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Distributions

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This script is done parallel to the lecture "Mathematical Continuum Mechanics" [4]. It is ment for the students of this lecture. Since "Distributions" are of use also outside of physical applications some topics in this script are of general interest. The script contains still texts in German, because this was the original language for certain parts, but in the long run everything will be translated into English. A German version is always available.

Skript ist noch in Bearbeitung.

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1 Introduction

This paper deals with the definition of distributions. This is a notion, which is used in the theory of partial differential equations. It includes a wide range from fundamental solutions to the study of the dynamics in particle physics to Fourier analysis. To handle distributions, it is not necessary to know the full topological definition (the toppology will be introduced in section 12). It is enough being able to manipulate distributions, by this I mean knowing the basic rules (see 2.4), that is, the rule of taking a derivative of a distribution, and the rule of multiplying a distribution by a function.

Being concrete, the purpose is to define, for example, the differential equation

$$\mathrm{div} q = g \tag{1.1}$$

in an open set $\mathcal{U} \subset \mathbb{R}^N$ for quantities which are only $\mathcal{L}^1_{\mathrm{loc}}(\mathcal{U})$-functions. In order to do so we multiply the differential equation with a ***test function*** $\zeta \in \mathcal{C}^\infty_0(\mathcal{U})$ and obtain[1]

$$0 = \int_{\mathcal{U}} \zeta \cdot (-\,\mathrm{div} q + g)\,\mathrm{dL}^n = \int_{\mathcal{U}} (\nabla\zeta \bullet q + \zeta \cdot g)\,\mathrm{dL}^n$$

after integration by parts. The last integral exists, if the functions q_i und g are in $\mathcal{L}^1_{\mathrm{loc}}(\mathcal{U})$. Therefore we have the following two contributions

$$\zeta \mapsto \int_{\mathcal{U}} \nabla\zeta \bullet q\,\mathrm{dL}^n\,, \qquad \zeta \mapsto \int_{\mathcal{U}} \zeta \cdot g\,\mathrm{dL}^n\,,$$

which are linear in the test function ζ. Defining linear mappings

$$Q_i, G : \mathcal{C}^\infty_0(\mathcal{U}) \to \mathbb{R}\,,$$

i.e. ***distributions*** (see 2.2, we write $Q_i(\zeta) = \langle \boldsymbol{\zeta}\,, \boldsymbol{Q_i} \rangle$ and $G(\zeta) = \langle \boldsymbol{\zeta}\,, \boldsymbol{G} \rangle$),

$$\langle \boldsymbol{\zeta}\,, \boldsymbol{Q_i} \rangle := \int_{\mathcal{U}} \zeta \cdot q_i\,\mathrm{dL}^n\,, \qquad \langle \boldsymbol{\zeta}\,, \boldsymbol{G} \rangle := \int_{\mathcal{U}} \zeta \cdot g\,\mathrm{dL}^n\,,$$

the differential equation is equivalent to

$$0 = \sum_i \langle \boldsymbol{\partial_i\zeta}\,, \boldsymbol{Q_i} \rangle + \langle \boldsymbol{\zeta}\,, \boldsymbol{G} \rangle\,.$$

Defining now linear mappings $\partial_i Q_i : \mathcal{C}^\infty_0(\mathcal{U}) \to \mathbb{R}$ by

$$\langle \boldsymbol{\zeta}\,, \boldsymbol{\partial_i Q_i} \rangle := \langle \boldsymbol{-\partial_i\zeta}\,, \boldsymbol{Q_i} \rangle$$

(it is $\partial_i Q_i(\zeta) = Q_i(-\partial_i\zeta)$), the differential equation becomes

$$\begin{aligned} 0 &= \sum_i \langle \boldsymbol{\partial_i\zeta}\,, \boldsymbol{Q_i} \rangle + \langle \boldsymbol{\zeta}\,, \boldsymbol{G} \rangle = -\sum_i \langle \boldsymbol{\zeta}\,, \boldsymbol{\partial_i Q_i} \rangle + \langle \boldsymbol{\zeta}\,, \boldsymbol{G} \rangle \\ &= \Big\langle \boldsymbol{\zeta}\,, -\sum_{\boldsymbol{i}} \boldsymbol{\partial_i Q_i} + \boldsymbol{G} \Big\rangle\,, \end{aligned}$$

[1] we denote by L^n the n-dimensional Lebesgue measure

that is, in the space of distributions

$$\sum_i \partial_i Q_i = G \quad \text{or} \quad \mathrm{div} Q = G\,.$$

This is just one example, which shows how useful the notion of distributions is.

The notion of distributions has a long history, see the doctoral theses of Peters [11], and there has also been an effort from aplications to introduce distributions, see Bedeaux [5]. In fact, the notion of distribution for the first time was introduced by physisists, and later this notion was dressed with a mathematical coat. Mathematically there are two different and equivalent methods to introduce distributions. One is to define a topology in the space $\mathscr{D}(\mathcal{U}) := \mathcal{C}_0^\infty(\mathcal{U})$ and then to define the set of distributions as the dual space $\mathscr{D}(\mathcal{U})^*$ (see section 12). The second method is to define the set of distributions $\mathscr{D}'(\mathcal{U})$ as the set of linear mappings on $\mathcal{C}_0^\infty(\mathcal{U})$ satisfying (see 2.2) an estimate

$$|\langle \boldsymbol{\zeta}, \boldsymbol{T} \rangle| \le C_U \|\zeta\|_{C^{k_U}(\overline{U})}$$

for all $\zeta \in C_0^\infty(U)$, $U \subset\subset \mathcal{U}$.

We follow this last method and we do not use in sections 2 – 11 the topological results of section 12. However, the two methods are equivalent (see 12.5).

2 Distributions

Here we define the main subject of this work, which are distributions on an open set $\mathcal{U} \subset \mathbb{R}^n$. We are interested in the set of linear mappings

$$\{\, T : C_0^\infty(\mathcal{U}; Y_1) \to Y_0 \;;\; T \text{ is linear} \,\}, \tag{2.1}$$

where Y_0 and Y_1 are two Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

2.1 Remark (Vector valued cases). We let here $Y_0 = \mathbb{K}$ and $Y_1 = Y$, where Y is a Banach space. That is, we focus on $\mathbb{K}$-valued linear mappings. The other case $Y_0 = Y$ and $Y_1 = \mathbb{K}$ you find, for example, in [9], see also section 5. The scalar case $Y_0 = \mathbb{K}$ and $Y_1 = \mathbb{K}$ is the usual one in literature.

