Clifford analysis and monogenic signals

Uwe Kähler

Departamento de Matemática
Universidade de Aveiro
uwek@mat.ua.pt

09.03.2010 - DMV Tagung 2010, München

Analytic signals

- Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{y-x} dy, y \in \mathbb{R}.$$

- Consider the complex-valued function

$$Z(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $Z(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase
- $\omega(x) = \theta'(x)$ is called "instantaneous frequency"
- The pair (a, θ) is called "canonical modulation pair" of $f(x)$

Analytic signals

- Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{y-x} dy, y \in \mathbb{R}.$$

- Consider the complex-valued function

$$Z(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $Z(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase
- $\omega(x) = \theta'(x)$ is called "instantaneous frequency"
- The pair (a, θ) is called "canonical modulation pair" of $f(x)$

Analytic signals

- Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{y-x}dy, y \in \mathbb{R}.$$

- Consider the complex-valued function

$$Z(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $Z(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase
- $\omega(x) = \theta'(x)$ is called "instantaneous frequency"
- The pair (a, θ) is called "canonical modulation pair" of $f(x)$

Analytic signals

- Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{y-x} dy, y \in \mathbb{R}.$$

- Consider the complex-valued function

$$Z(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $Z(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase
- $\omega(x) = \theta'(x)$ is called “instantaneous frequency”
- The pair (a, θ) is called “canonical modulation pair” of $f(x)$

Analytic signals

- Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{y-x} dy, y \in \mathbb{R}.$$

- Consider the complex-valued function

$$Z(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $Z(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase
- $\omega(x) = \theta'(x)$ is called "instantaneous frequency"
- The pair (a, θ) is called "canonical modulation pair" of $f(x)$

Analytic signals

- Applying to a given signal $f(x)$ the Hilbert transform

$$Hf(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{y-x} dy, y \in \mathbb{R}.$$

- Consider the complex-valued function

$$Z(x) = f(x) + iHf(x) = a(x)e^{i\theta(x)}$$

- $Z(x)$ is the analytic signal, $a(x)$ the amplitude and $\theta(x)$ the phase
- $\omega(x) = \theta'(x)$ is called "instantaneous frequency"
- The pair (a, θ) is called "canonical modulation pair" of $f(x)$

Analytic signals - Mathematical background

- Riemann-Hilbert problem

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}} &= 0 && \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re} F(z) &= f(x) && \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im} F(z_0) &= c && \text{in } \mathrm{Im}(z_0) > 0 \end{aligned}$$

- Solution:
  - $\mathrm{tr} F = \frac{1}{2} f + i \frac{1}{2} Hf =: P_\Gamma f$ where $P_\Gamma^2 = P_\Gamma$ is the Hardy projection
  - $\tilde{F} = F_\Gamma \left(\frac{1}{2} f + i \frac{1}{2} Hf \right)$ where F_Γ is the Cauchy integral operator
  - F is obtained using the condition $\mathrm{Im} F(z_0) = \tilde{c} + \mathrm{Im} \tilde{F}(z_0) = c$
  - Finite energy $\Rightarrow \mathrm{tr} F \in H_p(\mathbb{R})$

Analytic signals - Mathematical background

- Riemann-Hilbert problem

$$\begin{array}{rl} \frac{\partial F}{\partial \bar{z}} = 0 & \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re}F(z) = f(x) & \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im}F(z_0) = c & \text{in } \mathrm{Im}(z_0) > 0 \end{array}$$

- Solution:
  - $\mathrm{tr}F = \frac{1}{2}f + i\frac{1}{2}Hf =: P_\Gamma f$ where $P_\Gamma^2 = P_\Gamma$ is the Hardy projection
  - $\tilde{F} = F_\Gamma\left(\frac{1}{2}f + i\frac{1}{2}Hf\right)$ where F_Γ is the Cauchy integral operator
  - F is obtained using the condition $\mathrm{Im}F(z_0) = \tilde{c} + \mathrm{Im}\tilde{F}(z_0) = c$
  - Finite energy $\Rightarrow \mathrm{tr}F \in H_p(\mathbb{R})$

Analytic signals - Mathematical background

- Riemann-Hilbert problem

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}} &= 0 && \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re}F(z) &= f(x) && \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im}F(z_0) &= c && \text{in } \mathrm{Im}(z_0) > 0 \end{aligned}$$

- Solution:
  - $\mathrm{tr}F = \frac{1}{2}f + i\frac{1}{2}Hf =: P_\Gamma f$ where $P_\Gamma^2 = P_\Gamma$ is the Hardy projection
  - $\tilde{F} = F_\Gamma\left(\frac{1}{2}f + i\frac{1}{2}Hf\right)$ where F_Γ is the Cauchy integral operator
  - F is obtained using the condition $\mathrm{Im}F(z_0) = \tilde{c} + \mathrm{Im}\tilde{F}(z_0) = c$
  - Finite energy $\Rightarrow \mathrm{tr}F \in H_p(\mathbb{R})$

Analytic signals - Mathematical background

- Riemann-Hilbert problem

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}} &= 0 && \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re}F(z) &= f(x) && \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im}F(z_0) &= c && \text{in } \mathrm{Im}(z_0) > 0 \end{aligned}$$

- Solution:
  - $\mathrm{tr}F = \frac{1}{2}f + i\frac{1}{2}Hf =: P_\Gamma f$ where $P_\Gamma^2 = P_\Gamma$ is the Hardy projection
  - $\tilde{F} = F_\Gamma\left(\frac{1}{2}f + i\frac{1}{2}Hf\right)$ where F_Γ is the Cauchy integral operator
  - F is obtained using the condition $\mathrm{Im}F(z_0) = \tilde{c} + \mathrm{Im}\tilde{F}(z_0) = c$
  - Finite energy $\Rightarrow \mathrm{tr}F \in H_p(\mathbb{R})$

Analytic signals - Mathematical background

- Riemann-Hilbert problem

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}} &= 0 && \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re}F(z) &= f(x) && \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im}F(z_0) &= c && \text{in } \mathrm{Im}(z_0) > 0 \end{aligned}$$

- Solution:
  - $\mathrm{tr}F = \frac{1}{2}f + i\frac{1}{2}Hf =: P_\Gamma f$ where $P_\Gamma^2 = P_\Gamma$ is the Hardy projection
  - $\tilde{F} = F_\Gamma\left(\frac{1}{2}f + i\frac{1}{2}Hf\right)$ where F_Γ is the Cauchy integral operator
  - F is obtained using the condition $\mathrm{Im}F(z_0) = \tilde{c} + \mathrm{Im}\tilde{F}(z_0) = c$
  - Finite energy $\Rightarrow \mathrm{tr}F \in H_p(\mathbb{R})$

