Properties of representative product systems on the complete product of non-commutative finite groups

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Several results in Fourier analysis with respect to Walsh functions are obtained viewing them as the characters of the dyadic group, i.e., the complete product of the discrete cyclic group of order 2 with the product of topologies and measures. Then we often enumerate the Walsh functions in the Paley's sence writing them as the finite product of the Rademacher functions. It is named the Walsh-Paley system.

The above structure was generalized by Vilenkin [10] in 1947 studying the complete product of arbitrary cyclic groups. The construction of the Vilenkin system is similar, taking the finite product of the characters of the cyclic groups as it Paley did. In Vilenkin groups the order of the cyclic groups appeared in the product can be unbounded. The methods applied in the study of these cases differ significantly from the bounded cases and in many instances we obtain different results for the same question.

A natural generalization of the Vilenkin group is the complete product of arbitrary groups, non necessarily commutative groups, denoted by G. In this case we use representation theory in order to obtain orthonormal systems. These systems are named representative product systems and Section 1 deals with the structure of them. Representative product systems can be represented on the interval $[0, 1]$, where this systems are also orthonormal under the Lebesgue measure (see [9]).

Some examples have been showed in Section 2. We can not only find there the definition of the Walsh-Paley and Vilenkin systems, but we show two simple non commutative cases too. Naturally, the represented systems φ are defined only on the finite groups and we have to take the finite product of them to obtain the system ψ. In the both non commutative cases we suppose the complete product of the same finite group. Notice that representative product systems are not uniformly bounded for non commutative cases. This fact encumbers the study of these systems. Moreover, the values of the system φ depend on the chosen basis if it is not formed only by characters.

Dirichlet kernels and Lebesgue constants play an important role in the study of Fourier series. Hence we mutate the properties of them in Section 3 and we illustrate these properties with the two non commutative examples given in Section 2, comparing these illustrations with the Lebesgue constants of the Walsh-Paley system.

Finally, in the last section we deal with the convergence in L^p-norm $(1 < p < \infty)$ of Fourier series, Fejér means and Cesàro means. We proved the fact that for certain bounded groups G the Fourier series of a functions belongs to $L^p(G)$ do not converge to the function in general, but for all bounded groups G the convergence holds for Fejér means and Cesàro means of order α, where $\frac{1}{2} \leq \alpha \leq 1$. To be more exact, there is a $0 \leq \alpha_0 < \frac{1}{2}$ which depends only on the systems ψ, such that the Cesàro means of order α of the function f belongs to $L^p(G)$ $(1 < p < \infty)$ converge to the function f in L^p-norm for all $\alpha_0 < \alpha \leq 1$.

1 Representative product systems

Let $m := (m_k,\ k \in \mathbf{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k be a finite group with order m_k, $(k \in \mathbf{N})$. Suppose that each group has discrete topology and normalized Haar measure. Let G be the compact group formed by the complete direct product of G_k with the product of the topologies, operations and measures. Thus each $x \in G$ consist of sequences $x := (x_0, x_1, ...)$, where $x_k \in G_k$, $(k \in \mathbf{N})$. We call this sequence the expansion of x. The compact totally disconnected group G is called a *bounded group* if the sequence m is bounded.

If $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbf{N}$, then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the expansion of n with respect to m. We often use the following notations: let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$, $n_{(k)} := \sum_{j=0}^{k-1} n_k M_k$ and $n^{(k)} := \sum_{j=k}^{\infty} n_k M_k$.

Denote by Σ_k the dual object of the finite group G_k $(k \in \mathbf{N})$. Thus each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$. Let d_σ be the dimension of its representation space and let $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space. The functions

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \zeta_i, \zeta_j \rangle \qquad (i, j \in \{1, \dots, d_\sigma\},\ x \in G_k)$$

are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\zeta_1, \zeta_2, \dots, \zeta_{d_\sigma}\}$. In this manner for each $\sigma \in \Sigma_k$ we obtain d_σ^2 number of coordinate functions, in total m_k number of functions for the whole dual object of G_k. The L^2-norm of these functions is $1/\sqrt{d_\sigma}$.

Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be a system of all *normalized coordinate functions* of the group G_k. We do not decide now the enumeration of the system φ, only suppose that φ_k^0 is always the character 1. Thus for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, ..., d_\sigma\}$ such that

$$\varphi_k^s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_k).$$

Let ψ be the product system of φ_k^s, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \qquad (x \in G),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, ...)$. Thus we say that ψ is the *representative product system* of φ. The Weyl-Peter's theorem (see [4]) secures that the system ψ is orthonormal and complete on $L^2(G)$.

