Inversion of the noisy Radon transform on $SO(3)$ by Gabor frames and sparse recovery principles

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joint work with P. Cerejeiras, M. Ferreira, and U. Kähler

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Crystallographic texture analysis through X-ray tomography:

- Assumption: orientation of crystal is unique and given by $q \in SO(3)$
- Radon transform relates orientation density function f and its experimentally accessible pole density functions:

$$P(x,y) = \frac{1}{2}(Rf(x,y) + Rf(-x,y)) \ , \ \textit{where}$$

$$Rf(x,y) = \frac{1}{2\pi}\int_{\{q\in SO(3): y=\bar{q}xq\}} f(q)dq \ , \quad x,y \in S^2$$

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- Questions:

  - how to expand f ?
  - how to compute/approximate R^{-1} ?
  - how to treat noisy data y^δ, $\|y - y^\delta\| \leq \delta$?

Selection of existing literature:

- S. Bernstein, and H. Schaeben, *A one-dimensional Radon transform on SO(3) and its application to texture goniometry*, Math. Methods Appl. Sci., **28** (2005), 126989.
- K.G. v.d. Boogaart, R. Hielscher, J. Prestin and H. Schaeben, *Kernel-based methods for inversion of the radon transform on SO(3) and their applications to texture analysis*, J. Comput. Appl. Math. **199** (2007), 122-40.
- H.J. Bunge, *Texture Analysis in Material Science* (London: Butterworths), 1982.
- P. Cerejeiras, H. Schaeben, and F. Sommen *The spherical x-ray transform*, Math. Methods Appl. Sci., **25** (2002), 1493507.
- R. Hielscher, *The Radon transform on the rotation group inversion and application to texture analysis* Dissertation, Department of Geosciences, University of Technology Freiberg (2007). https://fridolin.tufreiberg.de/archiv/html/MathematikHielscherRalf361401.html
- R. Hielscher, D. Potts, J. Prestin, H. Schaeben, and M. Schmalz. *The Radon transform on* $SO(3)$*: a Fourier slice theorem and numerical inversion*, Invers. Prob. **24** (2008), doi:10.1088/0266-5611/24/2/025011.
- L. Meister, H. Schaeben, *A concise quaternion geometry of rotations*, MMAS **28** (2004), 101126.
- D. I. Nikolayev, and H. Schaeben, *Characteristics of the ultra-hyperbolic differential equation governing pole density functions*, Inverse Probl., **15** (1999), 160319.
- H. Schaeben and K.G. v. d. Boogaart, *Spherical harmonics in texture analysis*, Tectonophysics **370** (2003), 253-68.

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- group-based interpretation:

$$U : G = R^n \times R^n \to \mathcal{U}(L_2(R^n)) \text{ via } U(\omega, b)\psi(x) = \psi(x-b)e^{-i\langle\omega,x\rangle}$$

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- example: $H = \{(0,(0,0,0,p_4)) \in G,\ p_4 \in R\}$
- Consider instead for $(s,p) \in X$,

$$V_\psi f(s,p) = \langle f, U(\sigma(s,p)^{-1})\psi\rangle$$

Lemma (admissibility and isometry)

Assume $\psi \in L_1(S^3) \cap L_2(S^3)$ *is such that* $supp(\psi) \subseteq S^3_+$,

$$0 \neq C_\psi = 64\pi^5 \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi/2} \frac{|\psi(q(\theta,\alpha,\phi))|^2}{\cos\phi} \, d\phi \, d\alpha \, d\theta < \infty. \quad (1)$$

Then the map

$$f \in L^2(S^3) \mapsto \frac{1}{\sqrt{C_\psi}} V_\psi f \in L^2(Spin(4) \times \mathbb{R}^3)$$

is an isometry, i.e.

$$\int_{Spin(4)\times\mathbb{R}^3} |V_\psi f(s,p)|^2 \, d\mu(s) \, dp = C_\psi \int_{S^3} |f(q)|^2 dS_q.$$

Corollary (reconstruction)

Any $f \in L^2(S^3)$ can be reconstructed by

$$f(q) = \frac{1}{C_\psi} \int_{Spin(4)} \int_{\mathbb{R}^3} V_\psi f(s,p) e^{-i\langle \bar{s}ps, q\rangle} \psi(sq\bar{s})\, dp\, d\mu(s).$$

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No discrete expansion formula

Rough sketch of discretization procedure:

