, the functions $S^{-1}\theta_{j,0}$ do not have compact support. $\square$

Fom general frame theory:

Two ways to awoid inconvenient frame expansions:

- Look for tight frames;
- Look for convenient dual frame pairs.

Some more aspects are relevant in the wavelet case:

The popular wavelet bases are based on multiresolution analysis, which leads to a very convenient algorithmic structure. This implies a special form for the function ψ generating the wavelet basis:

$$\psi = \sum_{k\in\mathbb{Z}} c_k D T_k \phi$$

for a certain function ϕ satisfying a scaling equation,

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma)$$

for some 1-periodic function H_0. The algorithmic structure offered by a multiresolution analysis is a great advantage compared to the use of general wavelet orthonormal bases. Thus:

While constructing wavelet frames, it is desirable to maintain the important aspects of the multiresolution analysis.

Therefore, it is natural to require the generator ψ for a tight frame $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ to have the form

$$\psi = \sum_{k\in\mathbb{Z}} c_k DT_k\phi$$

for some function ϕ; and, if we want to construct two dual wavelet frames $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\tilde{\psi}\}_{j,k\in\mathbb{Z}}$, it is natural to require that both ψ and $\tilde{\psi}$ have that form.

We want the coefficients in these formulas to be finite sequences.

The B-splines B_m are obvious candidates for the function ϕ. However, we can not obtain all of these properties simultaneously:

Theorem 9.10 *Consider B_m for $m > 1$. Then there does not exist pairs of dual wavelet frames $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\tilde{\psi}\}_{j,k\in\mathbb{Z}}$ for which ψ and $\tilde{\psi}$ are finite linear combinations of functions $DT_kB_m,\ j,k\in\mathbb{Z}$.*

Thus, neither the approach of looking at tight frames, nor the idea of considering wavelet frame pairs work if we want the generator (respectively, generators) to have the form

$$\psi = \sum_{k \in \mathbb{Z}} c_k D T_k B_m$$

for a finite sequence $\{c_k\}$.

Solution: we will consider systems of the wavelet-type, but generated by more than one function.

Definition 9.11 *Consider two sequences of functions*

$$\psi_1, \ldots, \psi_n \in L^2(\mathbb{R}) \text{ and } \tilde{\psi}_1, \ldots, \tilde{\psi}_n \in L^2(\mathbb{R}).$$

We say that $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *and* $\{D^j T_k \tilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *are a pair of dual multiwavelet frames if both are Bessel sequences and*

$$f = \sum_{\ell=1}^{n} \sum_{j,k\in\mathbb{Z}} \langle f, D^j T_k \psi_\ell \rangle D^j T_k \tilde{\psi}_\ell, \ \forall f \in L^2(\mathbb{R}).$$

A pair of dual multiwavelet frames is also called *sibling frames* or *bi-frames*. The frame $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ itself is called a *multiwavelet frame.*

Characterization of all dual wavelet frame pairs:

Theorem 9.12 *Assume that* $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ *and* $\{D^jT_k\widetilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ *are Bessel sequences. Then* $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ *and* $\{D^jT_k\widetilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ *are a pair of dual wavelet frames if and only if the two equations*

$$\begin{cases} \displaystyle\sum_{\ell=1}^{n}\sum_{j\in\mathbb{Z}}\widehat{\psi}_\ell(2^j\gamma)\overline{\widehat{\widetilde{\psi}}_\ell(2^j\gamma)}=1, \\ \displaystyle\sum_{\ell=1}^{n}\sum_{j=0}^{\infty}\widehat{\psi}_\ell(2^j\gamma)\overline{\widehat{\widetilde{\psi}}_\ell(2^j(\gamma+q))}=0,\ \ q\in 2\mathbb{N}+1 \end{cases}$$

hold for a.e. $\gamma\in\mathbb{R}$.

Characterization of tight wavelet frames generated by one generator ψ:

Theorem 9.13 *A function $\psi \in L^2(\mathbb{R})$ generates a tight wavelet frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ with frame bound A if and only if the equations*

$$\begin{cases} \displaystyle\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\gamma)|^2 = A, \\ \displaystyle\sum_{j=0}^{\infty} \hat{\psi}(2^j\gamma)\overline{\hat{\psi}\left(2^j(\gamma+q)\right)} = 0, \quad q \in 2\mathbb{N}+1 \end{cases}$$

hold for a.e. $\gamma \in \mathbb{R}$.

10 The unitary extension principle

We will state the unitary extension principle of Ron and Shen, which enables us to construct tight frames for $L^2(\mathbb{R})$ of the form

$$\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}.$$

Terminology:

The interval $]-\frac{1}{2},\frac{1}{2}[$ is identified with the torus $\mathbb{T}$

The class of 1-periodic functions on $\mathbb{R}$ whose restriction to $]-\frac{1}{2},\frac{1}{2}[$ belongs to $L^p(-\frac{1}{2},\frac{1}{2})$ is denoted by $L^p(\mathbb{T})$.

The functions $\psi_1,\ldots,\psi_n$ will be constructed on the basis of a function satisfying a refinement equation. It is convenient to denote the refinable function by ψ_0 instead of ϕ.

Standing assumptions and conventions:

Let $\psi_0 \in L^2(\mathbb{R})$ and assume that

(i) There exists a function $H_0 \in L^\infty(\mathbb{T})$ such that
$$\widehat{\psi}_0(2\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma).$$

(ii) $\lim_{\gamma\to 0} \widehat{\psi}_0(\gamma) = 1.$

Further, let $H_1, \dots, H_n \in L^\infty(\mathbb{T})$, and define $\psi_1, \dots, \psi_n \in L^2(\mathbb{R})$ by
$$\widehat{\psi}_\ell(2\gamma) = H_\ell(\gamma)\widehat{\psi}_0(\gamma),\ \ell = 1, \dots, n.$$

Finally, let H denote the $(n+1)\times 2$ matrix-valued function defined by
$$H(\gamma) = \begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_n(\gamma) & T_{1/2}H_n(\gamma) \end{pmatrix},\ \gamma \in \mathbb{R}.$$

We want to find conditions on the functions $H_1, \ldots, H_n$ such that $\psi_1, \ldots, \psi_n$ generate a multiwavelet frame for $L^2(\mathbb{R})$.

Explicit expression for ψ_ℓ: expanding H_ℓ in a Fourier series,

$$H_\ell(\gamma) = \sum_{k \in \mathbb{Z}} c_{k,\ell} e^{2\pi i k \gamma},$$

we have

$$\begin{aligned} \psi_\ell(x) &= \sqrt{2} \sum_{k \in \mathbb{Z}} c_{k,\ell} D T_{-k} \psi_0(x) \\ &= 2 \sum_{k \in \mathbb{Z}} c_{k,\ell} \psi_0(2x + k). \end{aligned}$$

Note: If H_ℓ are trigonometric polynomials, the sums are finite. Therefore the functions ψ_ℓ have compact support if ψ_0 has compact support.

The general setup presented here preserves the algorithmic structure of a multiresolution analysis: by Theorem 9.5, the spaces

$$V_j := \overline{\text{span}}\{D^j T_k \psi_0\}_{k\in\mathbb{Z}}, \ j \in \mathbb{Z},$$

satisfy the conditions for a multiresolution analysis in Definition 9.1, except (v). Also,

$$\psi_1, \ldots, \psi_n \in V_1.$$

The unitary extension principle:

Theorem 10.1 *Let* $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ *be as in the general setup on page 113, and assume that* $H(\gamma)^* H(\gamma) = I$ *for a.e.* $\gamma \in \mathbb{T}$. *Then the multiwavelet system* $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *constitutes a tight frame for* $L^2(\mathbb{R})$ *with frame bound equal to* 1.