The functions ψ_n $(n \in \mathbf{N})$ are not uniformly bounded if G contains infinitely many non commutative groups in the product, so define

$$\Psi_k = \prod_{i=0}^{k-1} \max_{s < m_i} \|\varphi_i^s\|_1 \|\varphi_i^s\|_\infty \qquad (k \in \mathbf{N}).$$

It seems that the boundedness of the sequence Ψ plays an important role in the convergence in L^p-norm of Fourier series.

2 Examples

The representative product systems are the generalization of the well known *Walsh-Paley system*. Indeed, we obtain this system if $m_k = 2$ and $G_k := \mathcal{Z}_2$, the cyclic group of order 2 for all $k \in \mathbf{N}$. The characters of $\mathcal{Z}_2$ are the Rademacher functions:

$$\varphi^s(x) = (-1)^{sx} \qquad (s \in \{0,1\},\ x \in \mathcal{Z}_2).$$

Moreover, we obtain the *Vilenkin systems* if the sequence m is an arbitrary sequence of integers greater than 1 and $G_k := \mathcal{Z}_{m_k}$, the cyclic group of order m_k for all $k \in \mathbf{N}$. The characters of $\mathcal{Z}_{m_k}$ are the generalized Rademacher functions:

$$\varphi_k^s(x) = \exp(2\pi \imath s x/m_k) \qquad (s \in \{0, \dots m_k - 1\},\ x \in \mathcal{Z}_{m_k},\ \imath^2 = -1).$$

Consider now the complete product of $\mathcal{S}_3$ where $\mathcal{S}_3$ is the *symmetric group on 3 elements*. Thus $m_k = 6$ for all $k \in \mathbf{N}$. $\mathcal{S}_3$ has two characters and a 2-dimensional representation.

	e	(12)	(13)	(23)	(123)	(132)	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1
φ^1	1	-1	-1	-1	1	1	1	1
φ^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^3	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{2\sqrt{2}}{3}$	$\sqrt{2}$
φ^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$
φ^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{3}$	$\frac{\sqrt{6}}{2}$

Table 1: The system φ for $\mathcal{S}_3$

Notice that $\max_{0 \le s < 6} \|\varphi^s\|_1 \|\varphi^s\|_\infty = \frac{4}{3} > 1$ implies $\Psi_k = \left(\frac{4}{3}\right)^k \to \infty$.

Let $m_k = 8$ for all $k \in \mathbf{N}$ and $\mathcal{Q}_2$ be the the *quaternion group of order 8*, i.e. $\mathcal{Q}_2 := \{[a,b] : a^4 = e,\ b^2 = a^2,\ bab^{-1} = a^3\}$. Let $G_k = \mathcal{Q}_2$ for all $k \in \mathbf{N}$. $\mathcal{Q}_2$ has four characters and a 2-dimensional representation.

	e	a	a^2	a^3	b	ab	a^2b	a^3b	$\|\varphi^s\|_1$	$\|\varphi^s\|_\infty$
φ^0	1	1	1	1	1	1	1	1	1	1
φ^1	1	1	1	1	-1	-1	-1	-1	1	1
φ^2	1	-1	1	-1	1	-1	1	-1	1	1
φ^3	1	-1	1	-1	-1	1	-1	1	1	1
φ^4	$\sqrt{2}$	$\sqrt{2}\imath$	$-\sqrt{2}$	$-\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^5	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	0	0	0	0	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^6	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}\imath$	$-\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
φ^7	0	0	0	0	$-\sqrt{2}$	$-\sqrt{2}\imath$	$\sqrt{2}$	$\sqrt{2}\imath$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$

Table 2: The system φ for $\mathcal{Q}_2$

Notice that $|\varphi^s|$ can only take the values 0 or the square root of the corresponding dimension because this representation is *monomial*. In this case we have $\Psi_k = 1$, since $\max_{0 \le s < 8} \|\varphi^s\|_1 \|\varphi^s\|_\infty = 1$ for $k \in \mathbf{N}$, but the group G is not abelian.

3 Fourier series and Dirichlet kernels

For $f \in L^1(G)$ we define the *Fourier coefficients* $\widehat{f}_k$, *the n-th partial sums of Fourier series* $S_n f$ and the *Dirichlet kernels* D_n by

$$\widehat{f}_k := \int_G f\overline{\psi}_k d\mu, \qquad S_n f := \sum_{k=0}^{n-1} \widehat{f}_k \psi_k, \qquad D_n(x,y) := \sum_{k=0}^{n-1} \psi_k(x)\overline{\psi}_k(y)$$

for all $k, n \in \mathbf{N}$. The equality $S_n f(x) = \int_G f(y) D_n(x,y)\, dy$ shows the importance of the Dirichlet kernels in the study of the convergence of Fourier series. The lemmas below are used in this regard.

