MAMEBIA
Mathematical Methods in Biological Image Analysis

Wavelets for Multiscale Amplitude and Phase Decomposition

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Monogenic Signal

The monogenic signal allows a decomposition of multidimensional signals $f \in L^2(\mathbb{R}^d)$ into phase and amplitude. It is a natural extension of the one dimensional analytic signal.

- The *monogenic signal* is the $d+1$ dimensional vector
$$f_m(x) := (f(x), \mathscr{R}_1 f(x), \ldots, \mathscr{R}_d f(x)),$$
where $\mathscr{R}_i$ denotes the Riesz transform to the coordinate i.
- The *Riesz transform* is the singular integral
$$\mathscr{R}_j f(x) := C \lim_{\varepsilon \to 0} \int_{0<\varepsilon\leq|y|} \frac{y_j}{|y|^{d+1}} f(x-y)\, dy.$$
- The *amplitude* is defined as
$$A_f := \sqrt{f^2 + (\mathscr{R}_1 f)^2 + \ldots + (\mathscr{R}_d f)^2}$$
and contains information about the signal's energy.
- The *phase* is defined as
$$\alpha_f := \arctan \frac{\sqrt{(\mathscr{R}_1 f)^2 + \ldots + (\mathscr{R}_d f)^2}}{f}.$$
and contains information about the signal structure.