Analytic signals - Mathematical background

- Riemann-Hilbert problem

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}} &= 0 && \text{in } \mathrm{Im}(z) > 0 \\ \mathrm{Re}F(z) &= f(x) && \text{in } \mathrm{Im}(z) = 0 \\ \mathrm{Im}F(z_0) &= c && \text{in } \mathrm{Im}(z_0) > 0 \end{aligned}$$

- Solution:
  - $\mathrm{tr}F = \frac{1}{2}f + i\frac{1}{2}Hf =: P_\Gamma f$ where $P_\Gamma^2 = P_\Gamma$ is the Hardy projection
  - $\tilde{F} = F_\Gamma\left(\frac{1}{2}f + i\frac{1}{2}Hf\right)$ where F_Γ is the Cauchy integral operator
  - F is obtained using the condition $\mathrm{Im}F(z_0) = \tilde{c} + \mathrm{Im}\tilde{F}(z_0) = c$
  - Finite energy $\Rightarrow \mathrm{tr}F \in H_p(\mathbb{R})$

Why does it work for signal analysis?

- Basic idea: Poisson kernel is also low-pass filter!
- Multi-scale analysis by Gaussian kernel can be replaced by multi-scale analysis using Poisson kernel
- Weyl-relation: $[x, D] = i$ with $D = -i\partial_x$
- $f(x) = a(x)e^{i\theta(x)}$
- $< \omega > = \int \omega |\mathcal{F}f(\omega)|^2 d\omega = \int \theta'(x) a(x)^2 dx$

Why does it work for signal analysis?

- Basic idea: Poisson kernel is also low-pass filter!
- Multi-scale analysis by Gaussian kernel can be replaced by multi-scale analysis using Poisson kernel
- Weyl-relation: $[x, D] = i$ with $D = -i\partial_x$
- $f(x) = a(x)e^{i\theta(x)}$
- $<\omega>= \int \omega|\mathcal{F}f(\omega)|^2 d\omega = \int \theta'(x)a(x)^2 dx$

Why does it work for signal analysis?

- Basic idea: Poisson kernel is also low-pass filter!
- Multi-scale analysis by Gaussian kernel can be replaced by multi-scale analysis using Poisson kernel
- Weyl-relation: $[x, D] = i$ with $D = -i\partial_x$
- $f(x) = a(x)e^{i\theta(x)}$
- $< \omega >= \int \omega|\mathcal{F}f(\omega)|^2 d\omega = \int \theta'(x)a(x)^2 dx$

Why does it work for signal analysis?

- Basic idea: Poisson kernel is also low-pass filter!
- Multi-scale analysis by Gaussian kernel can be replaced by multi-scale analysis using Poisson kernel
- Weyl-relation: $[x, D] = i$ with $D = -i\partial_x$
- $f(x) = a(x)e^{i\theta(x)}$
- $< \omega >= \int \omega |\mathcal{F}f(\omega)|^2 d\omega = \int \theta'(x) a(x)^2 dx$

Why does it work for signal analysis?

- Basic idea: Poisson kernel is also low-pass filter!
- Multi-scale analysis by Gaussian kernel can be replaced by multi-scale analysis using Poisson kernel
- Weyl-relation: $[x, D] = i$ with $D = -i\partial_x$
- $f(x) = a(x)e^{i\theta(x)}$
- $< \omega >= \int \omega |\mathcal{F}f(\omega)|^2 d\omega = \int \theta'(x) a(x)^2 dx$

Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stopping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by "Empirical Mode Decomposition" (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by "Empirical Mode Decomposition" (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by "Empirical Mode Decomposition" (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

Huang-Hilbert transform

- The problem of the constant is overcome by "Empirical Mode Decomposition" (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

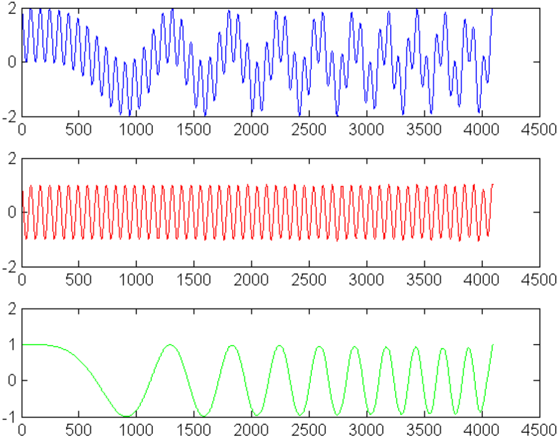
Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

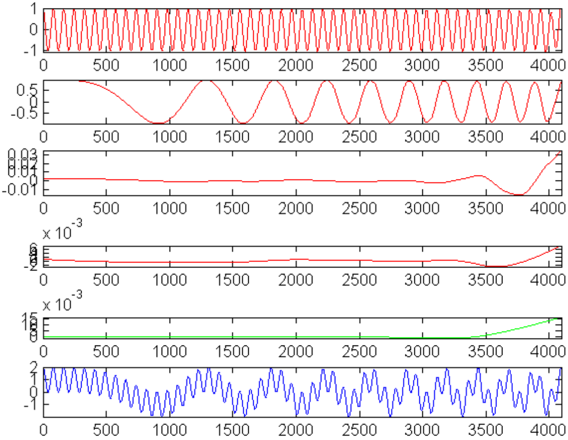
Huang-Hilbert transform

- The problem of the constant is overcome by “Empirical Mode Decomposition” (EMD):
  1. Single step:
     1. identify the extrema of the signal $x(t)$
     2. interpolate between the maxima to obtain the upper envelope $\overline{e} = \overline{e}(t)$
     3. interpolate between the minima to obtain the lower envelope $\underline{e} = \underline{e}(t)$
     4. compute the average $m(t) = \frac{\overline{e}(t)+\underline{e}(t)}{2}$
     5. compute the detail $d(t) = x(t) - m(t)$
     6. iterate upon $d(t)$ until it has *local zero-mean* under a given stoping criteria
  2. iterate on the residual $m = m(t)$
- Apply the Hilbert transform to $d(t)$
- Re-initiate with $m(t)$