You can take, for the first reading, the case that these Banach spaces are equal to $\mathbb{K}$, that is $Y_0 = \mathbb{K}$ and $Y_1 = Y = \mathbb{K}$, where you might consider the case $\mathbb{K} = \mathbb{R}$. Or you take the more advanced case $Y_0 = \mathbb{K}$ and $Y_1 = Y = \mathbb{K}^N$, where again you might consider the case $\mathbb{K} = \mathbb{R}$. In general we define distributions for $Y_0 = \mathbb{K}$ and $Y_1 = Y$ (where Y a Banach space, e.g. $Y = \mathbb{K}$ or $Y = \mathbb{K}^N$).

2.2 Definition of distributions. Let Y be a Banach spaces over $\mathbb{K}$ and denote the set $C_0^\infty(\mathcal{U}; Y)$ of ***test functions*** by

$$\mathscr{D}(\mathcal{U}; Y) := C_0^\infty(\mathcal{U}; Y).$$

Then the set of ***Distributions*** on $\mathcal{U}$ is

$$\mathscr{D}'(\mathcal{U}; Y) := \{\, T : \mathscr{D}(\mathcal{U}; Y) \to \mathbb{K} \;;\; T \text{ is linear and satisfies (2.3)} \,\}.$$

Here the estimate is

$$\begin{aligned} &\forall\, U \subset\subset \mathcal{U} :\ \exists\, C_U \geq 0,\ k_U \in \mathbb{N}_0 : \\ &\forall\, \zeta \in C_0^\infty(U; Y) :\ |\langle \boldsymbol{\zeta}, \boldsymbol{T} \rangle| \leq C_U \|\zeta\|_{C^{k_U}(\overline{U}; Y)} \end{aligned} \tag{2.2}$$

This is the definition of distributions. To try a definition in words: $T \in \mathscr{D}'(\mathcal{U}; Y)$ is a distribution if and only if T is a linear $\mathbb{K}$-valued map on the space of test functions $\mathscr{D}(\mathcal{U}; Y)$ such that for every set $U \subset\subset \mathcal{U}$ there is a constant $C_U \geq 0$ and an order $k_U \in \mathbb{N}_0$ with

$$|\langle \boldsymbol{\zeta}, \boldsymbol{T} \rangle| \leq C_U \|\zeta\|_{C^{k_U}(\overline{U}; Y)} \text{ for all } \zeta \in C_0^\infty(U; Y). \tag{2.3}$$

This is the same definition of a distribution. One writes $\mathscr{D}(\mathcal{U}) := \mathscr{D}(\mathcal{U}; \mathbb{K})$ and $\mathscr{D}'(\mathcal{U}) := \mathscr{D}'(\mathcal{U}; \mathbb{K})$. (We mention that in formulas we use $\mathscr{D}(\mathcal{U})$ instead of $\mathscr{D}(\mathcal{U}; Y)$ in order to make things shorter.)
Notation: It is $\langle \boldsymbol{\zeta}, \boldsymbol{T} \rangle := \langle \boldsymbol{\zeta}, \boldsymbol{T} \rangle_{\mathscr{D}(\mathcal{U})} := T(\zeta)$. And $U \subset\subset \mathcal{U}$ says that $\overline{U}$ is a compact set contained in $\mathcal{U}$. The prime in $\mathscr{D}'(\mathcal{U}; Y)$ has, for the moment, no meaning, see section 12 for an interpretation.
Notice: The estimate in the definition holds only for a particular set of test functions ζ, namely that its support is contained in $U \subset\subset \mathcal{U}$, as it is said. But U is arbitrary so that altogether all smooth functions ζ with compact support occur in the definition.

Later in section 12 we will give an equivalent definition with the help of a topoloy, therefore this is the topological definition. But here we rely on the definition using (2.3) (or equivalently (2.3)). We have the following

2.3 Property. Let $(\zeta_m)_{m\in\mathbb{N}}$ be a sequence in $\mathscr{D}(\mathcal{U};Y)$ satisfying the following:

(1) There is an $U \subset\subset \mathcal{U}$ with $\operatorname{supp}\zeta_m \subset U$ for all $m \in \mathbb{N}$.

(2) For every $k \in \mathbb{N}$ with this U it holds $\|\zeta_m\|_{C^k(\overline{U};Y)} \to 0$ as $m \to \infty$.

Then for any distribution $T \in \mathscr{D}'(\mathcal{U};Y)$

$$\langle \zeta_m , T \rangle \to 0 \text{ as } m \to \infty.$$

Notice: With the topology in section 12 the assumption on the sequence, that is (1) and (2), reads (see 12.4)

$$\zeta_m \to 0 \text{ in } \mathscr{D}(\mathcal{U};Y) \text{ as } m \to \infty\,. \tag{2.4}$$

Then the statement $\langle \zeta_m , T \rangle \to 0$ as $m \to \infty$, if it is true for every such sequence $(\zeta_m)_{m\in\mathbb{N}}$, means that T is sequentially continuous. But here we did not introduce a topology yet, for a topology see section 12. Consequently, here no convergence in the space $\mathscr{D}(\mathcal{U};Y)$ is defined. We mention that in the literature it is often said that (2.4) holds, if the properties (1) and (2) are satisfied.

Proof. This follows immediately from the inequality (2.3). □

The main property of distributions is that derivatives of an arbitrary order again defines a distribution. Therefore if T is a distribution then also $\partial^\alpha T$ (see (2.5)) is a distrubution.

2.4 Derivative and multiplication. Let $T \in \mathscr{D}'(\mathcal{U};Y)$, that is, T is a distribution over $\mathcal{U}$.

(1) For $j = 1,\dots,N$ is $\partial_j T \in \mathscr{D}'(\mathcal{U};Y)$ defined by

$$\langle \zeta , \partial_j T \rangle_{\mathscr{D}(\mathcal{U})} := \langle -\partial_j \zeta , T \rangle_{\mathscr{D}(\mathcal{U})}\,.$$

Note: Correspondingly we define higher derivatives, see (2.5).