The matrix $H(\gamma)^* H(\gamma)$ has four entries, but it is enough to verify two sets of equations:

Corollary 10.2 *Let* $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ *be as in the general setup on page 113, and assume that*

$$\begin{cases} \displaystyle\sum_{\ell=0}^{n} |H_\ell(\gamma)|^2 = 1, \\ \displaystyle\sum_{\ell=0}^{n} \overline{H_\ell(\gamma)} T_{1/2} H_\ell(\gamma) = 0, \end{cases}$$

for a.e. $\gamma \in \mathbb{T}$*. Then* $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ *constitutes a tight frame for* $L^2(\mathbb{R})$ *with frame bound equal to* 1*.*

Example 10.3 For any $m = 1, 2, \ldots$, we consider the B-spline

$$\psi_0 := B_{2m}$$

of order $2m$. Then

$$\widehat{\psi}_0(\gamma) = \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^{2m}.$$

It is clear that $\lim_{\gamma\to 0} \widehat{\psi}_0(\gamma) = 1$, and by direct calculation,

$$\begin{aligned} \widehat{\psi}_0(2\gamma) &= \left(\frac{\sin(2\pi\gamma)}{2\pi\gamma}\right)^{2m} \\ &= \left(\frac{2\sin(\pi\gamma)\cos(\pi\gamma)}{2\pi\gamma}\right)^{2m} \\ &= \cos^{2m}(\pi\gamma)\widehat{\psi}_0(\gamma). \end{aligned}$$

Thus ψ_0 satisfies a refinement equation with two-scale symbol

$$H_0(\gamma) = \cos^{2m}(\pi\gamma).$$

Now, consider the binomial coefficient

$$\begin{pmatrix} 2m \\ \ell \end{pmatrix} := \frac{(2m)!}{(2m-\ell)!\ell!},$$

and define the functions $H_1, \ldots, H_{2m} \in L^\infty(\mathbb{T})$ by

$$H_\ell(\gamma) = \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \sin^\ell(\pi\gamma) \cos^{2m-\ell}(\pi\gamma).$$

Using that $\cos(\pi(\gamma - 1/2)) = \sin(\pi\gamma)$ and $\sin(\pi(\gamma - 1/2)) = -\cos(\pi\gamma)$,

$$T_{1/2} H_\ell(\gamma) = \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} (-1)^\ell \cos^\ell(\pi\gamma) \sin^{2m-\ell}(\pi\gamma).$$

Thus, the matrix H is given by

$$H(\gamma)=\begin{pmatrix} H_0(\gamma) & T_{1/2}H_0(\gamma) \\ H_1(\gamma) & T_{1/2}H_1(\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ H_{2m}(\gamma) & T_{1/2}H_{2m}(\gamma) \end{pmatrix}=$$

$$\begin{pmatrix} \cos^{2m}(\pi\gamma) & \sin^{2m}(\pi\gamma) \\ \sqrt{\begin{pmatrix} 2m \\ 1 \end{pmatrix}}\sin(\pi\gamma)\cos^{2m-1}(\pi\gamma) & -\sqrt{\begin{pmatrix} 2m \\ 1 \end{pmatrix}}\cos(\pi\gamma)\sin^{2m-1}(\pi\gamma) \\ \sqrt{\begin{pmatrix} 2m \\ 2 \end{pmatrix}}\sin^2(\pi\gamma)\cos^{2m-2}(\pi\gamma) & \sqrt{\begin{pmatrix} 2m \\ 2 \end{pmatrix}}\cos^2(\pi\gamma)\sin^{2m-2}(\pi\gamma) \\ \cdot & \cdot \\ \cdot & \cdot \\ \sqrt{\begin{pmatrix} 2m \\ 2m \end{pmatrix}}\sin^{2m}(\pi\gamma) & \sqrt{\begin{pmatrix} 2m \\ 2m \end{pmatrix}}\cos^{2m}(\pi\gamma) \end{pmatrix}.$$

We now verify the conditions in Corollary 10.2. Using the binomial formula

$$(x+y)^{2m} = \sum_{\ell=0}^{2m} \binom{2m}{\ell} x^\ell y^{2m-\ell},$$

$$\begin{aligned}\sum_{\ell=0}^{2m} |H_\ell(\gamma)|^2 &= \sum_{\ell=0}^{2m} \binom{2m}{\ell} \sin^{2\ell}(\pi\gamma)\cos^{2(2m-\ell)}(\pi\gamma) \\ &= \left(\sin^2(\pi\gamma)+\cos^2(\pi\gamma)\right)^{2m} \\ &= 1, \ \gamma \in \mathbb{T}.\end{aligned}$$

Using the binomial formula with $x=-1, y=1$,

$$\begin{aligned}&\sum_{\ell=0}^{2m} \overline{H_\ell(\gamma)} T_{1/2} H_\ell(\gamma) \\ &= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma) \sum_{\ell=0}^{2m} (-1)^\ell \binom{2m}{\ell} \\ &= \sin^{2m}(\pi\gamma)\cos^{2m}(\pi\gamma)(1-1)^{2m} \\ &= 0.\end{aligned}$$

Now Corollary 10.2 implies that the $2m$ functions $\psi_1, \ldots, \psi_{2m}$ defined by

$$\begin{aligned}\widehat{\psi_\ell}(\gamma) &= H_\ell(\gamma/2)\psi_0(\gamma/2) \\ &= \sqrt{\binom{2m}{\ell}}\frac{\sin^{2m+\ell}(\pi\gamma/2)\cos^{2m-\ell}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}}\end{aligned}$$

generate a tight frame $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,2m}$ for $L^2(\mathbb{R})$. $\square$

A small modification:

Example 10.4 We continue Example 10.3, but now we define

$$H_\ell(\gamma) \;=\; i^\ell \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \sin^\ell(\pi\gamma)\cos^{2m-\ell}(\pi\gamma).$$

H_ℓ only differs from the choice in Example 10.3 by a constant of absolute value 1, so the functions $\psi_1, \ldots, \psi_{2m}$ given by

$$\widehat{\psi_\ell}(2\gamma) = H_\ell(\gamma)\widehat{\psi_0}(\gamma), \;\; \ell = 1, \ldots, 2m,$$

also generate a tight multiwavelet frame.

Rewrite $H_\ell(\gamma)$ using Euler's formula:

$$\begin{aligned} & H_\ell(\gamma) \\ = \;& i^\ell \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \left(\frac{e^{\pi i\gamma} - e^{-\pi i\gamma}}{2i}\right)^\ell \left(\frac{e^{\pi i\gamma} + e^{-\pi i\gamma}}{2}\right)^{2m-\ell} \\ = \;& 2^{-2m} \sqrt{\begin{pmatrix} 2m \\ \ell \end{pmatrix}} \left(e^{\pi i\gamma} - e^{-\pi i\gamma}\right)^\ell \left(e^{\pi i\gamma} + e^{-\pi i\gamma}\right)^{2m-\ell}. \end{aligned}$$

Via the binomial formula we see that $H_\ell(\gamma)$ is a finite linear combination of terms

$$e^{-2\pi i m\gamma}, e^{-2\pi i(m-1)\gamma}, \ldots, e^{2\pi i(m-1)\gamma}, e^{2\pi i m\gamma}.$$

All coefficients in the linear combination are real. Writing

$$H_\ell(\gamma) = \sum_{k=-m}^{m} c_{k,\ell} e^{2\pi i k\gamma},$$

Lemma 9.2 shows that

$$\psi_\ell = \sqrt{2} \sum_{k=-m}^{m} c_{k,\ell} D T_{-k} \psi_0.$$

That is, ψ_ℓ is a real-valued spline. Since $DT_m\psi_0$ has support in $[0, m]$ and $DT_{-m}\psi_0$ has support in $[-m, 0]$, the spline ψ_ℓ has support in $[-m, m]$. The splines ψ_ℓ inherit other properties from ψ_0: they have degree $2m-1$, belong to $C^{2m-2}(\mathbb{R})$, and have knots at $\mathbb{Z}/2$. $\square$

Explicitly, in the case $m = 1$:

Example 10.5 In the case $m = 1$, the construction in Example 10.4 leads to two generators ψ_1 and ψ_2. Via the expression for H_1,

$$\begin{aligned} H_1(\gamma) &= \frac{1}{4}\sqrt{\binom{2m}{1}}(e^{\pi i\gamma} - e^{-\pi i\gamma})(e^{\pi i\gamma} + e^{-\pi i\gamma}) \\ &= \frac{1}{2\sqrt{2}}(e^{2\pi i\gamma} - e^{-2\pi i\gamma}). \end{aligned}$$