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- Define the oscillation kernel

$$osc(l, h) := \sup_{u\in\mathcal{U}} |\langle\psi, U(\sigma(l)\sigma(h)^{-1})\psi - U(u^{-1}\sigma(l)\sigma(h)^{-1})\psi\rangle_{L_2(S^3)}|$$

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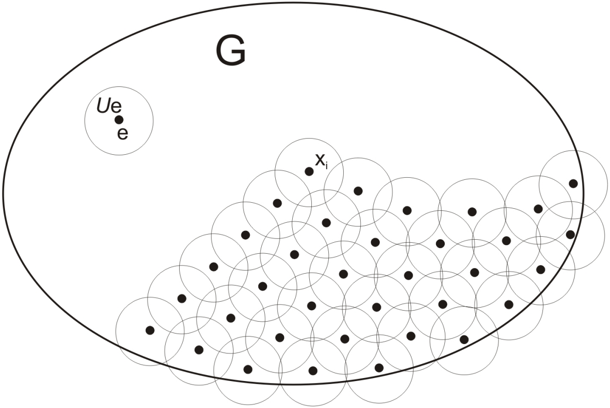

Theorem (frames)

Assume

$$\int_X osc(l,h)d\mu(l) < \frac{\eta}{C_\psi} \quad \text{and} \quad \int_X osc(l,h)d\mu(h) < \frac{\eta}{C_\psi} \tag{2}$$

with $\eta < 1$*. Then the set* $\{\psi_i := U(\sigma(x_i))\psi : i \in I\}$ *is a frame for* $L_2(S^3)$*. This means that*

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1. $f \in L_2(S^3) \Leftrightarrow \{\langle f, \psi_i\rangle\}_{i\in I} \in \ell_2$,
2. *there exists constants* $0 < A \leq B < \infty$ *such that*

$$A\|f\|_{L_2(S^3)} \leq \|\{\langle f, \psi_i\rangle\}_{i\in I}\|_{\ell_2} \leq B\|f\|_{L_2(S^3)} ,$$

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3. *there exists a bounded, linear synthesis operator* $S : \ell_2 \to L_2(S^3)$ *such that* $S(\{\langle f, \psi_i \rangle\}_{i\in I}) = f$.

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- $R : L_2(S^3) \longrightarrow L_2(S^2 \times S^2)$ is **ill-posed** $\longrightarrow$ regularization
- Assume, that f has a **sparse expansion** within $\{\psi_i : i \in I\}$

$$f(p) = (Fc)(p) = \sum_{i \in J \subset I, |J| \text{ small}} c_i \psi_i(p)$$

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- Minimization through projected steepest descent with step length control

$$c_i^{n+1} = P_K \left(c_i^n + \frac{\beta^n}{r} (F^* R^* (y - R(Fc^n)))_i \right).$$

Given	operator R, some initial guess c^0, and K (sparsity constraint ℓ_1-ball B_K)
Initialization	$\|RF\|^2 \leq r$, set $q = 0.9$ (as an example)
Iteration	**for** $n = 0, 1, 2, \ldots$ **until** a preassigned precision / maximum number of iterations **1.** $\beta^n = C \cdot \sqrt{\frac{D(x^0)}{D(x^n)}}$, $C \geq 1$ (greedy guess) **2.** $c^{n+1} = P_K \left(c^n + \frac{\beta^n}{r} F^* R^* (y - R(F(c^n))) \right)$; **3.** verify (B2): $\beta^n \|R(Fc^{n+1}) - R(Fc^n)\|^2 \leq r \|c^{n+1} - c^n\|^2$ **if** (B2) is satisfied increase n and go to **1.** **otherwise** set $\beta^n = q \cdot \beta^n$ and go to **2.** **end**

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- Reconstruction/approximation of c by

$$c^{n+1} = P_K\left(c^n + \frac{\beta^n}{r} F^* R^*(y - R(F(c^n)))\right)$$

- Simple analyzing atom:

$$\psi(q) = \cos^6(2.6 \arccos(q_0)), \quad \frac{\sqrt{3}}{2} \leq q_0 \leq 1,$$

if $q = \Lambda(\theta, \alpha, \phi)$, $\theta \in [0, 2\pi[$, $\alpha \in [0, \pi[$ and $\phi \in [0, \pi]$, then the Gabor atom reads as