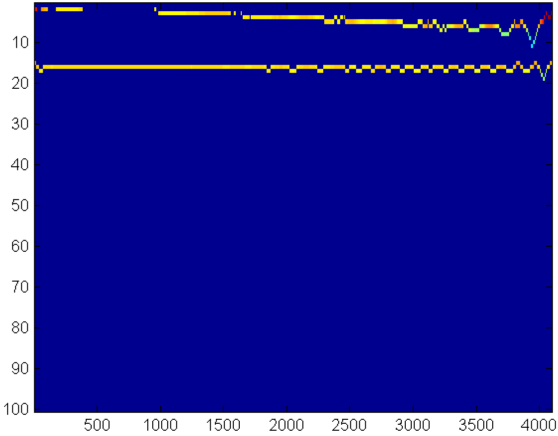
Example - 1.IMF



Example - all IMF’s



Example - Hilbert spectrum



Concept of IMF's

- Basic idea (assumption) of HHT:

$$f(x) = \sum_{i=1}^{n} a_i(x) \cos(\theta_i(x)) + r(x)$$

- IMF is a function of type $a(x)\cos(\theta(x))$ with local mean zero
- In practice: Bedrosian Identity $H(a\cos(\theta)) = aH\cos(\theta)$ plus $H\cos(\theta) = \sin(\theta)$
- Good counter-example: $f(x) = \cos x^2$

Concept of IMF's

- Basic idea (assumption) of HHT:

$$f(x) = \sum_{i=1}^{n} a_i(x) \cos(\theta_i(x)) + r(x)$$

- IMF is a function of type $a(x)\cos(\theta(x))$ with local mean zero
- In practice: Bedrosian Identity $H(a\cos(\theta)) = aH\cos(\theta)$ plus $H\cos(\theta) = \sin(\theta)$
- Good counter-example: $f(x) = \cos x^2$

Concept of IMF's

- Basic idea (assumption) of HHT:

$$f(x) = \sum_{i=1}^{n} a_i(x) \cos(\theta_i(x)) + r(x)$$

- IMF is a function of type $a(x)\cos(\theta(x))$ with local mean zero
- In practice: Bedrosian Identity $H(a\cos(\theta)) = aH\cos(\theta)$ plus $H\cos(\theta) = \sin(\theta)$
- Good counter-example: $f(x) = \cos x^2$

Concept of IMF's

- Basic idea (assumption) of HHT:

$$f(x) = \sum_{i=1}^{n} a_i(x) \cos(\theta_i(x)) + r(x)$$

- IMF is a function of type $a(x)\cos(\theta(x))$ with local mean zero
- In practice: Bedrosian Identity $H(a\cos(\theta)) = aH\cos(\theta)$ plus $H\cos(\theta) = \sin(\theta)$
- Good counter-example: $f(x) = \cos x^2$

Concept of IMF's

- Basic idea (assumption) of HHT:

$$f(x) = \sum_{i=1}^{n} a_i(x)\cos(\theta_i(x)) + r(x)$$

- IMF is a function of type $a(x)\cos(\theta(x))$ with local mean zero
- In practice: Bedrosian Identity $H(a\cos(\theta)) = aH\cos(\theta)$ plus $H\cos(\theta) = \sin(\theta)$
- Good counter-example: $f(x) = \cos x^2$

An interesting example

However, local mean condition does not imply uniqueness:

$$f(x) = \left(1 + \cos\left(\frac{\theta_1(x) - \theta_2(x)}{2}\right)\right) \cos\left(\frac{\theta_1(x) + \theta_2(x)}{2}\right)$$

$$= 1 \cos\left(\frac{\theta_1(x) + \theta_2(x)}{2}\right) + \frac{1}{2}\cos(\theta_1(x)) + \frac{1}{2}\cos(\theta_2(x)).$$

- Which line is the result of the HHT?
- Let's see:

An interesting example

However, local mean condition does not imply uniqueness:

$$f(x) = \left(1 + \cos\left(\frac{\theta_1(x) - \theta_2(x)}{2}\right)\right) \cos\left(\frac{\theta_1(x) + \theta_2(x)}{2}\right)$$

$$= 1 \cos\left(\frac{\theta_1(x) + \theta_2(x)}{2}\right) + \frac{1}{2} \cos(\theta_1(x)) + \frac{1}{2} \cos(\theta_2(x)).$$

- Which line is the result of the HHT?
- Let's see:

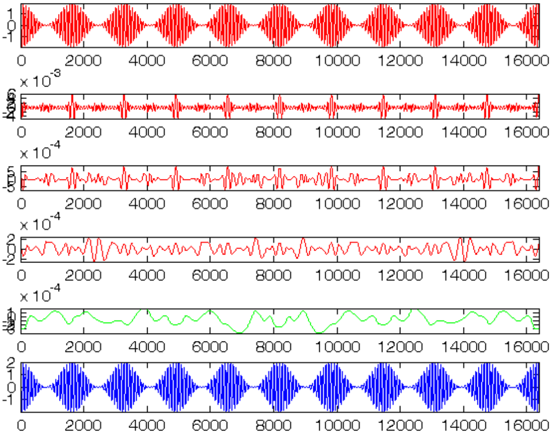
An interesting example

However, local mean condition does not imply uniqueness:

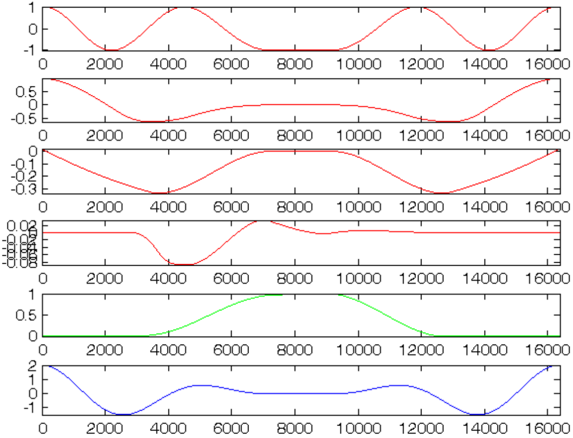
$$\begin{aligned} f(x) &= \left(1+\cos\left(\frac{\theta_1(x)-\theta_2(x)}{2}\right)\right)\cos\left(\frac{\theta_1(x)+\theta_2(x)}{2}\right) \\ &= 1\cos\left(\frac{\theta_1(x)+\theta_2(x)}{2}\right)+\frac{1}{2}\cos(\theta_1(x))+\frac{1}{2}\cos(\theta_2(x)). \end{aligned}$$

- Which line is the result of the HHT?
- Let's see:

Decomposition for frequencies 90π and 100π



Decomposition for frequencies 10π and 20π



Clifford Analysis

- $X = x_0 + e_1 x_1 + e_2 x_2 + e_{12} x_3$
- Multiplication rule: $e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1 = e_{12}$
- Dirac operator:

$$Df = \sum_{i=1}^{3} \partial_{x_i} f$$

- $D^2 = -\Delta$
- Weyl-relation: $[x_i, \partial_{x_i}] = 1$
- Monogenic function: $Df = 0$