By Lemma 9.2 we conclude that

$$\psi_1(x) = \frac{1}{\sqrt{2}}(B_2(2x+1) - B_2(2x-1)).$$

Similarly,

$$\psi_2(x) = \frac{1}{2}\left(B_2(2x+1) - 2B_2(2x) + B_2(2x-1)\right).$$

$\square$

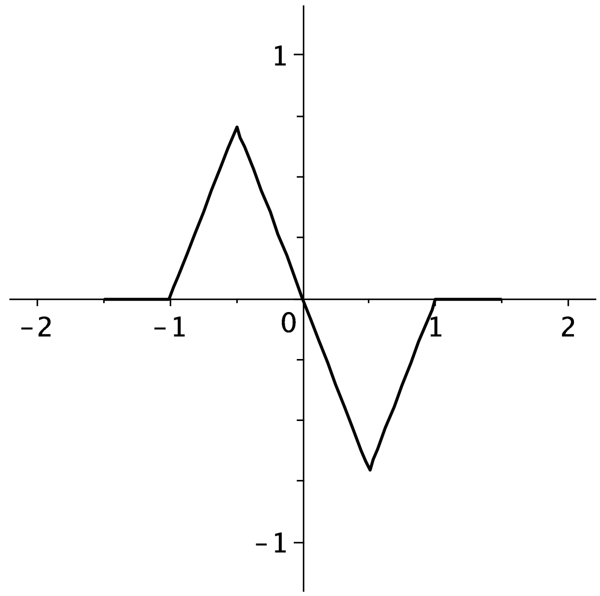


Figure 10: The function ψ_1.

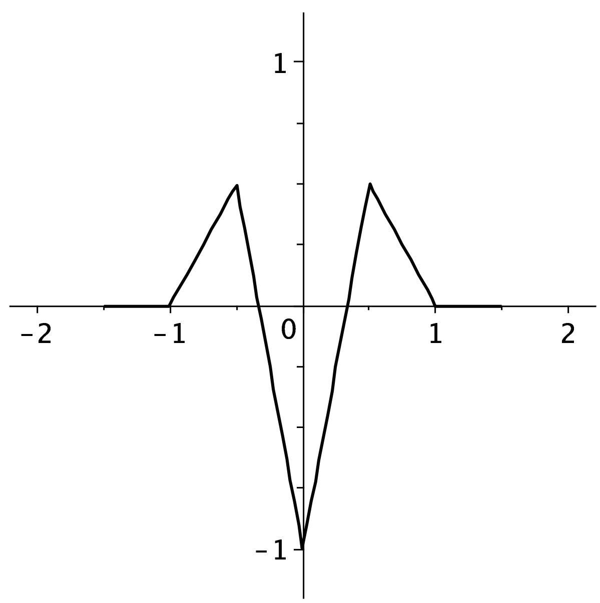


Figure 11: The function ψ_2.

Example 10.6 Can also construct spline frames with support on $[0, 2m]$. Letting $\psi_0 := N_{2m}$, one can prove that

$$\widehat{\psi_0}(2\gamma) = H_0(\gamma)\widehat{\psi_0}(\gamma)$$

with

$$H_0(\gamma) = \left(\frac{1 + e^{-2\pi i\gamma}}{2}\right)^{2m} = e^{-2\pi i m\gamma}\cos^{2m}(\pi\gamma).$$

Since H_0 appears from the corresponding function for B_{2m} simply by multiplication with $e^{-2\pi i m\gamma}$, the functions

$$H_\ell(\gamma) = e^{-2\pi i m\gamma}\sqrt{\binom{2m}{\ell}}\sin^\ell(\pi\gamma)\cos^{2m-\ell}(\pi\gamma)$$

satisfy the conditions in the unitary extension principle. We prefer to consider

$$H_\ell(\gamma) = i^\ell e^{-2\pi i m\gamma}\sqrt{\binom{2m}{\ell}}\sin^\ell(\pi\gamma)\cos^{2m-\ell}(\pi\gamma);$$

with this choice, the functions $\psi_1, \dots, \psi_{2m}$

defined by

$$
\begin{aligned}
&\widehat{\psi_\ell}(\gamma) \\
&= H_\ell(\gamma/2)\psi_0(\gamma/2) \\
&= i^\ell e^{-2\pi i m\gamma}\sqrt{\binom{2m}{\ell}}\frac{\sin^{2m+\ell}(\pi\gamma/2)\cos^{2m-\ell}(\pi\gamma/2)}{(\pi\gamma/2)^{2m}}
\end{aligned}
$$

generate a tight frame for $L^2(\mathbb{R})$. The spline functions $\psi_1, \ldots, \psi_{2m}$ now have support on $[0, 2m]$.

□

Shortcomings for the unitary extension principle:

- The computational effort increases with the order of the B-spline B_{2m}: For higher orders, we need more generators, and more non-zero coefficients appear in ψ_ℓ.
- There is a limitation on the possible number of vanishing moments ψ_ℓ can have.

If $\{\psi_\ell\}_{\ell=1}^n$ is constructed via the unitary extension principle,then

$$\widehat{\psi_\ell}(\gamma) = H_\ell(\gamma/2)\widehat{\psi_0}(\gamma/2)$$

and

$$\widehat{\psi_0}(0) = 1.$$

Thus, the number of vanishing moments for the function ψ_ℓ is equal to the order of zero for H_ℓ at $\gamma = 0$.

Example 10.7 Consider B_{2m}; it satisfies a refinement equation with two-scale symbol

$$H_0(\gamma) = \cos^{2m}(\pi\gamma).$$

If we want to construct a frame via the unitary extension principle, then

$$1 = \sum_{\ell=0}^{n} |H_\ell(\gamma)|^2,$$

i.e.,

$$\sum_{\ell=1}^{n} |H_\ell(\gamma)|^2 = 1 - \cos^{4m}(\pi\gamma). \qquad (11)$$

The order of the zero at $\gamma = 0$ for the function $1 - \cos^{4m}(\pi\gamma)$ is 2, so also on the left-hand side of (11) we can only factor γ^2 out; this implies that at least one of the functions $|H_\ell|^2$ can at most have a zero at $\gamma = 0$ of order 2.

Therefore at least one of the functions ψ_ℓ can at most have one vanishing moment. □

11 The oblique extension principle

An important reformulation of Theorem 10.1 was simultaneously obtained by Daubechies, Han, Ron and Shen in [4] and Chui, He and Stöckler in [3]. It gives a more flexible recipe for construction of frames than Theorem 10.1, and is called the *oblique extension principle:*

Theorem 11.1 *Let* $\{\psi_\ell, H_\ell\}_{\ell=0}^n$ *be as in the general setup. Assume that there exists a strictly positive function* $\theta \in L^\infty(\mathbb{T})$ *for which*

$$\lim_{\gamma\to 0} \theta(\gamma) = 1$$

and such that for a.e. $\gamma \in \mathbb{T}$,

$$H_0(\gamma)\overline{H_0(\gamma+\nu)}\theta(2\gamma) \;+\; \sum_{\ell=1}^n H_\ell(\gamma)\overline{H_\ell(\gamma+\nu)} = \begin{cases} \theta(\gamma) & \text{if } \nu = 0, \\ 0 & \text{if } \nu = \frac{1}{2}. \end{cases}$$

Then the functions $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ *constitute a tight frame for* $L^2(\mathbb{R})$ *with frame bound equal to* 1.