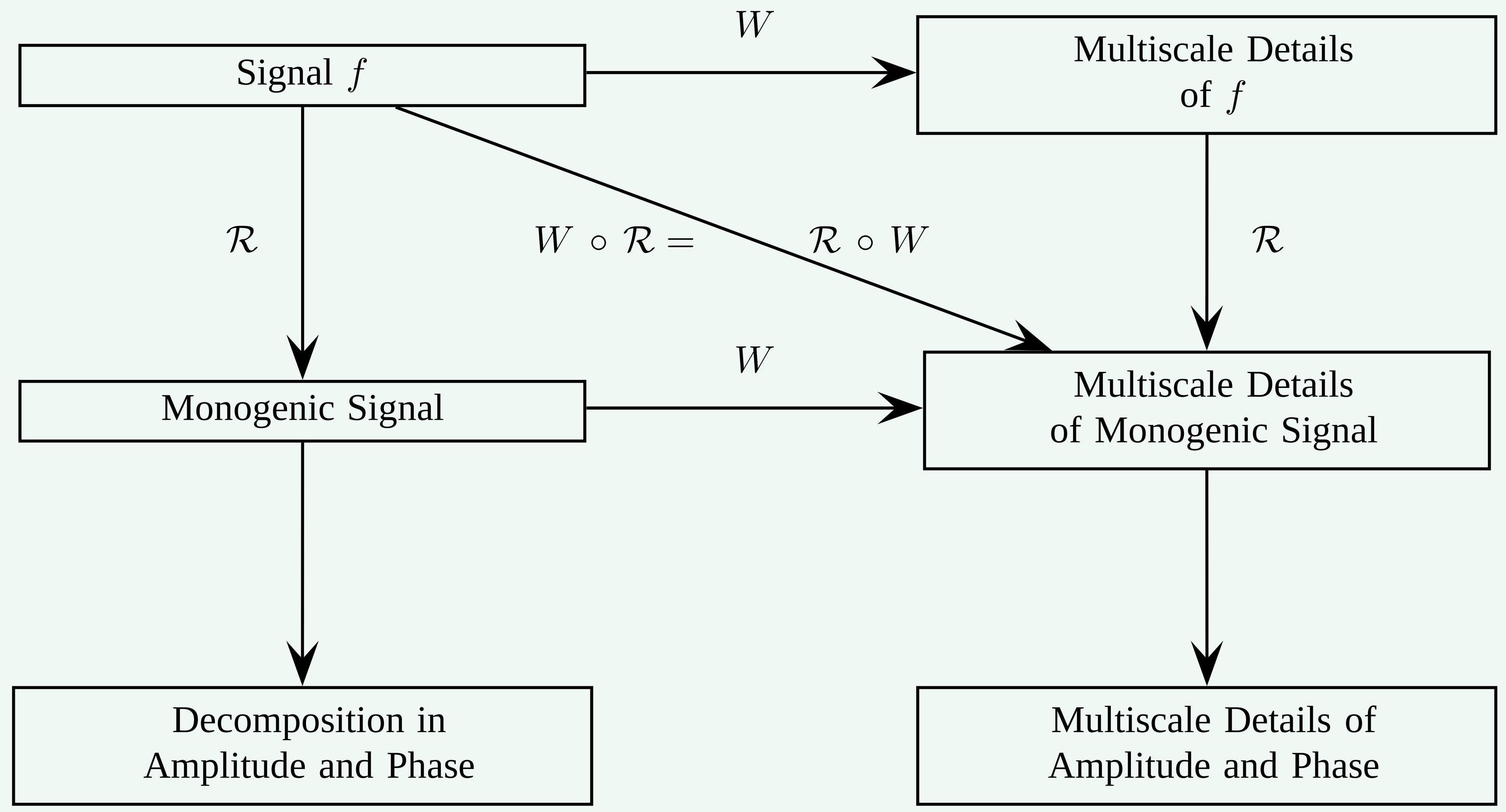
Isotropic Tight Wavelet Frames

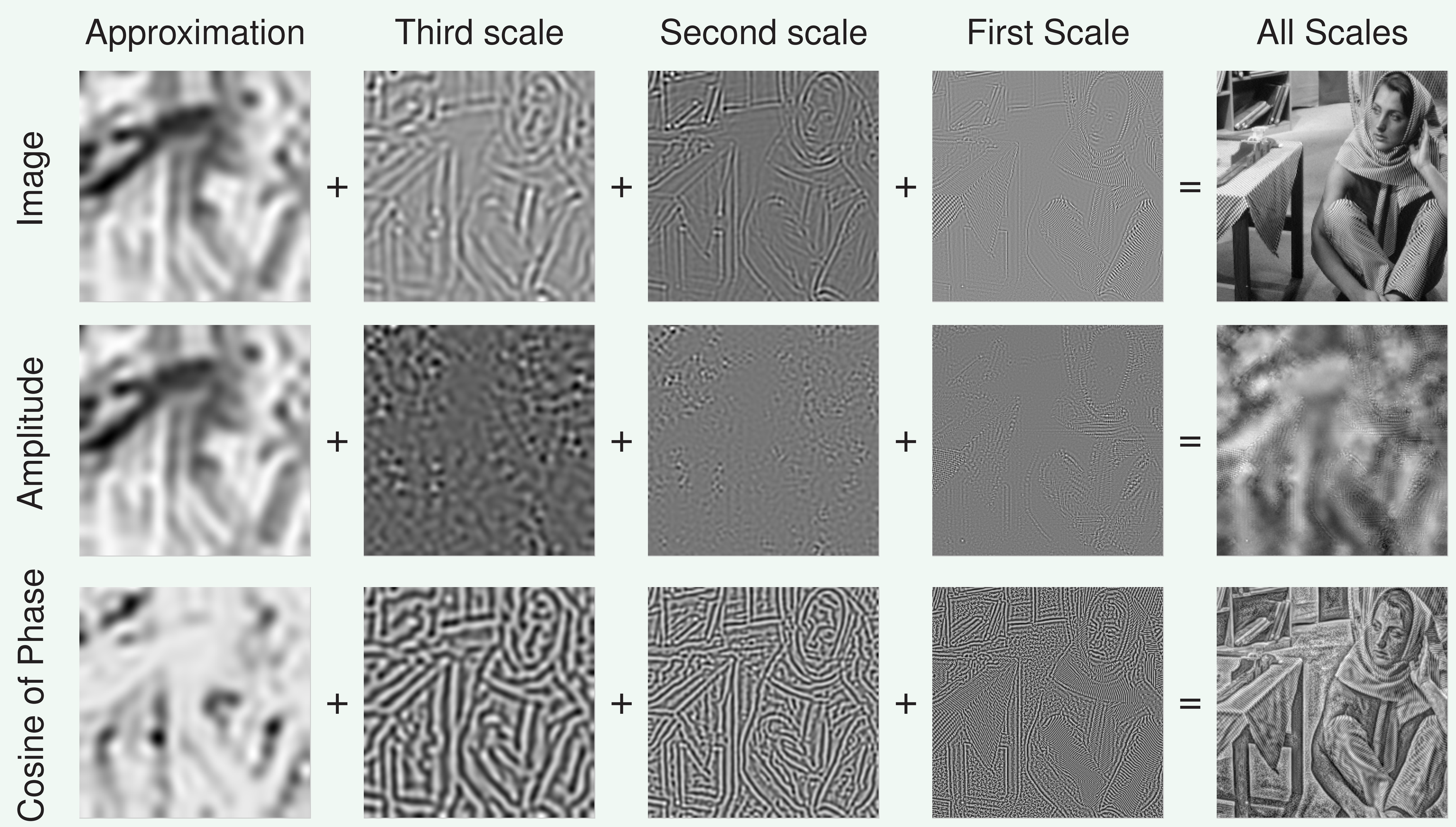
Tight wavelet frames allow for a continuous decomposition of a signal $f \in L^2(\mathbb{R}^d)$ into different detail levels.