Clifford Analysis

- $X = x_0 + e_1 x_1 + e_2 x_2 + e_{12} x_3$
- Multiplication rule: $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1 = e_{12}$
- Dirac operator:

$$Df = \sum_{i=1}^{3} \partial_{x_i} f$$

- $D^2 = -\Delta$
- Weyl-relation: $[x_i, \partial_{x_i}] = 1$
- Monogenic function: $Df = 0$

Clifford Analysis

- $X = x_0 + e_1 x_1 + e_2 x_2 + e_{12} x_3$
- Multiplication rule: $e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1 = e_{12}$
- Dirac operator:

$$Df = \sum_{i=1}^{3} \partial_{x_i} f$$

- $D^2 = -\Delta$
- Weyl-relation: $[x_i, \partial_{x_i}] = 1$
- Monogenic function: $Df = 0$

Clifford Analysis

- $X = x_0 + e_1 x_1 + e_2 x_2 + e_{12} x_3$
- Multiplication rule: $e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1 = e_{12}$
- Dirac operator:

$$Df = \sum_{i=1}^{3} \partial_{x_i} f$$

- $D^2 = -\Delta$
- Weyl-relation: $[x_i, \partial_{x_i}] = 1$
- Monogenic function: $Df = 0$

Clifford Analysis

- $X = x_0 + e_1 x_1 + e_2 x_2 + e_{12} x_3$
- Multiplication rule: $e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1 = e_{12}$
- Dirac operator:

$$Df = \sum_{i=1}^{3} \partial_{x_i} f$$

- $D^2 = -\Delta$
- Weyl-relation: $[x_i, \partial_{x_i}] = 1$
- Monogenic function: $Df = 0$

Clifford Analysis

- $X = x_0 + e_1 x_1 + e_2 x_2 + e_{12} x_3$
- Multiplication rule: $e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1 = e_{12}$
- Dirac operator:

$$Df = \sum_{i=1}^{3} \partial_{x_i} f$$

- $D^2 = -\Delta$
- Weyl-relation: $[x_i, \partial_{x_i}] = 1$
- Monogenic function: $Df = 0$

Important Formulae

- Cauchy-kernel: $C(x) = x/|x|^3, x \in \mathbb{R}^3$
- Cauchy formulae: If f monogenic then

$$f(x) = F_\Gamma f = \int_\Gamma \frac{x-y}{|x-y|^3} n(y) f(y) dy$$

- Plemelj-Sokhotzki formulae:

$$\mathrm{tr} F_\Gamma f = P_\Gamma f = \frac{1}{2} f + \frac{1}{2} H_\Gamma f$$

with $H_\Gamma f = e_1 R_1 f + e_2 R_2 f$, R_i - Riesz transform

- Decomposition into Hardy spaces:

$$L_p(\mathbb{R}^n) = H_p^+(\mathbb{R}^n) \oplus H_p^-(\mathbb{R}^n)$$

Important Formulae

- Cauchy-kernel: $C(x) = x/|x|^3, x \in \mathbb{R}^3$
- Cauchy formulae: If f monogenic then

$$f(x) = F_\Gamma f = \int_\Gamma \frac{x-y}{|x-y|^3} n(y) f(y) dy$$

- Plemelj-Sokhotzki formulae:

$$\mathrm{tr} F_\Gamma f = P_\Gamma f = \frac{1}{2} f + \frac{1}{2} H_\Gamma f$$

with $H_\Gamma f = e_1 R_1 f + e_2 R_2 f$, R_i - Riesz transform

- Decomposition into Hardy spaces:

$$L_p(\mathbb{R}^n) = H_p^+(\mathbb{R}^n) \oplus H_p^-(\mathbb{R}^n)$$

Important Formulae

- Cauchy-kernel: $C(x) = x/|x|^3, x \in \mathbb{R}^3$
- Cauchy formulae: If f monogenic then

$$f(x) = F_\Gamma f = \int_\Gamma \frac{x-y}{|x-y|^3} n(y) f(y) dy$$

- Plemelj-Sokhotzki formulae:

$$\mathrm{tr} F_\Gamma f = P_\Gamma f = \frac{1}{2} f + \frac{1}{2} H_\Gamma f$$

with $H_\Gamma f = e_1 R_1 f + e_2 R_2 f$, R_i - Riesz transform

- Decomposition into Hardy spaces:

$$L_p(\mathbb{R}^n) = H_p^+(\mathbb{R}^n) \oplus H_p^-(\mathbb{R}^n)$$

Important Formulae

- Cauchy-kernel: $C(x) = x/|x|^3, x \in \mathbb{R}^3$
- Cauchy formulae: If f monogenic then

$$f(x) = F_\Gamma f = \int_\Gamma \frac{x-y}{|x-y|^3} n(y) f(y) dy$$

- Plemelj-Sokhotzki formulae:

$$\mathrm{tr} F_\Gamma f = P_\Gamma f = \frac{1}{2} f + \frac{1}{2} H_\Gamma f$$

with $H_\Gamma f = e_1 R_1 f + e_2 R_2 f$, R_i - Riesz transform

- Decomposition into Hardy spaces:

$$L_p(\mathbb{R}^n) = H_p^+(\mathbb{R}^n) \oplus H_p^-(\mathbb{R}^n)$$

Monogenic signal for images

G. Sommer, M. Felsberg 2001

- Given signal f take Riesz transforms $R_i f$
- Monogenic signal: $f + \sum e_i R_i f$
- Amplitude: $a = \sqrt{f^2 + \sum (R_i f)^2}$
- Local phase direction: $u = \frac{\sum e_i R_i f}{|\sum e_i R_i f|}$
- Local phase angle: $\tan \alpha = \frac{\sqrt{\sum (R_i f)^2}}{f}$
- Local phase: $\Phi = \alpha u$

Monogenic signal for images

G. Sommer, M. Felsberg 2001

- Given signal f take Riesz transforms $R_i f$
- Monogenic signal: $f + \sum \mathrm{e}_i R_i f$
- Amplitude: $a = \sqrt{f^2 + \sum (R_i f)^2}$
- Local phase direction: $u = \frac{\sum \mathrm{e}_i R_i f}{|\sum \mathrm{e}_i R_i f|}$
- Local phase angle: $\tan \alpha = \frac{\sqrt{\sum (R_i f)^2}}{f}$
- Local phase: $\Phi = \alpha u$

Monogenic signal for images

G. Sommer, M. Felsberg 2001

- Given signal f take Riesz transforms $R_i f$
- Monogenic signal: $f + \sum \mathrm{e}_i R_i f$
- Amplitude: $a = \sqrt{f^2 + \sum (R_i f)^2}$
- Local phase direction: $u = \frac{\sum \mathrm{e}_i R_i f}{|\sum \mathrm{e}_i R_i f|}$
- Local phase angle: $\tan\alpha = \frac{\sqrt{\sum (R_i f)^2}}{f}$
- Local phase: $\Phi = \alpha u$