(2) Let a function $a \in C^\infty_{\mathrm{loc}}(\mathcal{U};\mathbb{K})$ be given. Then $aT \in \mathscr{D}'(\mathcal{U};Y)$ is defined by

$$\langle \zeta , aT \rangle_{\mathscr{D}(\mathcal{U})} := \langle a\zeta , T \rangle_{\mathscr{D}(\mathcal{U})}\,.$$

Proof (1). It is with (2.3)

$$\big|\langle \zeta , \partial_j T \rangle\big| = \big|\langle \partial_j\zeta , T \rangle\big| \le C_U \|\partial_j \zeta\|_{C^{k_U}(\overline{U})} \le C_U \|\zeta\|_{C^{k_U+1}(\overline{U})}\,.$$

Hence $\partial_j T$ is a distribution. □

Proof (2). It is with (2.3)

$$\begin{aligned} |\langle \zeta , aT \rangle| = |\langle a\zeta , T \rangle| &\le C_U \|a\zeta\|_{C^{k_U}(\overline{U})} \\ &\le C_U C_{n,k_U} \|a\|_{C^{k_U}(\overline{U})} \cdot \|\zeta\|_{C^{k_U}(\overline{U})}\,. \end{aligned}$$

Remark: It is used that $\|uv\|_{C^k(\overline{U})} \le C_{n,k} \|u\|_{C^k(\overline{U})} \|v\|_{C^k(\overline{U})}$. □

The derivatives are commutative, that is, $\partial_i\partial_j T = \partial_j\partial_i T$. This follows from the fact that $\partial_i\partial_j\zeta = \partial_j\partial_i\zeta$ for test functions, which are C^∞ (so at least C^2). Therefore this implies that for ***multiindices*** $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0$ we are able to define

$$\partial^\alpha T := \partial_{i_1} \cdots \partial_{i_k} T \text{ if } \alpha = \textstyle\sum_{l=1}^k \mathbf{e}_{i_l} \,. \tag{2.5}$$

The essential examples of distributions are given by measures and functions.

2.5 Measures and functions as distributions.

(1) Let μ be a measure on $\mathcal{U}$ such that measurable sets w.r.t. μ are induced by Borel sets. Then C_0^0-functions are integrable and $[\mu] \in \mathscr{D}'(\mathcal{U})$ is defined for test functions ζ by

$$\langle \zeta , [\mu] \rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \zeta \, \mathrm{d}\mu \,.$$

(2) Let $g \in L^1_{\mathrm{loc}}(\mathcal{U})$. Then $[g] \in \mathscr{D}'(\mathcal{U})$ is defined for test functions ζ by

$$\langle \zeta , [g] \rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \zeta \cdot g \, \mathrm{dL}^n \,.$$

Remark: It is $[g] = [g\mathrm{L}^n]$. *Information:* The notation $[\mu]$ and $[f]$ usually one does not find in literature. We consequently use this notation in order to avoid misunderstandings. In [9] the notation $<\mu>$ is used.

Proof (1). For $\operatorname{supp}\zeta \subset U$

$$\Big| \langle \zeta , [\mu] \rangle_{\mathscr{D}(\mathcal{U})} \Big| = \Big| \int_{\mathcal{U}} \zeta \, \mathrm{d}\mu \Big| \le \mu(\overline{U}) \|\zeta\|_{C^0(\overline{U})} \,.$$

□

A family of particular measures are considered in section 6. The simplest version of a measure one can think of is “Dirac’s measure”, sometimes called “Dirac function” although it does not exist as a function.

2.6 Dirac distribution and Heaviside step function.

(1) If $x_0 \in \mathbb{R}^n$, then $\boldsymbol{\delta}_{x_0} \in \mathscr{D}'(\mathbb{R}^n)$ defined by

$$\langle \zeta , \boldsymbol{\delta}_{x_0} \rangle := \zeta(x_0)$$

is called ***Dirac distribution*** at x_0.

(2) We let $n = 1$ and set

$$h(x) = \begin{cases} 0 \text{ for } x < 0 \,, \\ \text{any fixed value for } x = 0 \,, \\ 1 \text{ for } x > 0 \,. \end{cases}$$

The function h is called ***Heaviside function***. It follows

$$[h]' = \boldsymbol{\delta}_0 \text{ in } \mathscr{D}'(\mathbb{R}) \,.$$

Remark: Here the prime in $[h]'$ denotes the first derivative.

(3) We take the Heaviside function in n-dimensional space

$$h(x) = \begin{cases} 0 \text{ for } x \bullet \mathbf{e}_1 < 0\,, \\ \text{any value for } x \bullet \mathbf{e}_1 = 0\,, \\ 1 \text{ for } x \bullet \mathbf{e}_1 > 0\,. \end{cases}$$

Then, if $\Gamma = \{x\,;\ x \bullet \mathbf{e}_1 = 0\}$, in $\mathscr{D}'(\mathbb{R}^n)$

$$\partial_1[h] = [\mathrm{H}^{n-1} \llcorner \Gamma]\,, \quad \partial_i[h] = 0 \text{ for } i = 2, \dots, n.$$

Definition: If $\Gamma \subset \mathcal{U} \subset \mathbb{R}$ is a smooth m-dimensional surface, then the m-dimensional surface measure is denoted by $\mathrm{H}^m \llcorner \Gamma$. More general, if $\Gamma \subset \mathcal{U} \subset \mathbb{R}$ is a Borel set, e.g. a locally closed set, then $\mathrm{H}^m \llcorner \Gamma(E) := \mathrm{H}^m(\Gamma \cap E)$, where H^m is the m-dimensional Hausdorff measure. See 6.1.