- A *tight frame* is a family of functions $\{f_j\}_{j\in\mathbb{Z}} \subset L^2(\mathbb{R}^d)$, such that
$$\|f\|^2 = A \sum_{j\in\mathbb{Z}} |\langle f, f_j \rangle|^2$$
with some $A > 0$. This assures the continuous expansion of $f \in L^2(\mathbb{R}^d)$ into its atomic frame elements
$$f = \sum_{j\in\mathbb{Z}} \langle f, f_j \rangle f_j.$$
- A *wavelet frame* is a frame that has a particularly easy structure, i.e. the family is generated by the translations and dilations of a single function ψ.
- *Isotropic* means invariant under rotations. Isotropy is necessary for the combination of wavelet frames with the monogenic signal.

Combination of the Monogenic Signal and Wavelet Frames

The combination of monogenic signal and wavelet frame decomposition allows the analysis of amplitude and phase on different detail levels. The monogenic signal of a tight wavelet frame is also a tight wavelet frame, yielding continuous decomposition and perfect reconstruction.

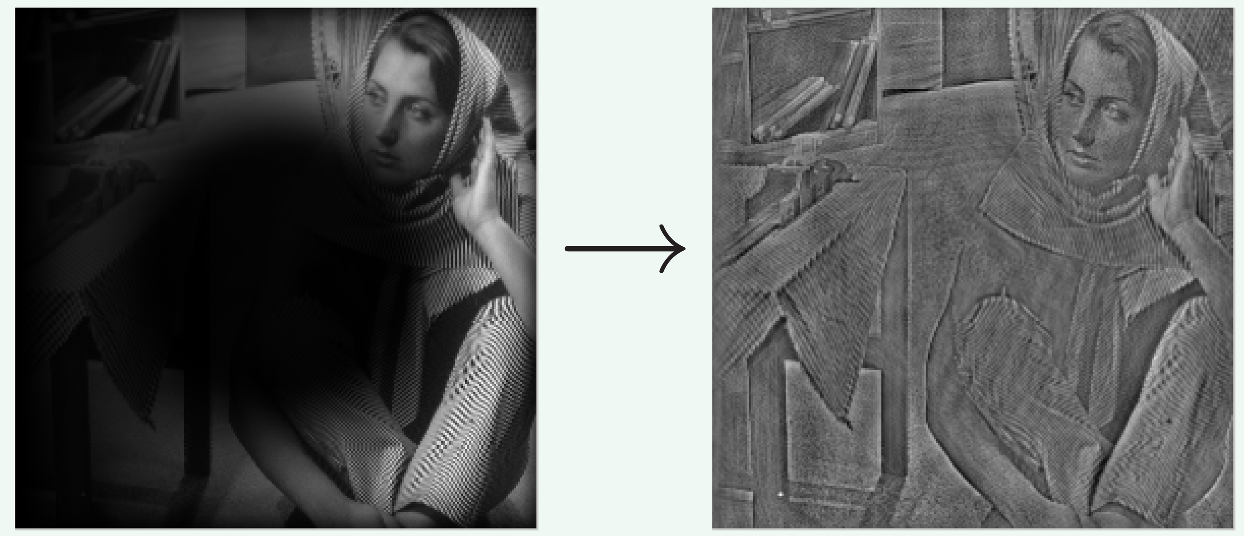




An image (upper right) is decomposed into its fine to coarse details (first row). The reconstruction algorithm restores the original image (first row, last image). The amplitude of the multiscale decomposition has higher response on regular structures (second row). The reconstruction from the amplitude only does not have an obvious interpretation. The phase contains nearly all structural information of the image (third row). The reconstruction from phase only maintains the image structure completely.

Application 1: Equalization of Brightness

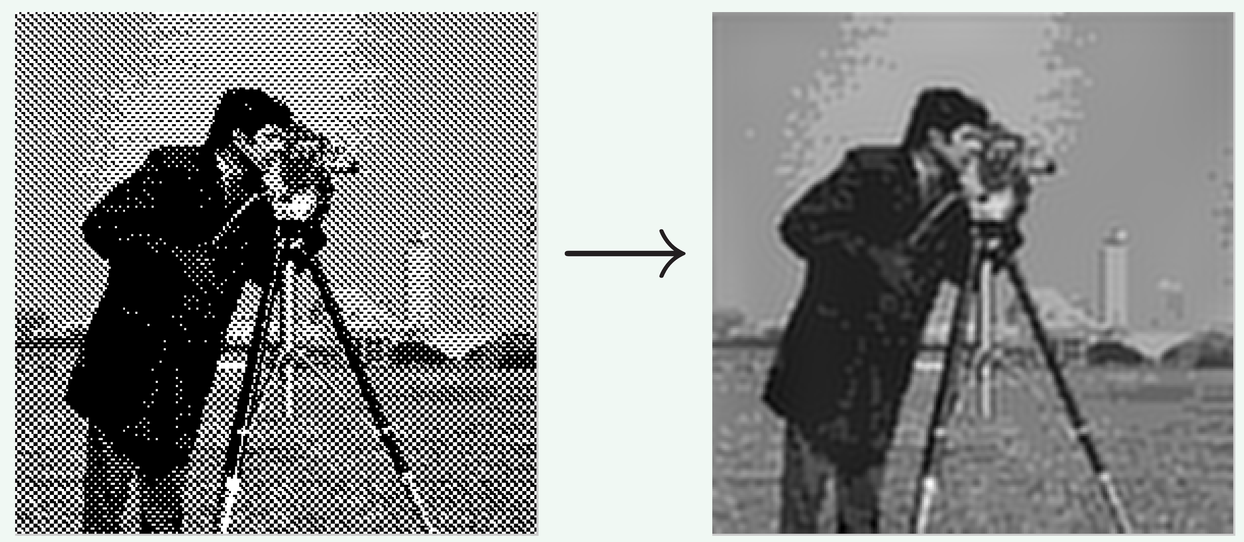
Reconstruction only from phase delivers an image with balanced brightness.



Left: Artificially corrupted image, multiplied by a ramp function and by 0.05 in the center. Right: Result of phase only reconstruction. The brightness is now equalized.

Application 2: Parameter Free Descreening

The screening effect is removed by setting detail levels of maximal amplitude to zero. This algorithm is completely parameter free.



Left: Image with screening effect (16 colors). Right: Result of descreening. The screened image is interpolated.

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