Monogenic signal for images

G. Sommer, M. Felsberg 2001

- Given signal f take Riesz transforms $R_i f$
- Monogenic signal: $f + \sum \mathrm{e}_i R_i f$
- Amplitude: $a = \sqrt{f^2 + \sum (R_i f)^2}$
- Local phase direction: $u = \frac{\sum \mathrm{e}_i R_i f}{|\sum \mathrm{e}_i R_i f|}$
- Local phase angle: $\tan \alpha = \frac{\sqrt{\sum (R_i f)^2}}{f}$
- Local phase: $\Phi = \alpha u$

Monogenic signal for images

G. Sommer, M. Felsberg 2001

- Given signal f take Riesz transforms $R_i f$
- Monogenic signal: $f + \sum \mathrm{e}_i R_i f$
- Amplitude: $a = \sqrt{f^2 + \sum (R_i f)^2}$
- Local phase direction: $u = \frac{\sum \mathrm{e}_i R_i f}{|\sum \mathrm{e}_i R_i f|}$
- Local phase angle: $\tan \alpha = \frac{\sqrt{\sum (R_i f)^2}}{f}$
- Local phase: $\Phi = \alpha u$

Monogenic signal for images

G. Sommer, M. Felsberg 2001

- Given signal f take Riesz transforms $R_i f$
- Monogenic signal: $f + \sum \mathrm{e}_i R_i f$
- Amplitude: $a = \sqrt{f^2 + \sum (R_i f)^2}$
- Local phase direction: $u = \frac{\sum \mathrm{e}_i R_i f}{|\sum \mathrm{e}_i R_i f|}$
- Local phase angle: $\tan \alpha = \frac{\sqrt{\sum (R_i f)^2}}{f}$
- Local phase: $\Phi = \alpha u$

Link with monogenic functions

- Hypercomplex signals: T. Bülow, G. Sommer, M. Felsberg, R. Baraniuk, T. Qian, L. Peng, ...
- Hilbert transform = conjugate harmonic functions
- Not really different when based on the Riesz transform
- Other terms: complex analytic signals, quaternionic analytic signals, ...
- Main point: f - monogenic signal $\Leftrightarrow f \in H_p^+$
- Monogenic signal is boundary value of a monogenic function (in the upper half-space)

Link with monogenic functions

- Hypercomplex signals: T. Bülow, G. Sommer, M. Felsberg, R. Baraniuk, T. Qian, L. Peng, . . .
- Hilbert transform = conjugate harmonic functions
- Not really different when based on the Riesz transform
- Other terms: complex analytic signals, quaternionic analytic signals, . . .
- Main point: f - monogenic signal $\Leftrightarrow f \in H_p^+$
- Monogenic signal is boundary value of a monogenic function (in the upper half-space)

Link with monogenic functions

- Hypercomplex signals: T. Bülow, G. Sommer, M. Felsberg, R. Baraniuk, T. Qian, L. Peng, ...
- Hilbert transform = conjugate harmonic functions
- Not really different when based on the Riesz transform
- Other terms: complex analytic signals, quaternionic analytic signals, ...
- Main point: f - monogenic signal $\Leftrightarrow f \in H_p^+$
- Monogenic signal is boundary value of a monogenic function (in the upper half-space)

Link with monogenic functions

- Hypercomplex signals: T. Bülow, G. Sommer, M. Felsberg, R. Baraniuk, T. Qian, L. Peng, . . .
- Hilbert transform = conjugate harmonic functions
- Not really different when based on the Riesz transform
- Other terms: complex analytic signals, quaternionic analytic signals, . . .
- Main point: f - monogenic signal $\Leftrightarrow f \in H_p^+$
- Monogenic signal is boundary value of a monogenic function (in the upper half-space)

Link with monogenic functions

- Hypercomplex signals: T. Bülow, G. Sommer, M. Felsberg, R. Baraniuk, T. Qian, L. Peng, ...
- Hilbert transform = conjugate harmonic functions
- Not really different when based on the Riesz transform
- Other terms: complex analytic signals, quaternionic analytic signals, ...
- Main point: f - monogenic signal $\Leftrightarrow f \in H_p^+$
- Monogenic signal is boundary value of a monogenic function (in the upper half-space)

Link with monogenic functions

- Hypercomplex signals: T. Bülow, G. Sommer, M. Felsberg, R. Baraniuk, T. Qian, L. Peng, . . .
- Hilbert transform = conjugate harmonic functions
- Not really different when based on the Riesz transform
- Other terms: complex analytic signals, quaternionic analytic signals, . . .
- Main point: f - monogenic signal $\Leftrightarrow f \in H_p^+$
- Monogenic signal is boundary value of a monogenic function (in the upper half-space)

Observations

- Monogenic signal as boundary value of monogenic function is not unique!
- Example: Quaternionic analytic signal - L. Peng
- $f_3 = R_1 R_2 f$
- $g = f + e_1(R_1 + R_1 R_2 R_1)f + e_2(R_2 - R_1 R_1 R_2)f + e_1 e_2 R_1 R_2 f$
- Again boundary value of a monogenic function (Cauchy-Riemann operator)
- Hilbert pairs = conjugate harmonic functions

Observations

- Monogenic signal as boundary value of monogenic function is not unique!
- Example: Quaternionic analytic signal - L. Peng
- $f_3 = R_1 R_2 f$
- $g = f + e_1(R_1 + R_1 R_2 R_1)f + e_2(R_2 - R_1 R_1 R_2)f + e_1 e_2 R_1 R_2 f$
- Again boundary value of a monogenic function (Cauchy-Riemann operator)
- Hilbert pairs = conjugate harmonic functions

Observations

- Monogenic signal as boundary value of monogenic function is not unique!
- Example: Quaternionic analytic signal - L. Peng
- $f_3 = R_1 R_2 f$
- $g = f + e_1(R_1 + R_1 R_2 R_1)f + e_2(R_2 - R_1 R_1 R_2)f + e_1 e_2 R_1 R_2 f$
- Again boundary value of a monogenic function (Cauchy-Riemann operator)
- Hilbert pairs = conjugate harmonic functions

Observations

- Monogenic signal as boundary value of monogenic function is not unique!
- Example: Quaternionic analytic signal - L. Peng
- $f_3 = R_1 R_2 f$
- $g = f + e_1(R_1 + R_1 R_2 R_1)f + e_2(R_2 - R_1 R_1 R_2)f + e_1 e_2 R_1 R_2 f$
- Again boundary value of a monogenic function (Cauchy-Riemann operator)
- Hilbert pairs = conjugate harmonic functions