Proof (2). It is

$$\begin{aligned} \langle \zeta\,, [h]' \rangle &= \langle -\zeta'\,, [h] \rangle = -\int_{\mathbb{R}} \zeta'(x) h(x)\,\mathrm{d}x \\ &= -\int_0^\infty \zeta'(x)\,\mathrm{d}x = \zeta(0) = \langle \zeta\,, \boldsymbol{\delta}_0 \rangle\,, \end{aligned}$$

that is, $[h]' = \boldsymbol{\delta}_0$. □

Proof (3). It is

$$\begin{aligned} \langle \zeta\,, \partial_1[h] \rangle &= \langle -\partial_1 \zeta\,, [h] \rangle = -\int_{\mathbb{R}^n} \partial_1 \zeta(x) h(x)\,\mathrm{d}x \\ &= -\int_{\{x\,;\ x \bullet \mathbf{e}_1 > 0\}} \partial_1 \zeta(x)\,\mathrm{d}\mathrm{L}^n(x) = \int_{\mathbb{R}^{n-1}} \zeta(y, 0)\,\mathrm{d}\mathrm{L}^{n-1}(y) \\ &= \int_\Gamma \zeta(x)\,\mathrm{d}\mathrm{H}^{n-1}(x) = \int_{\mathbb{R}^n} \zeta(x)\,\mathrm{d}(\mathrm{H}^{n-1} \llcorner \Gamma)(x) = \langle \zeta\,, [\mathrm{H}^{n-1} \llcorner \Gamma] \rangle\,, \end{aligned}$$

that is, $\partial_1[h] = [\mathrm{H}^{n-1} \llcorner \Gamma]$, and for $i \geq 2$

$$\begin{aligned} \langle \zeta\,, \partial_i[h] \rangle &= \langle -\partial_i \zeta\,, [h] \rangle = -\int_{\mathbb{R}^n} \partial_i \zeta(x) h(x)\,\mathrm{d}x \\ &= -\int_{\{x\,;\ x \bullet \mathbf{e}_1 > 0\}} \partial_i \zeta(x)\,\mathrm{d}\mathrm{L}^n(x) = 0\,, \end{aligned}$$

that is, $\partial_i[h] = 0$, □

The Heaviside function is a fundamental solution of an ODE (see 7.9(1)). Another basic function is the fundamental solution of the Laplace equation (see 8.3). Fundamental solutions are defined in section 7 as distributions which is one of the main applications.

3 Convolution

The convolution is usually defined between to $\mathcal{L}^1_{loc}(\mathbb{R}^n)$-functions, where one of these functions has to have compact support. The definition then is

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y\,. \tag{3.1}$$

If one drops the assumption on the support this identity is still valid for almost all x if $f, g \in \mathcal{L}^1(\mathbb{R}^n)$, and then $f * g \in \mathcal{L}^1(\mathbb{R}^n)$ with the estimate

$$\|f * g\|_{\mathcal{L}^1(\mathbb{R}^n)} \le \|f\|_{\mathcal{L}^1(\mathbb{R}^n)} \cdot \|g\|_{\mathcal{L}^1(\mathbb{R}^n)}\,. \tag{3.2}$$

We consider here the convolution of a distribution with a smooth function. By this way we approximate arbitrary distributions by smooth functions, and this clarifies the question what the entirety of distributions $\mathscr{D}'(\mathcal{U})$ is.

3.1 Convolution of a distribution. Let $T \in \mathscr{D}'(\mathcal{U})$ and $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ and define [2]

$$U_\varphi := \{x \in \mathcal{U}\,;\ \operatorname{supp}\varphi(x - \sqcup) \subset \mathcal{U}\}\,.$$

Then U_φ is an open set and

$$(\varphi * T)(x) := \langle\, \varphi(x-\sqcup)\,, T\,\rangle \text{ for } x \in U_\varphi \tag{3.3}$$

is well defined. It holds that:

(1) For $T = [f]$ with $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{U})$ it follows that

$$(\varphi * [f])(x) = (\varphi * f)(x) \text{ if } x \in U_\varphi\,.$$

(2) It is $\varphi * T \in \mathcal{C}^\infty(U_\varphi)$ with derivatives $\partial^\alpha(\varphi * T) = (\partial^\alpha\varphi) * T = \varphi * \partial^\alpha T$.

Proof (1). It holds that

$$(\varphi * [f])(x) = [f](\varphi(x-\sqcup)) = \int_{\mathcal{U}} \varphi(x-y)f(y)\,\mathrm{d}y = (\varphi * f)(x)\,,$$

since $\operatorname{supp}(\varphi(x-\sqcup)) \subset \mathcal{U}$ (formally set $f = 0$ in the exterior of $\mathcal{U}$). ☐

Proof (2). Let k_U be chosen for T and U as in (2.3). On introducing the difference quotients $\partial^h_i\psi(x) := \frac{1}{h}(\psi(x+h\mathbf{e}_i) - \psi(x))$, the linearity of T yields that

$$\partial^h_i(\varphi * T)(x) = \frac{1}{h}(\langle\, \varphi(x+h\mathbf{e}_i-\sqcup)\,, T\,\rangle - \langle\, \varphi(x-\sqcup)\,, T\,\rangle)$$
$$= \left\langle\, \frac{1}{h}(\varphi(x+h\mathbf{e}_i-\sqcup) - \varphi(x-\sqcup))\,, T\,\right\rangle = \langle\, \partial^h_i\varphi(x-\sqcup)\,, T\,\rangle\,.$$

As $h \to 0$ we have that $\partial^h_i\varphi(x-\sqcup) \to \partial_i\varphi(x-\sqcup)$ in $\mathcal{C}^{k_U}(\overline{U})$, and hence it follows from (2.3) for T that

[2] The blank $\sqcup$ denotes an empty position (that is a hole) for the argument of the mapping.

$$\langle\, \partial_i^h\varphi(x-\sqcup)\,,T\,\rangle \longrightarrow \langle\, \partial_i\varphi(x-\sqcup)\,,T\,\rangle = \big((\partial_i\varphi)*T\big)(x)\,.$$

This shows that the partial derivative $\partial_i(\varphi*T)(x) = \big((\partial_i\varphi)*T\big)(x)$ exists. The desired result for higher derivatives now follows by induction on the order of the derivative. □

3.2 Approximation of distributions. Let $T \in \mathscr{D}'(\mathcal{U})$ and $U \subset\subset \mathcal{U}$ and let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard Dirac sequence. For small ε we have that $\varphi_\varepsilon * T \in \mathcal{C}^\infty(U)$ and for all $\zeta \in \mathcal{C}_0^\infty(U)$

$$\langle\, \zeta\,,[\varphi_\varepsilon*T]\,\rangle \longrightarrow \langle\, \zeta\,,T\,\rangle \quad \text{as } \varepsilon\to 0.$$

Proof. We have that

$$\langle\, \zeta\,,[\varphi_\varepsilon*T]\,\rangle = \int_{\mathcal{U}} \zeta(x)\, \underbrace{(\varphi_\varepsilon*T)(x)}_{=T(\varphi_\varepsilon(x-\sqcup))}\, \mathrm{d}x\,.$$