Proof. Assume that the conditions in Theorem 11.1 are satisfied, and define the function $\widetilde{\psi_0} \in L^2(\mathbb{R})$ by

$$\widehat{\widetilde{\psi_0}}(\gamma) = \sqrt{\theta(\gamma)}\widehat{\psi_0}(\gamma).$$

Define the 1-periodic functions $\widetilde{H_0}, \dots, \widetilde{H_n}$ by

$$\widetilde{H_0}(\gamma) = \sqrt{\frac{\theta(2\gamma)}{\theta(\gamma)}} H_0(\gamma), \quad \widetilde{H_\ell}(\gamma) = \sqrt{\frac{1}{\theta(\gamma)}} H_\ell(\gamma).$$

We now prove that $\widetilde{\psi_0}, \widetilde{H_0}, \dots, \widetilde{H_n}$ satisfy the conditions in the general setup. First,

$$\begin{aligned}
\widehat{\widetilde{\psi_0}}(2\gamma) = \sqrt{\theta(2\gamma)}\widehat{\psi_0}(2\gamma) &= \sqrt{\theta(2\gamma)} H_0(\gamma)\widehat{\psi_0}(\gamma) \\
&= \sqrt{\frac{\theta(2\gamma)}{\theta(\gamma)}} H_0(\gamma)\widehat{\widetilde{\psi_0}}(\gamma) \\
&= \widetilde{H_0}(\gamma)\widehat{\widetilde{\psi_0}}(\gamma).
\end{aligned}$$

Also,

$$\lim_{\gamma \to 0} \widehat{\widetilde{\psi_0}}(\gamma) = \lim_{\gamma \to 0} \left(\sqrt{\theta(\gamma)}\widehat{\psi_0}(\gamma) \right) = 1.$$

Via the definition,

$$\begin{aligned}\sum_{\ell=0}^{n} |\widetilde{H_\ell}(\gamma)|^2 &= \frac{\theta(2\gamma)}{\theta(\gamma)}|H_0(\gamma)|^2 + \sum_{\ell=1}^{n} \frac{|H_\ell(\gamma)|^2}{\theta(\gamma)} \\ &= 1,\ a.e.\ \gamma \in \mathbb{T}.\end{aligned}$$

Thus, $\widetilde{H_0}, \ldots, \widetilde{H_n} \in L^\infty(\mathbb{T})$. Similarly,

$$\sum_{\ell=0}^{n} \widetilde{H_\ell}(\gamma)\overline{\widetilde{H_\ell}(\gamma + \frac{1}{2})} = 0,\ a.e.\ \gamma \in \mathbb{T}.$$

Defining the functions $\widetilde{\psi_1}, \ldots, \widetilde{\psi_n}$ by

$$\widehat{\widetilde{\psi}}_\ell(2\gamma) = \tilde{H}_\ell(\gamma)\widehat{\widetilde{\psi_0}}(\gamma),\ \ell = 1, \ldots, n,$$

it follows from Theorem 10.1 that the functions $\{D^j T_k \widetilde{\psi_\ell}\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ constitute a tight frame for $L^2(\mathbb{R})$ with frame bound equal to 1. Now,

$$\begin{aligned}\widehat{\psi_\ell}(2\gamma) &= H_\ell(\gamma)\widehat{\psi_0}(\gamma) \\ &= \sqrt{\theta(\gamma)}\widetilde{H_\ell}(\gamma)\frac{1}{\sqrt{\theta(\gamma)}}\widehat{\widetilde{\psi}}_0(\gamma) \\ &= \widehat{\widetilde{\psi}}_\ell(2\gamma),\end{aligned}$$

which shows that $\psi_\ell = \widetilde{\psi}_\ell$. □

The UEP and the OEP are equivalent:

Taking $\theta = 1$ in the OEP we obtain the UEP.

The proof of Theorem 11.1 shows that all OEP constructions can also be obtained via the UEP.

However, in practice the OEP is more flexible:

Suppose that

- ψ_0 is a compactly supported function such that for some function $H_0 \in L^\infty(\mathbb{R})$,
$$\widehat{\psi}_0(2\gamma) = H_0(\gamma)\widehat{\psi}_0(\gamma);$$
- the functions θ, H_ℓ are trigonometric polynomials satisfying the conditions in the OEP

Then the generators ψ_ℓ for the frame $\{D^j T_k \psi_\ell\}$ have the form
$$\psi_\ell(x) = \sum c_{k,\ell}\psi_0(2x - k)$$
(finite sum) and thus compact support.

The same frame can be constructed via the UEP: if we define $\widetilde{\psi_0}$ by

$$\widehat{\widetilde{\psi_0}}(\gamma) = \sqrt{\theta(\gamma)}\widehat{\psi_0}(\gamma),$$

then the functions $\widetilde{\psi_\ell}$ constructed in the proof of the UEP will satisfy the conditions in the unitary extension principle, and $\psi_\ell = \widetilde{\psi_\ell}$.

However, in general $\widetilde{\psi_0}$ is not compactly supported, and H_ℓ is not a trigonometric polynomial, so the fact that the resulting frame $\{D^j T_k \widetilde{\psi_\ell}\}$ is generated by compactly supported functions is somewhat miraculous and could certainly not be predicted in advance.

There are constructions which appear naturally via the OEP, but one would not even think about constructing them via the UEP.

In practice: desirable to have as few generators as possible.

Corollary 11.2 *Let ψ_0 and H_0 be as in the general setup. Let $\theta \in L^\infty(\mathbb{T})$ be a strictly positive function for which* $\lim_{\gamma\to 0}\theta(\gamma) = 1$, *chosen such that the function*

$$\eta(\gamma) := \theta(\gamma) - \theta(2\gamma)\left(|H_0(\gamma)|^2 + |H_0(\gamma+\frac{1}{2})|^2\right)$$

is positive as well. Fix an integer $n \geq 2$ and let $\{G_\ell\}_{\ell=2}^n$ be 1-periodic trigonometric polynomials for which

$$\sum_{\ell=2}^{n} |G_\ell(\gamma)|^2 = 1, \text{ and } \sum_{\ell=2}^{n} G_\ell(\gamma)\overline{G_\ell(\gamma+\frac{1}{2})} = 0.$$

Let ρ, σ be 1*-periodic functions such that*

$$|\rho(\gamma)|^2 = \theta(\gamma), \quad |\sigma(\gamma)|^2 = \eta(\gamma), \qquad (12)$$

and define the 1-periodic functions $\{H_\ell\}_{\ell=1}^n$ by

$$H_1(\gamma) = e^{2\pi i\gamma}\rho(2\gamma)\overline{H_0(\gamma+\frac{1}{2})}, \quad H_\ell(\gamma) = G_\ell(\gamma)\sigma(\gamma).$$

Then the functions $\{\psi_\ell\}_{\ell=1}^n$ in the general setup generate a tight frame $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$.

If the condition on η is satisfied, Corollary 11.2 makes it easy to obtain frames with for example three generators.

For example, the conditions on G_ℓ are satisfied with

$$G_2(\gamma) = \frac{1}{\sqrt{2}}, \quad G_3(\gamma) = \frac{1}{\sqrt{2}} e^{2\pi i \gamma}.$$

Thus, in order to apply Corollary 11.2, the remaining work consists in finding ρ, σ such that (12) is satisfied.

If the functions θ and η are trigonometric polynomials, then the functions ρ and σ can be chosen to be trigonometric polynomials as well (*spectral factorization*).

The assumption on η even implies that we can construct a frame generated by two functions:

Corollary 11.3 *Let* ψ_0 *and* H_0 *be as in the general setup. Let* $\theta \in L^\infty(\mathbb{T})$ *be a strictly positive function for which* $\lim_{\gamma\to 0}\theta(\gamma) = 1$, *chosen such that the function*

$$\eta(\gamma) := \theta(\gamma) - \theta(2\gamma)\left(|H_0(\gamma)|^2 + |H_0(\gamma + \frac{1}{2})|^2\right)$$

is positive as well. Define the functions ρ, σ *by*

$$|\rho(\gamma)|^2 = \theta(\gamma), \quad |\sigma(\gamma)|^2 = \eta(\gamma),$$

and let

$$\begin{aligned} H_1(\gamma) &= e^{2\pi i\gamma}\rho(2\gamma)\overline{H_0(\gamma + \frac{1}{2})}, \\ H_2(\gamma) &= H_0(\gamma)\sigma(2\gamma). \end{aligned}$$

Then the functions $\{\psi_\ell\}_{\ell=1}^2$ *generate a tight frame* $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,2}$ *for* $L^2(\mathbb{R})$.