Observations

- Monogenic signal as boundary value of monogenic function is not unique!
- Example: Quaternionic analytic signal - L. Peng
- $f_3 = R_1 R_2 f$
- $g = f + e_1(R_1 + R_1 R_2 R_1)f + e_2(R_2 - R_1 R_1 R_2)f + e_1 e_2 R_1 R_2 f$
- Again boundary value of a monogenic function (Cauchy-Riemann operator)
- Hilbert pairs = conjugate harmonic functions

Observations

- Monogenic signal as boundary value of monogenic function is not unique!
- Example: Quaternionic analytic signal - L. Peng
- $f_3 = R_1 R_2 f$
- $g = f + e_1(R_1 + R_1 R_2 R_1)f + e_2(R_2 - R_1 R_1 R_2)f + e_1 e_2 R_1 R_2 f$
- Again boundary value of a monogenic function (Cauchy-Riemann operator)
- Hilbert pairs = conjugate harmonic functions

Higher-dimensional extension of EMD

1. Approach: Scattered Data interpolation

- Interpolation by (compactly supported) Radial Basis Functions (RBF)

$$s(x) = \sum_{k=1}^{N} c_k \Phi(|x - x^{(k)}|)$$

- Micchelli 1986: Interpolation matrix

$$A_{X,\phi} = \left(\Phi(|x^{(l)} - x^{(k)}|) \right)_{l,k}$$

is positive definite

- For quality of approximation we have two terms:
  - Size of support σ
  - Minimum distance of two interpolation points q

Higher-dimensional extension of EMD

1. Approach: Scattered Data interpolation

- Interpolation by (compactly supported) Radial Basis Functions (RBF)

$$s(x) = \sum_{k=1}^{N} c_k \Phi(|x - x^{(k)}|)$$

- Micchelli 1986: Interpolation matrix

$$A_{X,\phi} = \left(\Phi(|x^{(l)} - x^{(k)}|)\right)_{l,k}$$

  is positive definite

- For quality of approximation we have two terms:
  - Size of support σ
  - Minimum distance of two interpolation points q

Higher-dimensional extension of EMD

1. Approach: Scattered Data interpolation

- Interpolation by (compactly supported) Radial Basis Functions (RBF)

$$s(x) = \sum_{k=1}^{N} c_k \Phi(|x - x^{(k)}|)$$

- Micchelli 1986: Interpolation matrix

$$A_{X,\phi} = \left(\Phi(|x^{(l)} - x^{(k)}|) \right)_{l,k}$$

  is positive definite
- For quality of approximation we have two terms:
  - Size of support σ
  - Minimum distance of two interpolation points q

Higher-dimensional extension of EMD

1. Approach: Scattered Data interpolation

- Interpolation by (compactly supported) Radial Basis Functions (RBF)

$$s(x) = \sum_{k=1}^{N} c_k \Phi(|x - x^{(k)}|)$$

- Micchelli 1986: Interpolation matrix

$$A_{X,\phi} = \left(\Phi(|x^{(l)} - x^{(k)}|)\right)_{l,k}$$

  is positive definite

- For quality of approximation we have two terms:
  - Size of support σ
  - Minimum distance of two interpolation points q

Higher-dimensional extension of EMD

1. Approach: Scattered Data interpolation

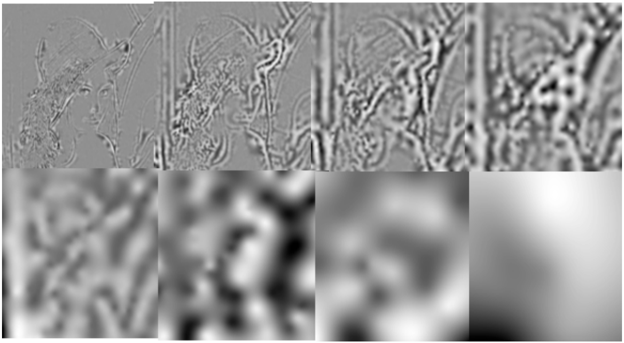
- Interpolation by (compactly supported) Radial Basis Functions (RBF)

$$s(x) = \sum_{k=1}^{N} c_k \Phi(|x - x^{(k)}|)$$

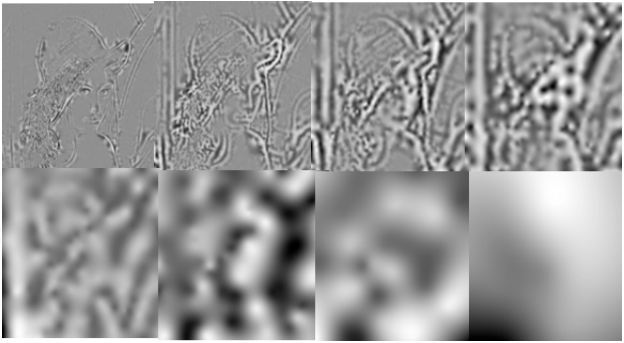
- Micchelli 1986: Interpolation matrix

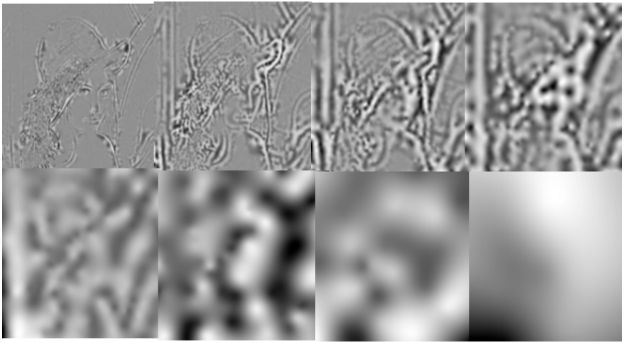
$$A_{X,\phi} = \left(\Phi(|x^{(l)} - x^{(k)}|)\right)_{l,k}$$

  is positive definite
- For quality of approximation we have two terms:
  - Size of support σ
  - Minimum distance of two interpolation points q

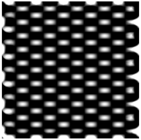
- Radial basis functions (RBF) and multi-level B-Splines
- $s(x) = \sum_{k=1}^{N} [g_k(x) + c_k]\Phi(|x - x^{(k)}|)$
- 

- Radial basis functions (RBF) and multi-level B-Splines
- $s(x) = \sum_{k=1}^{N} [g_k(x) + c_k] \Phi(|x - x^{(k)}|)$



- Radial basis functions (RBF) and multi-level B-Splines
- $s(x) = \sum_{k=1}^{N} [g_k(x) + c_k] \Phi(|x - x^{(k)}|)$
- 