Now it holds that (the proof is given below)

$$\int_{\mathcal{U}} \zeta(x) T\big(\varphi_\varepsilon(x-\sqcup)\big)\,\mathrm{d}x = \Big\langle \int_{\mathcal{U}} \zeta(x)\varphi_\varepsilon(x-\sqcup)\,\mathrm{d}x\,,T \Big\rangle. \tag{3.4}$$

The argument of T on the right-hand side is $\zeta_\varepsilon(\sqcup)$, if $\zeta_\varepsilon := \varphi_\varepsilon^- * \zeta$ with $\varphi_\varepsilon^-(y) := \varphi_\varepsilon(-y)$. If k_U for T and U is chosen as in (2.3), it follows that $\zeta_\varepsilon \to \zeta$ in $\mathcal{C}^{k_U}(\overline{U})$ as $\varepsilon \to 0$, hence $\langle\, \zeta_\varepsilon\,,T\,\rangle \to \langle\, \zeta\,,T\,\rangle$, and so we have shown that

$$[\varphi_\varepsilon * T](\zeta) = \langle\, \zeta_\varepsilon\,,T\,\rangle \to \langle\, \zeta\,,T\,\rangle \quad \text{as } \varepsilon\to 0.$$

The proof of identity (3.4): Approximate ζ uniformly by step functions ζ_j with a common compact support in U. Then (3.4) holds for ζ_j because of the linearity of T. The left-hand side converges as $j\to\infty$, since $x \mapsto T(\varphi_\varepsilon(x-\sqcup))$ is continuous, recall 3.1(2). The right-hand side converges using the same argument as above, since $\varphi_\varepsilon^- * \zeta_j \to \varphi_\varepsilon^- * \zeta$ in $\mathcal{C}^{k_U}(\overline{U})$. □

Usually one finds in the literature the convolution of two of distributions, see ????????. Since we do not use this general definition, it is not included here.

4 Other function spaces

Es sei $\mathcal{U}$ beschränkt und W ein normierter Funktionenraum mit

$$\text{Norm } v \mapsto \|v\|_W\,, \quad \text{clos}_W\,(\mathcal{C}_0^\infty(\mathcal{U})) = W \tag{4.1}$$

Dabe bedeuted $\text{clos}_W\,(\mathcal{C}_0^\infty(\mathcal{U}))$ den Abschluss bezüglich der Norm von W. Sei nun $T : \mathcal{C}_0^\infty(\mathcal{U}) \to \mathbb{R}$ eine lineare Abbildung und es gelte für $\zeta \in \mathcal{C}_0^\infty(\mathcal{U})$

$$|T(\zeta)| \le C_T\|\zeta\|_W \le C\|\zeta\|_{\mathcal{C}^k(\overline{\mathcal{U}})} \text{ für ein } C \text{ und } k \in \mathbb{N}\cup\{0\}. \tag{4.2}$$

Es sei bemerkt, dass es sich hier im zwei Ungleichungen handelt, die erste ist eine Eigenschaft von T und die zweite eine Eigenschaft von W.

4.1 Theorem. Es gelte (4.1) und (4.2). Dann ist $T \in \mathscr{D}'(\mathcal{U})$ eine Distribution und T lässt sich auf eindeutige Weise stetig auf W fortsetzen (die Fortsetzung wird wieder mit T bezeichnet) mit

$$|T(w)| \le C_T\|w\|_W \text{ für } w \in W\,,$$

also ist $T \in W'$, wobei W' der Dualraum von W ist. Wir schreiben dann auch

$$\langle w\,, T\rangle_W := T(w) \text{ für } w \in W.$$

Hinweis: Es ist $C_T = \|T\|$ die Operatornorm von T auf W.

Proof. Wegen der ersten Ungleichung von (4.2) ist T auf $\mathcal{C}_0^\infty(\mathcal{U}) \subset W$ mit der Norm von W stetig, also da linear auch gleichmäßig stetig. Also lässt sich T auf eindeutige Weise unter Beibehaltung der Abschätzung stetig auf W fortsetzen und zwar (benutze (4.1)) eindeutig. Die zweite Ungleichung von (4.2) besagt, dass T eine Distribution ist. □

Dieser Satz lässt sich anwenden auf Distributionen, welche eine Abschätzung wie in (4.2) erfüllen. Wir wenden dies nun zum Beispiel an auf Maße μ und Funktionen g für die $[\mu] \in \mathscr{D}'(\mathcal{U})$ und $[g] \in \mathscr{D}'(\mathcal{U})$ in 2.5 definiert waren.

4.2 Maße als Funktionale auf $\mathcal{C}^0(\mathcal{U})$. Sei $\mu \ge 0$ eine Maß, für das $\mathcal{C}_0^0$-Funktionen integrierbar sind und für das $\mu(\mathcal{U}) < \infty$. Definiere $W := \mathcal{C}^0(\mathcal{U})$ .... Für eine Maß μ für das $\mathcal{C}_0^0$-Funktionen integrierbar sind und $\mu(\mathcal{U}) < \infty$ ist sei $W := \{\zeta \in \mathcal{C}^0(\mathcal{U})\,;\ \zeta = 0 \text{ auf } \partial\mathcal{U}\}$, so dass für $\zeta \in \mathcal{C}_0^\infty(\mathcal{U})$

$$|\langle \zeta\,, \mu\rangle_W| = \left|\int_{\mathcal{U}} \zeta\,\mathrm{d}\mu\right| \le \mu(\mathcal{U})\|\zeta\|_W = \mu(\mathcal{U})\|\zeta\|_{C^0(\overline{\mathcal{U}})}\,,$$

also eine Abschätzung (4.2) mit $k = 0$.

4.3 $\mathcal{L}^p(\mathcal{U})$-Funktionen. Sei $g \in \mathcal{L}^p(\mathcal{U})$ mit $\mathrm{L}^n(\mathcal{U}) < \infty$. Dann gilt für den Raum $W := \mathcal{L}^{p'}(\mathcal{U})$ und $\zeta \in \mathcal{C}_0^\infty(\mathcal{U})$

$$|\langle \zeta\,, g\rangle_W| = \left|\int_{\mathcal{U}} \zeta g\,\mathrm{dL}^n\right| \le \|g\|_{\mathcal{L}^p(\mathcal{U})}\|\zeta\|_W \le \|g\|_{\mathcal{L}^p(\mathcal{U})}\|1\|_W\|\zeta\|_{C^0(\overline{\mathcal{U}})}\,,$$

also eine Abschätzung (4.2) mit $k = 0$. Beachte, dass nur für $p > 1$ (also $p' < \infty$) aus $T \in W'$ folgt, dass $T = [g]$ für ein $g \in \mathcal{L}^p(\mathcal{U})$.