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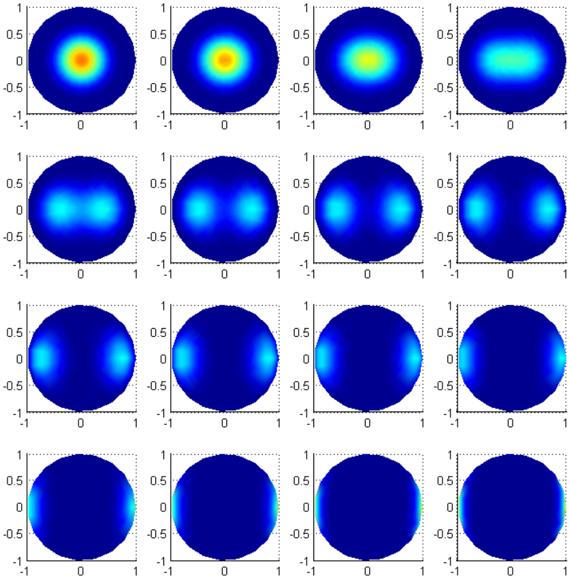
- Three cases: no noise, 5% rel. noise, and 10% rel. noise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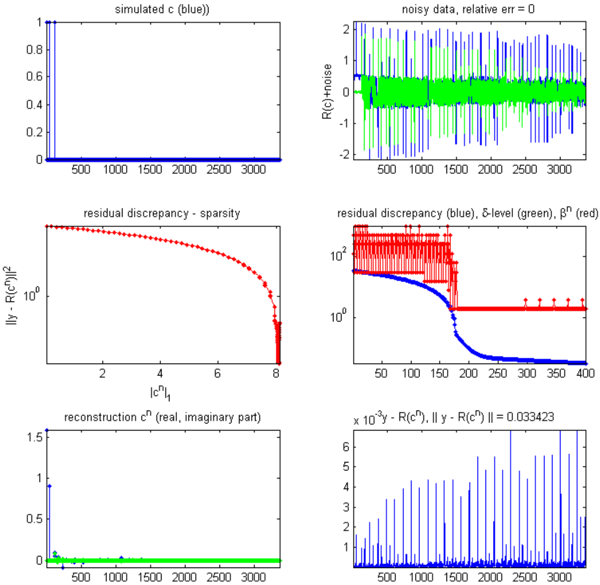
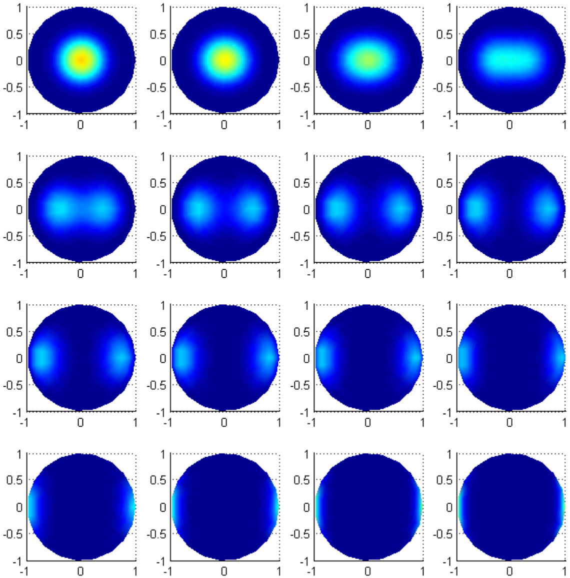