Example of IMF



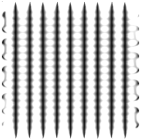
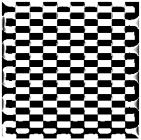


Figure: 1: IMF 2: Amplitude



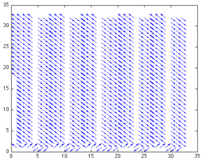


Figure: 3: Phase angle 4: Phase direction

Higher-dimensional extension of EMD

2. Approach: Radon transform

- Link between Radon and Riesz transforms in $\mathbb{R}^n$:

$$\mathcal{R}(H_{n-1}f)(\omega, s) = \omega H_1(\mathcal{R}f(\omega, \cdot))(s), \quad (\omega, s) \in S^{n-1} \times \mathbb{R}$$

- Apply Radon transform to the image
- Use classic EMD for each direction ω
- Apply inverse Radon transform
- Depends on fast (stable) inversion of Radon transform

Higher-dimensional extension of EMD

2. Approach: Radon transform

- Link between Radon and Riesz transforms in $\mathbb{R}^n$:

$$\mathcal{R}(H_{n-1}f)(\omega, s) = \omega H_1(\mathcal{R}f(\omega,\cdot))(s), \quad (\omega, s) \in S^{n-1} \times \mathbb{R}$$

- Apply Radon transform to the image
- Use classic EMD for each direction ω
- Apply inverse Radon transform
- Depends on fast (stable) inversion of Radon transform

Higher-dimensional extension of EMD

2. Approach: Radon transform

- Link between Radon and Riesz transforms in $\mathbb{R}^n$:

$$\mathcal{R}(H_{n-1}f)(\omega, s) = \omega H_1(\mathcal{R}f(\omega, \cdot))(s), \quad (\omega, s) \in S^{n-1} \times \mathbb{R}$$

- Apply Radon transform to the image
- Use classic EMD for each direction ω
- Apply inverse Radon transform
- Depends on fast (stable) inversion of Radon transform

Higher-dimensional extension of EMD

2. Approach: Radon transform

- Link between Radon and Riesz transforms in $\mathbb{R}^n$:

$$\mathcal{R}(H_{n-1}f)(\omega, s) = \omega H_1(\mathcal{R}f(\omega, \cdot))(s), \quad (\omega, s) \in S^{n-1} \times \mathbb{R}$$

- Apply Radon transform to the image
- Use classic EMD for each direction ω
- Apply inverse Radon transform
- Depends on fast (stable) inversion of Radon transform

Higher-dimensional extension of EMD

2. Approach: Radon transform

- Link between Radon and Riesz transforms in $\mathbb{R}^n$:

$$\mathcal{R}(H_{n-1}f)(\omega, s) = \omega H_1(\mathcal{R}f(\omega, \cdot))(s), \quad (\omega, s) \in S^{n-1} \times \mathbb{R}$$

- Apply Radon transform to the image
- Use classic EMD for each direction ω
- Apply inverse Radon transform
- Depends on fast (stable) inversion of Radon transform

Conformal monogenic signal

- Idea of G. Sommer, L. Wietzke, and O. Fleischmann
- Embed image f on S^2 conformally
- Apply a Hilbert transform (connected to the Cauchy integral) on f
- Embed S^2 in S^3 for better decay
- Boundary value of monogenic function in B^3

Conformal monogenic signal

- Idea of G. Sommer, L. Wietzke, and O. Fleischmann
- Embed image f on S^2 conformally
- Apply a Hilbert transform (connected to the Cauchy integral) on f
- Embed S^2 in S^3 for better decay
- Boundary value of monogenic function in B^3

Conformal monogenic signal

- Idea of G. Sommer, L. Wietzke, and O. Fleischmann
- Embed image f on S^2 conformally
- Apply a Hilbert transform (connected to the Cauchy integral) on f
- Embed S^2 in S^3 for better decay
- Boundary value of monogenic function in B^3

Conformal monogenic signal

- Idea of G. Sommer, L. Wietzke, and O. Fleischmann
- Embed image f on S^2 conformally
- Apply a Hilbert transform (connected to the Cauchy integral) on f
- Embed S^2 in S^3 for better decay
- Boundary value of monogenic function in B^3

Conformal monogenic signal

- Idea of G. Sommer, L. Wietzke, and O. Fleischmann
- Embed image f on S^2 conformally
- Apply a Hilbert transform (connected to the Cauchy integral) on f
- Embed S^2 in S^3 for better decay
- Boundary value of monogenic function in B^3

Conformal group of the ball

- Lorentz group Spin$(1, n)$ - conformal group of the ball.
- It can be decomposed (Cartan decomposition Spin$(1, n)$=Spin$(n)\times$Spin$(1, 1)\times$Spin(n).
- For $n \geq 3$ representations are not square-integrable $\Rightarrow$ need to factorize
- Dilations given by Spin$(n)\times$Spin$(1, 1)$ (Ferreira 2007)
- For EMD: Interpolation by SFB's (RFB's on the sphere), with isotropic dilation (Antoine/Vandergheynst 1999) or anisotropic dilation (Ferreira 2007)

Conformal group of the ball

- Lorentz group Spin$(1, n)$ - conformal group of the ball.
- It can be decomposed (Cartan decomposition Spin$(1, n)$=Spin$(n)\times$Spin$(1, 1)\times$Spin(n).
- For $n \geq 3$ representations are not square-integrable $\Rightarrow$ need to factorize
- Dilations given by Spin$(n)\times$Spin$(1, 1)$ (Ferreira 2007)
- For EMD: Interpolation by SFB's (RFB's on the sphere), with isotropic dilation (Antoine/Vandergheynst 1999) or anisotropic dilation (Ferreira 2007)

Conformal group of the ball

- Lorentz group Spin$(1, n)$ - conformal group of the ball.
- It can be decomposed (Cartan decomposition Spin$(1, n)$=Spin$(n)\times$Spin$(1, 1)\times$Spin(n).
- For $n \geq 3$ representations are not square-integrable $\Rightarrow$ need to factorize
- Dilations given by Spin$(n)\times$Spin$(1, 1)$ (Ferreira 2007)
- For EMD: Interpolation by SFB's (RFB's on the sphere), with isotropic dilation (Antoine/Vandergheynst 1999) or anisotropic dilation (Ferreira 2007)

Conformal group of the ball

- Lorentz group Spin$(1, n)$ - conformal group of the ball.
- It can be decomposed (Cartan decomposition Spin$(1, n)$=Spin$(n)\times$Spin$(1, 1)\times$Spin(n).
- For $n \geq 3$ representations are not square-integrable $\Rightarrow$ need to factorize
- Dilations given by Spin$(n)\times$Spin$(1, 1)$ (Ferreira 2007)
- For EMD: Interpolation by SFB's (RFB's on the sphere), with isotropic dilation (Antoine/Vandergheynst 1999) or anisotropic dilation (Ferreira 2007)