Die Sobolev-Funktionen in $W^{m,p}(\mathcal{U})$ und Funktionen in $BV(\mathcal{U})$ sind weitere Beispiele.
kommt später

5 Other Definitions

We now give some generalizations of the notion of distribution. It is based on the general definition (2.1) (see also 2.1).

5.1 Generalization. We consider, see (2.1),

$$\{\, T : C_0^\infty(\mathcal{U}; Y_1) \to Y_0 \ ; \ T \text{ is linear } \}$$

where T satisfies (2.3) with appropriate norms. Besides Banach spaces Y_0 and Y_1 we assume that a Banach space Y_2 is given. Furthermore, let a bilinear (if $\mathbb{K} = \mathbb{R}$) map

$$b : Y_1 \times Y_2 \to Y_0$$

be given satisfying the property of a ***Banach product***

$$\|b(y_1, y_2)\|_{Y_0} \le \|y_1\|_{Y_1} \cdot \|y_2\|_{Y_2} .$$

We consider a map

$$f \in L^1_{loc}(\mathcal{U}; Y_2) \ \longmapsto \ [f] : C_0^\infty(\mathcal{U}; Y_1) \to Y_0$$

given by

$$\langle \zeta , [f] \rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} b(\zeta(x), f(x))\, \mathrm{d}x \in Y_0.$$

We have several choices for the map b:

- b scalar multiplication $(Y_0, Y_1, Y_2) = (Y, \mathbb{K}, Y)$ and $(y_1, y_2) \mapsto b(y_1, y_2) := y_1 y_2$,
- b dual product $(Y_0, Y_1, Y_2) = (\mathbb{K}, Y, Y^*)$ and $(y_1, y_2) \mapsto b(y_1, y_2) := \langle y_1 , y_2 \rangle_Y$,
- b inner product $(Y_0, Y_1, Y_2) = (\mathbb{K}, Y, Y)$ and $(y_1, y_2) \mapsto b(y_1, y_2) := (\, y_1 \, , \, y_2 \,)_Y$.

If Y_0 is different from $\mathbb{K}$, then the proof of 11.1 does not apply, but it can be generalized to certain cases.

The “inner product” has also the names “scalar product” or “dot product”:

5.2 Beispiel mit Skalarmultiplikation. Sei Y ein beliebiger Banachraum und setze

$$(Y_0, Y_1, Y_2) = (Y, \mathbb{K}, Y).$$

Es ist dann

$$f \in L^1_{loc}(\mathcal{U}; Y) \longmapsto [f] : \mathscr{D}(\mathcal{U}; \mathbb{R}) \to Y$$

definiert und zwar für $\zeta \in \mathscr{D}(\mathcal{U}; \mathbb{R})$ durch

$$\langle \zeta , [f] \rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \zeta(x) f(x)\, \mathrm{d}x \ \in Y.$$

5.3 Beispiel mit dualem Produkt. Sei Y ein beliebiger Banachraum und setze für das Tripel $(Y_0, Y_1, Y_2) = (\mathbb{K}, Y, Y^*)$. Es ist dann

$$f \in L^1_{loc}(\mathcal{U}; Y^*) \longmapsto [f] \in \mathscr{D}'(\mathcal{U}; Y)$$

definiert und zwar für $\zeta \in \mathscr{D}(\mathcal{U}; Y)$ durch

$$\langle\, \zeta\,, [f]\,\rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} \langle\, \zeta(x)\,, f(x)\,\rangle_Y \,\mathrm{d}x \;\in \mathbb{K}.$$

Definition: Das ***duale Produkt*** ist definiert durch $(y, y') \mapsto \langle\, y\,, y'\,\rangle_Y := y'(y)$.

5.4 Beispiel mit Skalarprodukt. Nun sei Y ein Hilbertraum und setze für das Tripel $(Y_0, Y_1, Y_2) = (\mathbb{K}, Y, Y)$. Es ist dann

$$f \in L^1_{loc}(\mathcal{U}; Y) \longmapsto [f] \in \mathscr{D}'(\mathcal{U}; Y)$$

definiert und zwar für $\zeta \in \mathscr{D}(\mathcal{U}; Y)$ durch

$$\langle\, \zeta\,, [f]\,\rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U}} (\, \zeta(x)\,,\, f(x)\,)_Y \,\mathrm{d}x \;\in \mathbb{K}.$$

Definition: Das ***Hilbertraumprodukt*** ist definiert durch $(y_1, y_2) \mapsto (\, y_1\,,\, y_2\,)_Y$.

6 Surfaces

Here we define special distributions on a given surface. It means that the functions are defined on a manifold M, which is a submanifold $M \subset \mathcal{U}$ without boundary, where $\mathcal{U} \subset \mathbb{R}^n$ is an open set, the local test set.

6.1 Smooth surfaces. Consider a regular d-dimensional set $M \subset \mathcal{U}$ (for our purpose a $\mathcal{C}^2$-surface without boundary), where $0 \leq d \leq n$ is an integer. We denote its tangent space in x by $T_x(M)$. The measure on M is the d-dimensional Hausdorff measure [3]

$$\mathrm{H}^d \llcorner M$$

(which for $\mathcal{C}^2$ surfaces is the same as the usual surface measure). To this measure there exists a distribution $\boldsymbol{\mu}_M$ given by (see 2.5(1))

$$\langle \boldsymbol{\zeta}, \boldsymbol{\mu_M} \rangle := \int_M \zeta(y)\, \mathrm{dH}^d(y) \text{ for } \zeta \in \mathscr{D}(\mathcal{U})\,.$$

Proof. We have to show that $\boldsymbol{\mu}_M$ is a distribution in $\mathscr{D}'(\mathcal{U})$. For an open set $U \subset\subset \mathcal{U}$ and $\operatorname{supp}\zeta \subset U$ we compute

$$|\langle \boldsymbol{\zeta}, \boldsymbol{\mu_M} \rangle| = \left| \int_M \zeta \,\mathrm{dH}^d \right| \leq \int_M |\zeta| \,\mathrm{dH}^d \leq \|\zeta\|_{\mathcal{C}^0(\overline{U})} \cdot \mathrm{H}^d(M \cap U)\,,$$

i.e. $k_U = 0$ and $C_U = \mathrm{H}^d(M \cap U)$ in the definition 2.2. □

Hence the measure on surfaces is a distribution of order 0. This means that these distributions are defined for $\mathcal{C}^0$-functions, and first derivatives of it for $\mathcal{C}^1$-functions.