If θ and H_0 are trigonometric polynomials, then η is also a trigonometric polynomial.

The assumption that θ and η are positive implies that we can choose ρ, σ in (12) to be trigonometric polynomials.

In this case, the generators ψ_ℓ are finite linear combinations of functions $DT_k\psi_0$.

The conditions for reduction of the number of generators are satisfied for B-splines:

Theorem 11.4 *Let B_{2m} denote the B-spline of order $2m$ with two-scale symbol $H_0(\gamma) = \cos^{2m}(\pi\gamma)$. Then, for each positive integer $M \leq 2m$, there exists a trigonometric polynomial θ of the form*

$$\theta(\gamma) = 1 + \sum_{j=1}^{M-1} c_j \sin^{2j}(\pi\gamma),$$

for which the following hold:

(i) $c_j \geq 0$ *for all* $j = 1, \ldots, M-1$, *i.e.,* $\theta(\gamma) > 0$ *for all* $\gamma \in \mathbb{R}$*;*

(ii) The function η is positive;

(iii) The generators in the tight wavelet frames constructed via the oblique extension principle and its corollaries have M vanishing moments.

The coefficients $c_j,\ j = 1, \ldots, M-1$ can be determined via the requirement that

$$\left(1 + \sum_{j=1}^{\infty} \frac{(2j-1)!}{(2j)!\,(2j+1)} y^j\right)^{4m}$$

$$= \ 1 + \sum_{j=1}^{M-1} c_j y^j + O(|y|^M) \text{ as } y \to 0.$$

Example 11.5 Consider the B-spline B_{2m} and $M = 2$. Note that

$$\begin{aligned}
&\left(1 + \sum_{j=1}^{\infty} \frac{(2j-1)!}{(2j)!\,(2j+1)} y^j\right)^{4m} \\
&= \left(1 + \frac{1}{6}y + \frac{1}{20}y^2 + \cdots\right)^{4m} \\
&= 1 + \frac{2m}{3}y + O(|y|^2).
\end{aligned}$$

This proves that for $M = 2$, the condition is satisfied with $c_1 = 2m/3$. Thus, the desired trigonometric polynomial is

$$\begin{aligned}
\theta(\gamma) &= 1 + \frac{2m}{3}\sin^2(\pi\gamma) \\
&= 1 + \frac{2m}{3}\frac{1 - \cos 2\pi\gamma}{2} \\
&= \frac{3 + m}{3} - \frac{m}{3}\cos 2\pi\gamma.
\end{aligned}$$

□

Conclusion:

- We can construct multiwavelet frames with two generators based on *any* B-spline B_{2m}.
- If we choose ρ, σ in (12) to be trigonometric polynomials, then H_1, H_2 in Theorem 11.3 are trigonometric polynomials, and the associated frame generators ψ_1, ψ_2 are finite linear combinations of functions

$$B_{2m}(2x-k), k \in \mathbb{Z}.$$

- Choosing m large enough, we can obtain generators belonging to any prescribed smoothness class $C^N(\mathbb{R})$.

Frame constructions via the UEP and the OEP:

Example 11.6 Let $\psi_0 = N_2$. Then

$$\widehat{N_2}(2\gamma) = H_0(\gamma)\widehat{N_2}(\gamma),$$

where

$$H_0(\gamma) = \frac{(1+e^{-2\pi i\gamma})^2}{4} = e^{-2\pi i\gamma}\cos^2(\pi\gamma).$$

We first revisit Example 10.6 and then give constructions via the oblique extension principle and its corollaries.

(i) Defining H_1 and H_2 by

$$\begin{aligned} H_1(\gamma) &= ie^{-2\pi i\gamma}\sqrt{2}\sin(\pi\gamma)\cos(\pi\gamma) \\ &= \frac{\sqrt{2}}{4}(1-e^{-4\pi i\gamma}), \\ H_2(\gamma) &= -e^{-2\pi i\gamma}\sin^2(\pi\gamma) = \frac{(1-e^{-2\pi i\gamma})^2}{4}, \end{aligned}$$

it follows from Example 10.6 that the associated functions $\psi_1^{(i)} := \psi_1$ and ψ_2 generate a tight frame for $L^2(\mathbb{R})$. They are given by

$$\psi_1^{(i)}(x) \;=\; \frac{1}{\sqrt{2}}(N_2(2x) - N_2(2x-2)),$$
$$\psi_2(x) \;=\; \frac{1}{2}\left(N_2(2x) - 2N_2(2x-1) + N_2(2x-2)\right).$$

See Figures 9-10.

(ii) An alternative construction can be obtained via the oblique extension principle. Let

$$\theta(\gamma) := \frac{4 - \cos(2\pi\gamma)}{3}.$$

In this example we keep the choice of H_2 in (i). Thus, if we want to use the oblique extension principle, we have to choose H_1 such that

$$|H_1(\gamma)|^2 \;=\; \theta(\gamma) - |H_0(\gamma)|^2\theta(2\gamma) - |H_2(\gamma)|^2,$$

and

$$H_1(\gamma)\overline{H_1(\gamma + \frac{1}{2})}$$
$$= \; -H_0(\gamma)\overline{H_0(\gamma + \frac{1}{2})}\theta(2\gamma) - H_2(\gamma)\overline{H_2(\gamma + \frac{1}{2})}.$$

One can take

$$\begin{aligned}\psi_1^{(ii)}(\gamma) \;=\;& \frac{1}{2\sqrt{6}}\left(N_2(2\gamma-4)+2N_2(2\gamma-3)\right)\\ &-\frac{6}{2}N_2(2\gamma-2)\\ &+\frac{1}{2\sqrt{6}}\left(2N_2(2\gamma-1)+N_2(2\gamma)\right),\end{aligned}$$

which has support on $[0,3]$. The function $\psi_1^{(ii)}$ is shown in Figure 11. □

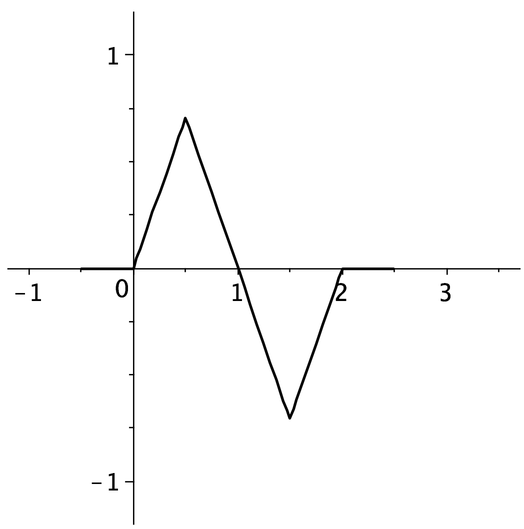


Figure 12: The function $\psi_1^{(i)}$.

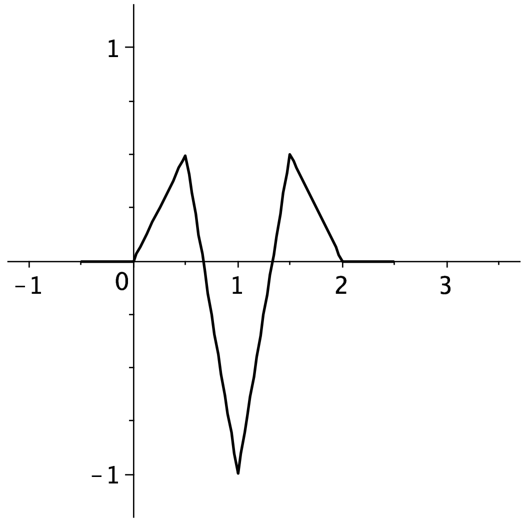


Figure 13: The function ψ_2.