Lemma 1. *If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_n(x,y) = \sum_{k=0}^{\infty} D_{M_k}(x,y) \left(\sum_{s=0}^{n_k-1} \varphi_k^s(x_k)\overline{\varphi}_k^s(y_k) \right) \psi_{n^{(k+1)}}(x)\overline{\psi}_{n^{(k+1)}}(y),$$

where $(n_0, n_1, \ldots)$ is the expansion of n and $x = (x_0, x_1, \ldots)$, $y = (y_0, y_1, \ldots)$.

The sets $I_n(x) := \{y \in G : y_k = x_k, \text{ for } 0 \le k < n\}$ are called *intervals* and they form a countable neighborhood base of the product topology on G.

Lemma 2. *(Paley lemma) If $n \in \mathbf{N}$ and $x, y \in G$, then*

$$D_{M_n}(x,y) = \begin{cases} M_n & \text{for } x \in I_n(y), \\ 0 & \text{for } x \notin I_n(y) \end{cases}$$

The Paley lemma is used to prove that the $S_{M_n} f$ partial sequence of Fourier sums converge to f in L^p-norm and a.e., if $f \in L^p(G)$, $p \ge 1$. This property mark a essential difference between the comportment of these systems and the trigonometric system.

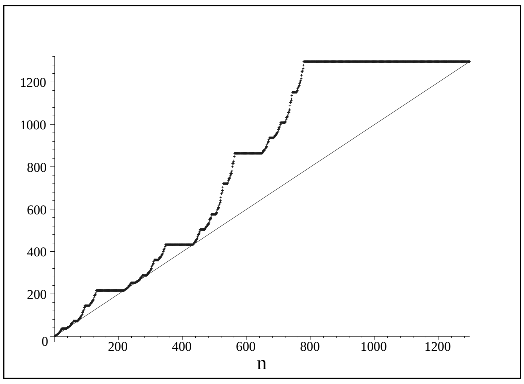
Define the *maximum of the Dirichlet kernels* by

$$\mathcal{D}_n := \sup_{x,y \in G} |D_n(x,y)| \qquad (n \in \mathbf{P}).$$

For abelian cases $\mathcal{D}_n = n$ for all $n \in \mathbf{P}$, but the general case is a bit more different.

Lemma 3. *If $n \in \mathbf{P}$ and $A := \max\{k \in \mathbf{N} : n_k \neq 0\}$, then $n \le \mathcal{D}_n \le M_{A+1}$.*

The figure below illustrates the statements of Lemma 3 with respect to the system φ on $\mathcal{S}_3$ appeared in Table 1.



Moreover, define the *Lebesgue constants* by

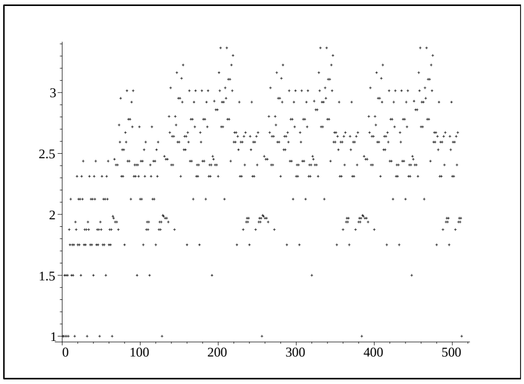
$$L_n := \sup_{x \in G} \int_G \sup_{k \le n} |D_k(x,y)| \, d\mu(y) \quad (n \in \mathbf{N})$$

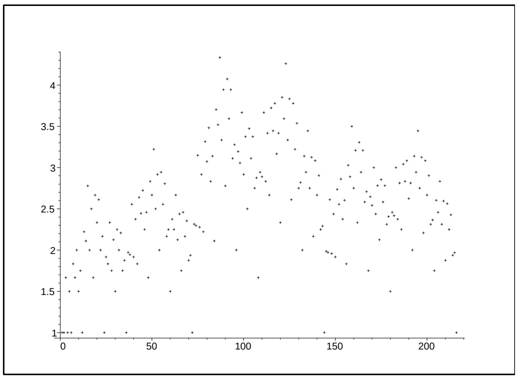
Theorem 1. *Let G be a bounded group.*

(a) If the sequence Ψ is bounded then $L_n = O(\log n)$.

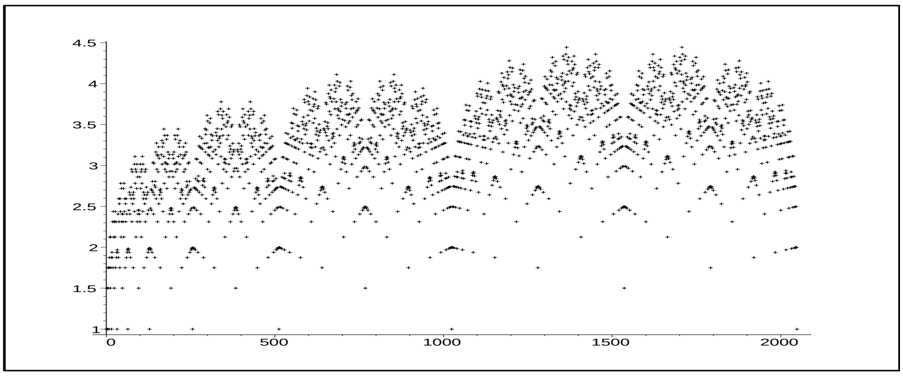
(b) If the sequence Ψ is not bounded and $M_k \le n < M_{k+1}$ then $L_n = O(\Psi_k)$.