6.2 Definition (of function spaces). We say $g \in \mathcal{C}^1(M)$, if locally $g \circ \chi \in \mathcal{C}^1(U_\chi)$, where $\chi : U_\chi \subset \mathbb{R}^d \to M$ is a local parametrization of M. We say $g \in \mathcal{L}^1_{loc}(M)$, if $g \in \mathcal{L}^1_{loc}(\mathrm{H}^d \llcorner M)$. *Remark:* $\mathcal{L}^1_{loc}(\mu)$ for measures μ is the original space.

6.3 Lemma. If $g \in \mathcal{L}^1_{loc}(M)$ then $g\boldsymbol{\mu}_M$ is a distribution, i.e. $g\boldsymbol{\mu}_M \in \mathscr{D}'(\mathcal{U})$.

Proof. For $U \subset\subset \mathcal{U}$ and $\operatorname{supp}\zeta \subset U$ we now compute

$$|\langle \boldsymbol{\zeta}, \boldsymbol{g\mu_M} \rangle| = \left| \int_M \zeta g \,\mathrm{dH}^d \right| \leq \|\zeta\|_{\mathcal{C}^0(\overline{U})} \int_{\overline{U}} |g| \,\mathrm{dH}^d\,,$$

i.e. $k_U = 0$ and

$$C_U = \int_{\overline{U}} |g| \,\mathrm{dH}^d.$$

□

We now consider a differential equation

$$\mathrm{div} Q = G \text{ in } \mathscr{D}'(\mathcal{U}) \tag{6.1}$$

[3] we denote by H^d the d-dimensional Hausdorff measure in $\mathbb{R}^n$

with given quantities

$$Q = q\boldsymbol{\mu}_M \quad G = g\boldsymbol{\mu}_M \,,$$

where $q_i, g \in \mathcal{L}^1_{loc}(M)$. By 6.3, Q_i and G are distributions (of order 0). Equation (6.1) is a distributional equation. We want to know how the strong version of this distributional equation reads. For this strong version we need the definitions of differential operators on M.

6.4 Definition. We define the following derivatives with respect to M. For this $\{\tau_1(x), \dots, \tau_d(x)\}$ is an orthonormal system of the tangent space $T_x(M)$ in x.

(1) ∂_τ is the directional derivative in tangential direction τ.

(2) $\nabla^M g := \sum_{k=1}^d (\partial_{\tau_k} g)\tau_k$.

(3) $\operatorname{div}^M q := \sum_{k=1}^d \tau_k \bullet \partial_{\tau_k} q$.

(4) $\kappa^M := \sum_{k=1}^d \partial_{\tau_k}\tau_k$ is a normel field, called the ***curvature*** of M.

These definitions are independent of the choice of $\{\tau_1(x), \dots, \tau_d(x)\}$.

We prove the following

6.5 Theorem. Let $q_i, g \in \mathcal{C}^1(M)$, $i = 1, \dots, n$. Then the following is equivalent:

(1) *Weak formulation:*

$$\operatorname{div}(q\boldsymbol{\mu}_M) = (\text{resp. } \leq)\ g\boldsymbol{\mu}_M \text{ in } \mathscr{D}'(\mathcal{U}).$$

(2) *Strong formulation:*

$$q \in T(M) \text{ and } \operatorname{div}^M q = (\text{resp. } \leq)\ g \text{ on } M.$$

Remark: This also applies under weaker assumptions, see 6.6. *Definition:* The set $T(M)$ consists of those functions q on M which satisfy $q(x) \in T_x(M)$ for all x.

The fact that q is a tangent vector field, is a consequence of the distributional differential equation 6.5(1).

Proof (2)⇒(1). The differential equation implies for nonnegative local test functions $\zeta \in \mathscr{D}(\mathcal{U})$, $\zeta \geq 0$,

$$\begin{aligned} 0 &\leq \int_M \zeta \cdot \big(-\operatorname{div}^M q + g\big)\, \mathrm{dH}^d \\ &= -\int_M \operatorname{div}^M(\zeta q)\, \mathrm{dH}^d + \int_M \Big((\nabla^M \zeta)\bullet q + \zeta \cdot g\Big)\, \mathrm{dH}^d \,. \end{aligned}$$

Since q, by (2), is a tangential vector field, the first integral is 0 by integration by parts on M, and the second integral equals

$$\begin{aligned} &= \int_M \Big(\nabla\zeta \bullet q + \zeta \cdot g\Big)\, \mathrm{dH}^d \\ &= \langle \nabla\zeta \,, q\boldsymbol{\mu}_M \rangle + \langle \zeta \,, g\boldsymbol{\mu}_M \rangle = \langle \zeta \,, -\operatorname{div}(q\boldsymbol{\mu}_M) + g\boldsymbol{\mu}_M \rangle \,. \end{aligned}$$

This is (1), that is $-\operatorname{div}(q\boldsymbol{\mu}_M) + g\boldsymbol{\mu}_M \geq 0$. □

Proof (1)⇒(2). The distributional inequality says

$$0 \leq \int_M (\nabla\zeta \bullet q + \zeta \cdot f)\, \mathrm{dH}^d \tag{6.2}$$

for all nonnegative $\zeta \in \mathscr{D}(\mathcal{U})$. This then also holds for all $\mathcal{C}^1$-functions ζ with compact support in M. Set

$$\zeta = \eta \cdot (1 + \sin(a\psi)) \geq 0\,,$$

where $\eta \geq 0$ is a nonnegative testfunction, $a \in \mathbb{R}$, and ψ is any $\mathcal{C}^1$-function vanishing on M. Then

$$\nabla\zeta = (1 + \sin(a\psi))\nabla\eta + \eta\cos(a\psi)a\nabla\psi\,.$$

Since $\psi = 0$ on M, this is equal to

$$\nabla\zeta = \nabla\eta + \eta a \nabla\psi \quad \text{ on } M\,.$$

Therefore (6.2) implies

$$\begin{aligned} 0 &\leq \int_M (\nabla\zeta \bullet q + \zeta \cdot f)\, \mathrm{dH}^d \\ &= \int_M (\nabla\eta \bullet q + \eta \cdot f)\, \mathrm{dH}^d + a \int_M \eta \nabla\psi \bullet q\, \mathrm{dH}^d\,. \end{aligned}$$