Conformal group of the ball

- Lorentz group Spin$(1, n)$ - conformal group of the ball.
- It can be decomposed (Cartan decomposition Spin$(1, n)$=Spin$(n)\times$Spin$(1, 1)\times$Spin(n).
- For $n \geq 3$ representations are not square-integrable $\Rightarrow$ need to factorize
- Dilations given by Spin$(n)\times$Spin$(1, 1)$ (Ferreira 2007)
- For EMD: Interpolation by SFB's (RFB's on the sphere), with isotropic dilation (Antoine/Vandergheynst 1999) or anisotropic dilation (Ferreira 2007)

Conformal Model

- Conformal embedding of $\mathbb{R}^{0,n}$ into $\mathbb{R}^{1,n+1}$ by means of
  $\underline{x} = x_1 e_1 + \cdots + x_n e_n$
  $\hookrightarrow \; X = X_1 e_1 + \cdots + X_n e_n + X_{n+1} e_{n+1} + X_{n+2} e_{n+2}$

$$= \underline{x} + \frac{1 - r^2}{2} e_{n+1} + \frac{1 + r^2}{2} e_{n+2}, \;\; r = |\underline{x}|.$$

- Bilinear form induced in $P\mathbb{R}^n$

$$Q(x) = X_{n+2}^2 - \sum_{i=1}^{n+1} X_j^2$$

- The application $\underline{x} \to X$ is (up to the signal) an isometry $[ds_E^2 = ds_H^2]$.

Conformal Model

- Conformal embedding of $\mathbb{R}^{0,n}$ into $\mathbb{R}^{1,n+1}$ by means of $\underline{x} = x_1 e_1 + \cdots + x_n e_n$
$\hookrightarrow \; X = X_1 e_1 + \cdots + X_n e_n + X_{n+1} e_{n+1} + X_{n+2} e_{n+2}$

$$= \underline{x} + \frac{1 - r^2}{2} e_{n+1} + \frac{1 + r^2}{2} e_{n+2}, \;\; r = |\underline{x}|.$$

- Bilinear form induced in $P\mathbb{R}^n$

$$Q(x) = X_{n+2}^2 - \sum_{i=1}^{n+1} X_j^2$$

- The application $\underline{x} \to X$ is (up to the signal) an isometry $[ds_E^2 = ds_H^2]$.

Conformal Model

- Conformal embedding of $\mathbb{R}^{0,n}$ into $\mathbb{R}^{1,n+1}$ by means of
$\underline{x} = x_1 e_1 + \cdots + x_n e_n$
$\hookrightarrow \ X = X_1 e_1 + \cdots + X_n e_n + X_{n+1} e_{n+1} + X_{n+2} e_{n+2}$

$$= \underline{x} + \frac{1 - r^2}{2} e_{n+1} + \frac{1 + r^2}{2} e_{n+2}, \ r = |\underline{x}|.$$

- Bilinear form induced in $P\mathbb{R}^n$

$$Q(x) = X_{n+2}^2 - \sum_{i=1}^{n+1} X_j^2$$

- The application $\underline{x} \to X$ is (up to the signal) an isometry $[ds_E^2 = ds_H^2]$.

$$
\begin{array}{rcl}
ray(\underline{0}, -1, +1) & \leftrightarrow & \infty \\
T & \leftrightarrow & NC_+ \cap \{X : X_{n+2} < 1\} \\
S^n = \mathbb{R}^n \cup \{\infty\} & \leftrightarrow & NC_+ \cap \{X : X_{n+2} = 1\}
\end{array}
$$

Projection Lorentz group $\mathrm{Spin}(1, n)$ is the automorphic group for $P\mathbb{R}^n$ (hyperbolic projective space)

$$\Downarrow$$

$$(X_1, \cdots, X_n, X_{n+1}, X_{n+2}) \rightarrow (X_1, \cdots, X_n, X_{n+2})$$

Spherical mean transform

- Spherical mean transform:

$$(Rf)(p) := \int_{p^\perp} f(q)dp^\perp \tag{1}$$

where the great geodesic circle $p^\perp = \{q \in S^3 : (q\bar{p})_0 = 0\} = S^2 p$ denotes the orthogonal complement of $p \in S^3$ and $dp^\perp$ is the normalized surface element.

- Can be rewritten as

$$Rf(y) = \int_{S^{n-1}} \delta(< x, y >) f(x) d\omega_{n-1}(x)$$

- Using conformal model we get ($y_m = 1$):

$$Rf(y) = 2\int_{\mathbb{R}^{n-1}+e_n} \delta\left(\sum_{i=1}^{n-1} x_i y_i + 1\right) f(x_1, \cdots, x_{n-1}, 1) dx_1 \cdots dx_{n-1}$$

Spherical mean transform

- Spherical mean transform:

$$(Rf)(p) := \int_{p^\perp} f(q)dp^\perp \tag{1}$$

where the great geodesic circle $p^\perp = \{q \in S^3 : (q\bar{p})_0 = 0\} = S^2 p$ denotes the orthogonal complement of $p \in S^3$ and $dp^\perp$ is the normalized surface element.

- Can be rewritten as

$$Rf(y) = \int_{S^{n-1}} \delta(<x,y>)f(x)d\omega_{n-1}(x)$$

- Using conformal model we get ($y_m = 1$):

$$Rf(y) = 2\int_{\mathbb{R}^{n-1}+e_n} \delta\left(\sum_{i=1}^{n-1} x_i y_i + 1\right) f(x_1, \cdots, x_{n-1}, 1)dx_1 \cdots dx_{n-1}$$

Spherical mean transform

- Spherical mean transform:

$$(Rf)(p) := \int_{p^\perp} f(q)dp^\perp \qquad (1)$$

where the great geodesic circle $p^\perp = \{q \in S^3 : (q\overline{p})_0 = 0\} = S^2 p$ denotes the orthogonal complement of $p \in S^3$ and $dp^\perp$ is the normalized surface element.

- Can be rewritten as

$$Rf(y) = \int_{S^{n-1}} \delta(< x, y >) f(x) d\omega_{n-1}(x)$$

- Using conformal model we get ($y_m = 1$):

$$Rf(y) = 2\int_{\mathbb{R}^{n-1}+e_n} \delta\left(\sum_{i=1}^{n-1} x_i y_i + 1\right) f(x_1, \cdots, x_{n-1}, 1) dx_1 \cdots dx_{n-1}$$

THAT'S ALL FOLKS!