simulated c (blue))
noisy data, relative err = 0
R(c)+noise
residual discrepancy - sparsity
||y - R(c^n)||^2
|c^n|_1
residual discrepancy (blue), δ-level (green), β^n (red)
reconstruction c^n (real, imaginary part)
x 10^-3 y - R(c^n), || y - R(c^n) || = 0.033423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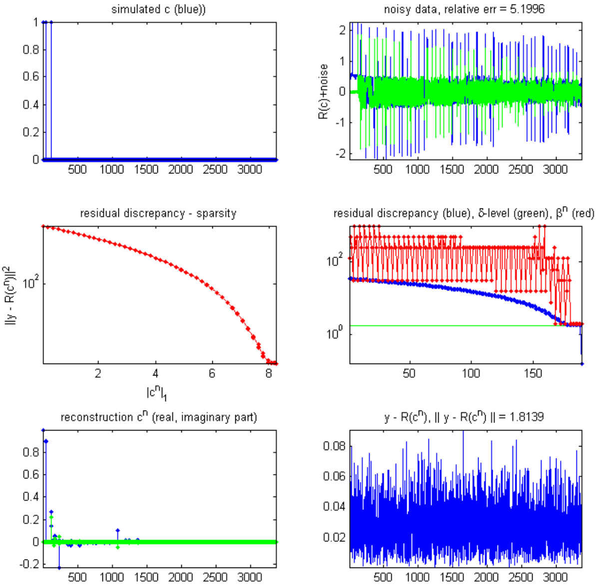
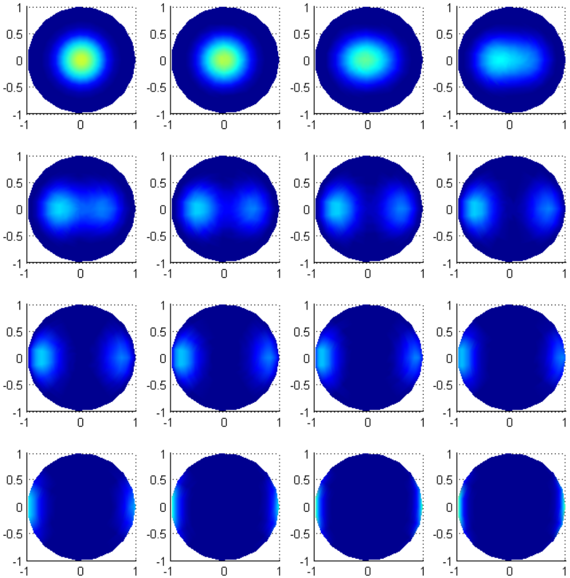
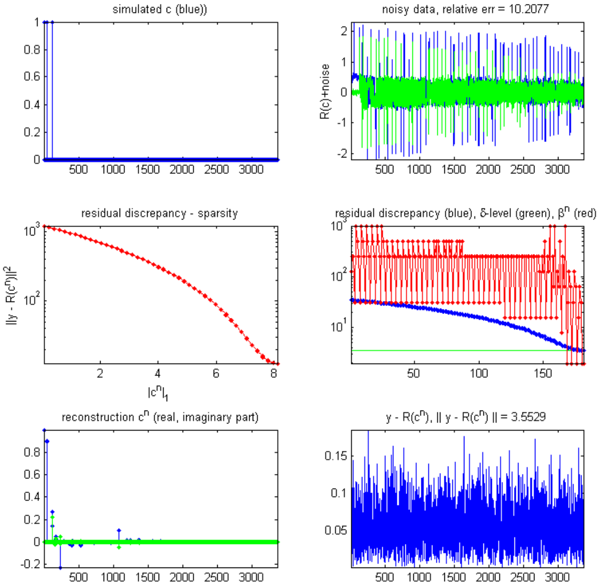
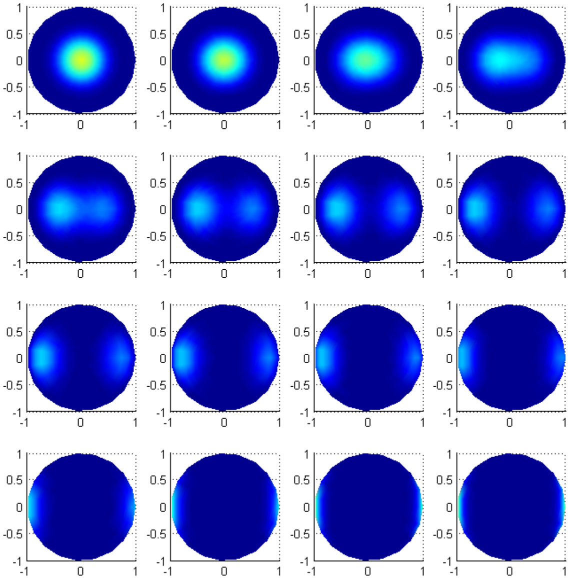

simulated c (blue))
noisy data, relative err = 5.1996
R(c)+noise
residual discrepancy - sparsity
||y - R(c^n)||^2
|c^n|_1
residual discrepancy (blue), δ-level (green), β^n (red)
reconstruction c^n (real, imaginary part)
y - R(c^n), || y - R(c^n) || = 1.8139




simulated c (blue))
noisy data, relative err = 10.2077
R(c)+noise
residual discrepancy - sparsity
||y - R(c^n)||^2
|c^n|_1
residual discrepancy (blue), δ-level (green), β^n (red)
reconstruction c^n (real, imaginary part)
y - R(c^n), || y - R(c^n) || = 3.5529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- $\Uparrow$ Sparse recovery algorithm to approximate f
- $\Downarrow$ Current approach: just linear approximation scheme
  - fast computation of matrix entries
  - but all entries must be computed
  - **goal:** adaptivity to overcome the curse of dimensionality

Thank you for your attention