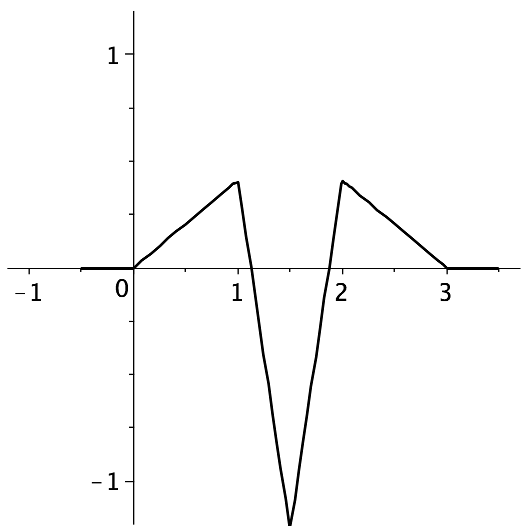


Figure 14: The function $\psi_1^{(ii)}$.

More reasons for constructing frames via the oblique extension principle:

Assume that $\{H_\ell, \psi_\ell\}_{\ell=0}^{n}$ is as in the general setup, and that $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\dots,n}$ is a tight frame constructed via the oblique extension principle and $\psi_0 = B_{2m}$ for some $m \in \mathbb{N}$.Let

$$V_j = \overline{\text{span}}\{D^j T_k \psi_0\}_{j,k\in\mathbb{Z}}.$$

For $s > 0$, consider the *Sobolev space*

$$H_s(\mathbb{R}) = \left\{ f \mid \int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2 (1+|\gamma|^2)^s d\gamma < \infty \right\}.$$

$H_s(\mathbb{R})$ is a Banach space with respect to the natural norm,

$$||f||_{H_s} = \left(\int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2 (1+|\gamma|^2)^s d\gamma \right)^{1/2}.$$

We say that ψ_0 *provides approximation order* s if for all f in the Sobolev space $H^s(\mathbb{R})$,

$$\operatorname{dist}(f, V_j) = O(2^{-js}),$$

i.e., if there exists a constant $C > 0$ such that

$$\operatorname{dist}(f, V_j) \leq C 2^{-js}, \ \forall j \in \mathbb{Z}.$$

For the tight frame $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$, we know that for all $f \in L^2(\mathbb{R})$,

$$f = \sum_{\ell=1}^{n} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, D^j T_k \psi_\ell \rangle D^j T_k \psi_\ell.$$

As an approximation of f we can use

$$Q_J f := \sum_{\ell=1}^{n} \sum_{j<J} \sum_{k\in\mathbb{Z}} \langle f, D^j T_k \psi_\ell \rangle D^j T_k \psi_\ell$$

for a reasonably large value of $J \in \mathbb{Z}$. We say that *the frame* $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *provides approximation order* s if for all $f \in H^s(\mathbb{R})$,

$$||f - Q_J f|| = O(2^{-sJ}).$$

We know that $\psi_1, \ldots, \psi_n \in V_1$, so $Q_J f \in V_J$ for all $J \in \mathbb{Z}$; thus, the approximation order of the frame $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ can not exceed the approximation order of the underlying refinable function ψ_0.

We usually want the approximation order of $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ to be as large as possible.

Can prove that

- The refinable function $\psi_0 = B_{2m}$ provides approximation order $2m$.
- With the function θ chosen as in the OEP, the approximation order of $\{D^j T_k \psi_\ell\}$ is

$$min(2m, 2M).$$

- Choosing M sufficiently large, we can obtain the approximation order $2m$, which is the best possible one can hope for with the given function $\psi_0 = B_{2m}$.

13 Construction of pairs of dual wavelet frames

Extension of the OEP to construction of dual multiwavelet pairs:

Theorem 13.1 *Let* $\{H_\ell, \psi_\ell\}_{\ell=0}^n$ *and* $\{K_\ell, \widetilde{\psi}_\ell\}_{\ell=0}^n$ *be two sets of functions, satisfying the conditions in the general setup on page 113, and such that for some* $C > 0$ *and* $\rho > \frac{1}{2}$,

$$|\widehat{\psi_0}(\gamma)|, |\widehat{\widetilde{\psi_0}}(\gamma)| \leq \frac{C}{|\gamma|^\rho}, \quad a.e.$$

Assume that there exists a function $\theta \in L^\infty(\mathbb{T})$ *such that* $\lim_{\gamma \to 0} \theta(\gamma) = 1$ *and*

$$H_0(\gamma)\overline{K_0(\gamma+\nu)}\theta(2\gamma) \ + \ \sum_{\ell=1}^n H_\ell(\gamma)\overline{K_\ell(\gamma+\nu)} = \begin{cases} \theta(\gamma) & \text{if } \nu = 0, \\ 0 & \text{if } \nu = \frac{1}{2}. \end{cases}$$

Then $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *and* $\{D^j T_k \widetilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\ell=1,\ldots,n}$ *are a pair of dual multiwavelet frames.*

Easier to find a pair of dual frames via Theorem 13.1 than to construct tight frames via the OEP:

- The function θ is not required to be positive.
- We have freedom to chose two sets of trigonometric polynomials H_ℓ and K_ℓ: in fact, the condition in the OEP corresponds exactly to the condition in Theorem 13.1 with $H_\ell = K_\ell$, and is more complicated to satisfy.

Setup for construction of pairs of dual wavelet frames:

Let $\{\psi_0, H_0\}, \{K_0, \widetilde{\psi_0}\}$ be as in the general setup. Let $\theta \in L^\infty(\mathbb{T})$ be a real-valued function for which $\lim_{\gamma\to 0}\theta(\gamma) = 1$, and assume that the function

$$\begin{aligned}\eta(\gamma) \;&:=\; \theta(\gamma)\\ &-\; \theta(2\gamma)\left(H_0(\gamma)\overline{K_0(\gamma)} + H_0(\gamma+\frac{1}{2})\overline{K_0(\gamma+\frac{1}{2}}\right)\end{aligned}$$

is real-valued and has a zero of order at least 2 at the origin. Choose real-valued functions $\eta_1, \eta_2 \in L^\infty(\mathbb{T})$ such that

$$\eta(\gamma) = 2\eta_1(\gamma)\eta_2(\gamma), \text{ and } \eta_1(0) = \eta_2(0) = 0,$$

and choose two $\frac{1}{2}$-periodic and real-valued functions θ_1, θ_2 such that

$$\theta(2\gamma) = \theta_1(\gamma)\theta_2(\gamma).$$

$\square$

In the construction of tight multiwavelet frames: we had to perform a spectral factorization of the functions θ and η.

The choices of the functions $\eta_1, \eta_2, \theta_1, \theta_2$ will replace the spectral factorization - much easier!

Corollary 13.2 *Assume the setup on page 153 and define* $\{H_\ell\}_{\ell=1}^3$ *and* $\{K_\ell\}_{\ell=1}^3$ *by*

$$\begin{aligned}
H_1(\gamma) &= e^{2\pi i\gamma}\theta_1(\gamma)\overline{K_0(\gamma+\frac{1}{2})},\\
H_2(\gamma) &= \eta_1(\gamma),\\
H_3(\gamma) &= e^{2\pi i\gamma}\eta_1(\gamma),\\
K_1(\gamma) &= e^{2\pi i\gamma}\theta_2(\gamma)\overline{H_0(\gamma+\frac{1}{2})},\\
K_2(\gamma) &= \eta_2(\gamma),\\
K_3(\gamma) &= e^{2\pi i\gamma}\eta_2(\gamma).
\end{aligned}$$

Define the associated functions $\{\psi_\ell\}_{\ell=1}^3$ *and* $\{\widetilde{\psi_\ell}\}_{\ell=1}^3$ *as in the general setup on page 113. Then* $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,2,3}$ *and* $\{D^jT_k\widetilde{\psi_\ell}\}_{j,k\in\mathbb{Z},\ell=1,2,3}$ *constitute a pair of dual multiwavelet frames.*

The number of generators can be reduced to two:

Corollary 13.3 *Assume the setup on page 153 and let*

$$\begin{aligned} H_1(\gamma) &= e^{2\pi i\gamma}\theta_1(\gamma)\overline{K_0(\gamma+\frac{1}{2})}, \\ H_2(\gamma) &= \eta_1(2\gamma)H_0(\gamma), \\ K_1(\gamma) &= e^{2\pi i\gamma}\theta_2(\gamma)\overline{H_0(\gamma+\frac{1}{2})}, \\ K_2(\gamma) &= \eta_2(2\gamma)K_0(\gamma). \end{aligned}$$

Then $\{D^jT_k\psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,2}$ *and* $\{D^jT_k\widetilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\ell=1,2}$ *constitute a pair of dual multiwavelet frames.*

If θ, H_0 and K_0 are trigonometric polynomials and η_1, η_2 and θ_1, θ_2 are real-valued trigonometric polynomials, then the frame generators $\{\psi_\ell\}_{\ell=1}^3$ and $\{\widetilde{\psi}_\ell\}_{\ell=1}^3$ are symmetric if the refinable functions ψ_0 and $\widetilde{\psi_0}$ are symmetric real-valued functions. Thus, the above process will lead to symmetric dual wavelet pairs when applied to even-order B-splines.