The figures below illustrate the statements of the theorem. The first one is with respect to the system φ on $\mathcal{Q}_2$ appeared in Table 2 and we can observe the property $L_n = O(\log n)$. The second one is with respect to the system φ on $\mathcal{S}_3$ appeared in Table 1 with property $L_n = O(\Psi_k)$.





Finally, we illustrate the Lebesgue constants for the classical Walsh-Paley system. This system is commutative, so it has the property $L_n = O(\log n)$.



4 Convergenge in L^p-norm

Paley proved that the partial sums of Fourier series are uniformly bounded, from L^p into itself, where $1 < p < \infty$. It is equivalent to the convergence of these operators in L^p-norm. This statement is known as the Paley's theorem. Paley's theorem was shown independently for arbitrary Vilenkin systems by Young, Schipp and Simon. Unfortunately, we cannot extend this statement for every non-abelian group.

Theorem 2. *If G is a bounded group with unbounded sequence Ψ, then the operators S_n are not uniformly bounded in L^p-norm for all $p \neq 2$.*

The statement of Paley's theorem has not been proved or refused yet for another groups G (obviously it always holds for $p = 2$). However, using the *Fejér means of the Fourier series*

$$\sigma_n f = \frac{1}{n} \sum_{k=0}^{n} S_k f \qquad (n \in \mathbf{P})$$

instead Fourier series we obtain uniformly boundedness in L^p-norm (see [2]).

Theorem 3. *If G is a bounded group and $f \in L^p(G)$ for $1 \leq p < \infty$, then $\sigma_n f \to f$ in L^p-norm, where $\sigma_n f$ are the Fejér means of the function f.*

This statement can be extended for Cesàro means of order α for certain values of α. The *Cesàro means of order α* are defined by

$$\sigma_n^\alpha f = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f, \quad \text{where } A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}, \qquad (n \in \mathbf{P}).$$

Assume that the number α_0 is the infimum of all $0 < \alpha < 1$ such that

$$\|\varphi_k^s\|_1 \|\varphi_k^s\|_\infty < m_k^\alpha \qquad (0 \leq s < m_k)$$

holds except finite numbers of $k \in \mathbf{N}$. Since $\|\varphi_k^s\|_\infty^2 < m_k$, the number α_0 exists and it less than $\frac{1}{2}$. Thus we have

Theorem 4. *Let G be a bounded group, $\alpha_0 < \alpha < 1$ and $f \in L^p(G)$ for $1 \leq p < \infty$. Then $\sigma_n^\alpha f \to f$ in L^p-norm, where $\sigma_n^\alpha f$ are the Cesàro means of order α of the function f.*

References

[1] G. Benke, *Trigonometric approximation theory in compact totally disconnected groups*, Pacific Jour. of Math. **77(1)** (1978), 23–32.

[2] G. Gát and R. Toledo, *L^p-norm convergence of series in compact totally disconected groups*, Anal. Math. **22** (1996), 13–24.

[3] ______, *Fourier coefficients and absolute convergence on compact totally disconnected groups*, Math. Pannonica **10/2** (1999), 223–233.

[4] E. Hewitt and K. Ross, *Abstract harmonic analysis I*, Springer-Verlag, Heidelberg, 1963.

[5] ______, *Abstract harmonic analysis II*, Springer-Verlag, Heidelberg, 1970.

[6] F. Schipp, *On l^p -norm convergence of series with respect to product systems*, Analysis Math. **2** (1976), 49–63.

[7] F. Schipp, W.R. Wade, P. Simon, and J. Pál, *Walsh series,"an introduction to dyadic harmonic analysis"*, Adam Hilger, Bristol and New York, 1990.

[8] R. Toledo, *On the convergence of Fourier series in CTD groups*, Functions, Series, Operators, Proceedings of the Alexits Memorial Conference, Budapest, August 9-14, 1999 (L. Leindler, F. Schipp, and J. Szabados, eds.), Coll. Soc. J. Bólyai, 2002, pp. 403–415.

[9] ______, *Representation of product systems on the interval* $[0, 1]$, Acta Acad. Paed. Nyíregyháza **(19/1)** (2003), 43–50.

[10] N. Ja. Vilenkin, *On a class of complete orthonormal system*, Izv. Akad. Nauk SSSR, ser. mat. **11** (1947), 363–400.