Example 13.4 Frame construction with two generators, based on $\psi_0 = \widetilde{\psi_0} = N_2$; the associated two-scale symbol is

$$H_0(\gamma) = \frac{(1+e^{-2\pi i\gamma})^2}{4} = e^{-2\pi i\gamma}\cos^2(\pi\gamma).$$

We again take

$$\theta(\gamma) = \frac{4-\cos(2\pi\gamma)}{3};$$

as proved in Example 11.6 this leads to

$$\begin{aligned} &\eta(\gamma) \\ = \ &\frac{2}{3}(8\cos^4(\pi\gamma)+1)(\cos(\pi\gamma)-1)^2(\cos(\pi\gamma)+1)^2. \end{aligned}$$

If we want to apply Corollary 13.3, we need to find functions $\eta_1, \eta_2, \theta_1, \theta_2$ such that

$$\begin{cases} \ \ \eta(\gamma) = 2\eta_1(\gamma)\eta_2(\gamma), \text{ and } \eta_1(0) = \eta_2(0) = 0, \\ \theta(2\gamma) = \theta_1(\gamma)\theta_2(\gamma) \end{cases}$$

Easy!

For example, we can take

$$\eta_1(\gamma)$$

$$= \frac{1}{3}(8\cos^4(\pi\gamma) + 1)(\cos(\pi\gamma) - 1)(\cos(\pi\gamma) + 1)^2,$$

$$\eta_2(\gamma) \;=\; (\cos(\pi\gamma) - 1),$$

$$\theta_1(\gamma) = 1, \quad \theta_2(\gamma) = \theta(2\gamma) = \frac{4 - \cos(4\pi\gamma)}{3}.$$

The functions in Corollary 13.3 are now

$$
\begin{aligned}
H_1(\gamma) &= e^{2\pi i\gamma}\theta_1(\gamma)\overline{K_0(\gamma+\frac{1}{2})} \\
&= e^{2\pi i\gamma}\frac{(1-e^{2\pi i\gamma})^2}{4}, \\
K_1(\gamma) &= e^{2\pi i\gamma}\theta_2(\gamma)\overline{H_0(\gamma+\frac{1}{2})} \\
&= e^{2\pi i\gamma}\left(\frac{4}{3}-\frac{e^{4\pi i\gamma}+e^{-4\pi i\gamma}}{6}\right)\frac{(1-e^{2\pi i\gamma})^2}{4}, \\
H_2(\gamma) &= \eta_1(2\gamma)H_0(\gamma) \\
&= \frac{1}{3}\left(8\left(\frac{e^{2\pi i\gamma}+e^{-2\pi i\gamma}}{2}\right)^4+1\right) \\
&\quad \times\left(\frac{e^{2\pi i\gamma}+e^{-2\pi i\gamma}}{2}-1\right) \\
&\quad \times\left(\frac{e^{2\pi i\gamma}+e^{-2\pi i\gamma}}{2}+1\right)^2\frac{(1+e^{-2\pi i\gamma})^2}{4}, \\
K_2(\gamma) &= \eta_2(2\gamma)K_0(\gamma) \\
&= \left(\frac{e^{2\pi i\gamma}+e^{-2\pi i\gamma}}{2}-1\right)\frac{(1+e^{-2\pi i\gamma})^2}{4}.
\end{aligned}
$$

With these choices, $\{D^j T_k \psi_\ell\}_{j,k\in\mathbb{Z},\ell=1,2}$ and $\{D^j T_k \widetilde{\psi}_\ell\}_{j,k\in\mathbb{Z},\ell=1,2}$ constitute a pair of dual multiwavelet frames. $\square$

14 The signal processing perspective

Goal: Formulate the UEP in signal processing terms.

Formally, the Z-transform of a sequences $\{h_k\}_{k\in\mathbb{Z}}$ is defined as the infinite series (depending on a variable $z \in \mathbb{C}$)

$$\tilde{H}(z) := \sum_{k\in\mathbb{Z}} h_k z^{-k}.$$

For finite sequences $\{h_k\}_{k\in\mathbb{Z}}$, the Z-transform is defined for all $z \neq 0$.

For $\ell^2(\mathbb{Z})$-sequences (Fourier coefficients), the Z-transform converges for a.e. $z \in \mathbb{C}$ with $|z| = 1$.

The sequence $\{h_k\}_{k\in\mathbb{Z}}$ is often called a *filter.*

Consider the 1-periodic functions $H_\ell,\ \ell = 0, \ldots, n$, in the general setup. Write H_ℓ in terms of Fourier series, with Fourier coefficients $h_{k,\ell}$:

$$H_\ell(\gamma) = \sum_{k\in\mathbb{Z}} h_{k,\ell} e^{2\pi i k\gamma}.$$

In terms of the Z-transform, this means that

$$H_\ell(\gamma) = \sum_{k\in\mathbb{Z}} h_{k,\ell} \left(e^{-2\pi i\gamma}\right)^{-k} = \widetilde{H_\ell}(e^{-2\pi i\gamma}).$$

Reformulation of the UEP:

Theorem 14.1 *Assume that the functions $H_\ell,\ \ell = 0, \ldots, n$, have real Fourier coefficients $h_{k,\ell},\ k \in \mathbb{Z}$. Then the UEP-conditions hold if and only if the equations*

$$\begin{cases} \displaystyle\sum_{\ell=0}^{n} \widetilde{H_\ell}(z)\widetilde{H_\ell}(z^{-1}) = 1, \\ \displaystyle\sum_{\ell=0}^{n} \widetilde{H_\ell}(z)\widetilde{H_\ell}(-z^{-1}) = 0 \end{cases}$$

hold for a.e. $z \in \mathbb{C}$ for which $|z| = 1$.

The polyphase decomposition

We can decompose a sequence $\{h_k\}_{k\in\mathbb{Z}}$ into “even“ and “odd“ parts:

$$\begin{aligned} & (\dots, h_{-2}, h_{-1}, h_0, h_1, h_2, \dots) \\ = & (\dots, h_{-2}, 0, h_0, 0, h_2, \dots) \\ + & (\dots, 0, h_{-1}, 0, h_1, 0, \dots). \end{aligned}$$

Thus, the Z-transformation of $\{h_k\}_{k\in\mathbb{Z}}$ is

$$\begin{aligned} \tilde{H}(z) &= \left[\cdots + h_{-2}z^2 + h_0 + h_2 z^{-2} + \cdots\right] \\ &\quad + \left[\cdots + h_{-1}z + h_1 z^{-1} + h_3 z^{-3} + \cdots\right] \\ &= \left[\cdots + h_{-2}z^2 + h_0 + h_2 z^{-2} + \cdots\right] \\ &\quad + z^{-1}\left[\cdots + h_{-1}z^2 + h_1 + h_3 z^{-2} + \cdots\right] \\ &= \sum_{k\in\mathbb{Z}} h_{2k} z^{-2k} + z^{-1} \sum_{k\in\mathbb{Z}} h_{2k+1} z^{-2k}. \end{aligned}$$

The *polyphase components* of $\tilde{H}(z)$ are now defined as the two functions

$$\widetilde{H_0}(z) := \sum_{k\in\mathbb{Z}} h_{2k} z^{-k}, \quad \widetilde{H_1}(z) = \sum_{k\in\mathbb{Z}} h_{2k+1} z^{-k}.$$

Thus, the Z-transformation has the *polyphase decomposition*

$$\tilde{H}(z) = \widetilde{H_0}(z^2) + z^{-1}\widetilde{H_1}(z^2).$$

Consider now a given sequence of 1-periodic functions H_ℓ, $\ell = 0, \ldots, n$, or, equivalently, a sequence of filters $\{h_{k,\ell}\}_{k\in\mathbb{Z}}$, $\ell = 0, \ldots n$. Associated to the filter $\{h_{k,\ell}\}_{k\in\mathbb{Z}}$, we denote the polyphase components of $\widetilde{H_\ell}$ by $\widetilde{H_{\ell,0}}$ and $\widetilde{H_{\ell,1}}$. Define the $(n+1)\times 2$ matrix of polyphase components H_p by

$$H_p(z) = \begin{pmatrix} \widetilde{H_{0,0}}(z) & \widetilde{H_{0,1}}(z) \\ \widetilde{H_{1,0}}(z) & \widetilde{H_{1,1}}(z) \\ \cdot & \cdot \\ \cdot & \cdot \\ \widetilde{H_{n,0}}(z) & \widetilde{H_{n,1}}(z) \end{pmatrix}.$$

The UEP in terms of the polyphase matrix:

Theorem 14.2 *Assume that the functions* H_ℓ, $\ell = 0, \ldots, n$, *have real Fourier coefficients* $h_{k,\ell}$, $k \in \mathbb{Z}$. *Then the UEP condition is satisfied if and only if*

$$H_p^T(z^{-1})H_p(z) = \frac{1}{2}I \tag{13}$$

for almost all $z \in \mathbb{C}$ *with* $|z| = 1$.

The condition (13) **is well known in the context of filter banks!**

An analysis filter bank is a “black box”, which performs operations on an incoming signal (i.e., a sequence of numbers).

Typically, a filter bank splits the incoming signal into certain subsignals, which contain particular information about the signal.

After processing the subsequences coming out of the analysis filter bank, engineers usually wish to get back to the original input sequence.

For this reason, an analysis filter bank is followed by another filter bank, which reconstructs the original signal from the subsignals; such a filter bank is called a *synthesis filter bank.*

In that case the entire system consisting of the two filter banks is said to have the *perfect reconstruction* property.

The filter banks considered here will contain three operations on the incoming sequence $\{x_k\}_{k\in\mathbb{Z}}$:

- **Convolution with a sequence $\{h_k\}_{k\in\mathbb{Z}}$:** The outcome is a new sequence, whose k-th coordinate is given by $\sum_{n\in\mathbb{Z}} h_n x_{k-n}$.
- **Downsampling:** The outcome is the sequence

  $$\downarrow \{x_k\}_{k\in\mathbb{Z}} := (\cdots x_{-2}, x_0, x_2, \cdots \}.$$

  Thus, downsampling removes each second element in the sequence.
- **Upsampling:** The outcome is the sequence

  $$\uparrow \{x_k\}_{k\in\mathbb{Z}} := (\cdots x_{-1}, 0, x_0, 0, x_1, \cdots \}.$$

  Thus, upsampling inserts zeroes between the elements in the sequence.

Note that downsampling is the left-inverse of upsampling, but not the right-inverse.

Description of particular filter bank:

The analysis filter bank splits the incoming signal $\{x_k\}_{k\in\mathbb{Z}}$ into $n+1$ subsignals: each of these signals is obtained by convolving $\{x_k\}_{k\in\mathbb{Z}}$ with a sequence $h_{k,\ell},\ \ell = 0, \dots, n$, followed by a downsampling.

The synthesis filter bank first upsamples each of the incoming $n+1$ subsignals, then convolves the resulting sequences with sequences $\{g_{k,\ell}\}_{k\in\mathbb{Z}},\ \ell = 0, \dots, n$, and finally add the outcoming $n+1$ signals.

See Figure 11.6.

We will assume that the sequences $\{h_{k,\ell}\}_{k\in\mathbb{Z}}$ and $\{g_{k,\ell}\}_{k\in\mathbb{Z}}, \ell = 0, \dots, n$, are related by

$$g_{k,\ell} = h_{-k,\ell},\ k \in \mathbb{Z}, \ell = 0, \dots, n.$$

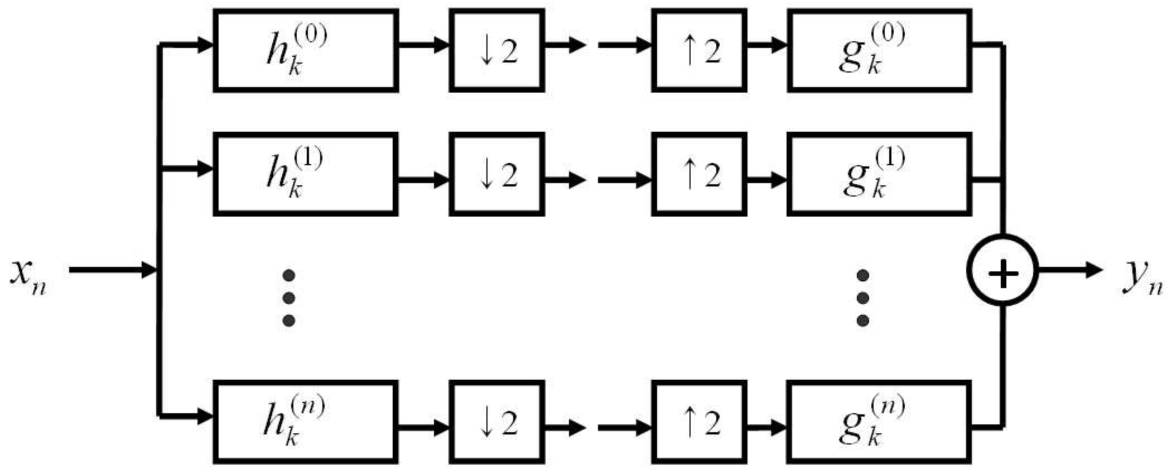


Figure 15:

For the above system , the perfect reconstruction property can be formulated in terms of the polyphase components associated to the filters $\{h_{k,\ell}\}_{k\in\mathbb{Z}}$:

Theorem 14.3 *For the considered filter bank, the perfect reconstruction property is equivalent to the condition*

$H_p^T(z^{-1})H_p(z) = I$ *for* $z \in \mathbb{C}$ *with* $|z| = 1.$

Note that the condition in Theorem 14.3 is "identical" to the UEP condition!

References

[1] Christensen, O.: *Frames and bases in mathematics and engineering. An introductory course.* Birkhäuser 2007.

[2] Christensen, O. and Kim, R. Y.: *On dual Gabor frame pairs generated by polynomials.* Preprint, 2007.

[3] Chui, C., He, W., and Stöckler, J.: *Compactly supported tight and sibling frames with maximum vanishing moments.* Appl. Comp. Harm. Anal. **13** no.3 (2002), 226-262.

[4] Daubechies, I., Han, B., Ron, A. and Shen, Z.: *Framelets: MRA-based constructions of wavelet frames.* Appl. Comp. Harm. Anal. **14** no.1 (2003), 1-46. 2001.

[5] Feichtinger, H. G. and Strohmer, T. (eds.): *Gabor Analysis and Algorithms: Theory and Applications.* Birkhäuser, Boston, 1998.

[6] Feichtinger, H. G. and Strohmer, T. (eds.): *Advances in Gabor Analysis.* Birkhäuser, Boston, 2002.

[7] Janssen, A.J.E.M.: *The duality condition for Weyl-Heisenberg frames.* In "Gabor analysis: theory and applications" (eds. H.G. Feichtinger and T. Strohmer). Birkhäuser, Boston, 1998.

[8] Gröchenig, K.: *Foundations of time-frequency analysis.* Birkhäuser, Boston, 2000.