Since a is an arbitrary number, it follows that the additional a-term has to vanish, that is

$$\int_M \eta \nabla\psi \bullet q\, \mathrm{dH}^d = 0\,.$$

Since this is true for all nonnegative test functions η, we conclude that

$$\nabla\psi \bullet q = 0 \quad \text{ on } M.$$

One can choose ψ so that $\mathbf{n} = \nabla\psi$ on M with a local $\mathcal{C}^1$ normal field $\mathbf{n}$ (not necessary a unit normal), hence $\mathbf{n}(x) \in T_x(M)^\perp$ at $x \in M$. Doing so one concludes that

$$q(x) \in T_x(M).$$

With this property (6.2) becomes for all nonnegative test functions ζ and with a tangential vector field q

$$\begin{aligned} 0 &\leq \int_M (\nabla\zeta \bullet q + \zeta \cdot f)\, \mathrm{dH}^d = \int_M (\nabla^M\zeta \bullet q + \zeta \cdot f)\, \mathrm{dH}^d \\ &= \underbrace{\int_M \mathrm{div}^M(\zeta q)\, \mathrm{dH}^d}_{=0} + \int_M \zeta(-\mathrm{div}^M q + f)\, \mathrm{dH}^d\,. \end{aligned}$$

Here we have used integration by parts on M. Since the nonnegative test function ζ is arbitrary, we conclude

$$\mathrm{div}^M q \leq f \text{ on } M. \tag{6.3}$$

This proves (2). □

6.6 Remark on Theorem. Let $q_i, g \in \mathcal{L}^1_{loc}(M)$. Then 6.5 is still true, if the equations in 6.5(2) are supposed to hold almost everywhere on M.

6.7 Appendix. The statement 6.5(1) holds only if the flux is a tangential vector field. For a general $q \in \mathcal{C}^1(M;\mathbb{R}^n)$ we have for $\zeta \in \mathcal{C}^1(M;\mathbb{R})$ that $f := \zeta q$ satisfies

$$\int_M \big(\operatorname{div}^M f + \kappa^M \bullet f\big)\,\mathrm{dH}^d = 0$$

Proof. We can assume that ζ is a local function and therefore we let $q = q_{tan} + r\mathbf{n}$ with a local unit normal vector $\mathbf{n}$. Then we compute

$$\begin{aligned}
&\nabla^M\zeta\bullet q = \nabla^M\zeta\bullet q_{tan} = \nabla\zeta\bullet q_{tan}\,, \\
&\operatorname{div}^M(r\mathbf{n}) + \kappa^M\bullet(r\mathbf{n}) = r(\operatorname{div}^M\mathbf{n} + \kappa^M\bullet\mathbf{n}) = 0\,,
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_M \big(\operatorname{div}^M(\zeta q) + \kappa^M\bullet(\zeta q)\big)\,\mathrm{dH}^d \\
&= \int_M \big(\nabla^M\zeta\bullet q + \zeta\big(\operatorname{div}^M q + \kappa^M\bullet q\big)\big)\,\mathrm{dH}^d \\
&= \int_M \big(\nabla\zeta\bullet q_{tan} + \zeta\big(\operatorname{div}^M q_{tan} + \operatorname{div}^M(r\mathbf{n}) + \kappa^M\bullet(r\mathbf{n})\big)\big)\,\mathrm{dH}^d \\
&= \int_M \big(\nabla\zeta\bullet q_{tan} + \zeta\operatorname{div}^M q_{tan}\big)\,\mathrm{dH}^d = 0
\end{aligned}$$

by the theorem 6.5. □

In Bearbeitung

7 Fundamental solutions

One of the main subjects in the field of distributions is the notion of a fundamental solution to differential operators with constant coefficients. These fundamental solutions are essential for the derivation of integral representations of solutions of the corresponding differential equation. For a given differential operator these fundamental solutions are functions (or distributions) with characteristic singularities.

Let us start with the definition of linear differential operators. We assume that $\mathcal{U} \subset \mathbb{R}^n$ is an open set and we remember that linear classical differential operators are functionals from $\mathcal{C}^m(\mathcal{U};\mathbb{R}^N)$ to $\mathcal{C}^0(\mathcal{U};\mathbb{R}^M)$:

7.1 Linear differential operators. Let N and M be integers. A (classical) ***linear differential operator*** of order $m \geq 0$ on $\mathcal{U}$ is a mapping

$$\begin{aligned} L\colon\ \mathcal{C}^m(\mathcal{U};\mathbb{R}^N) &\ \to\ \mathcal{C}^0(\mathcal{U};\mathbb{R}^M) \\ u &\ \mapsto\ L(u)\,, \end{aligned}$$

with the property that for $u \in \mathcal{C}^m(\Omega;\mathbb{R}^N)$ and $x \in \Omega$ the value $L(u)(x) \in \mathbb{R}^M$ is a linear combination of the partial deritives $\partial^\alpha u(x)$ for $|\alpha| \leq m$. Thus L has the following representation:

$$L(u)(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha u(x)\,.$$

Here $a_\alpha(x) \in \mathbb{R}^{M\times N}$ are $M \times N$-matrices. The international common short notation for this is

$$L(u) = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha u \quad \text{in } \mathcal{U}.$$

Assertion: The functions a_α are continuous and uniquely determined. This is a consequence of the above definition. *Definition:* The uniquely determined terms a_α are called ***coefficients*** of L. The operator L is called linear differential operator with ***constant coefficients*** (resp. $\mathcal{C}^\infty$-coefficients or analytic coefficienten, etc.), if the coefficients $x \mapsto a_\alpha(x)$ are independent of x (resp. infinitely often differentiable or real analytic, etc.).

Proof of continuity. Let $x_0 \in \mathcal{U}$. Consider for arbitrary multiindex the ***monom***

$$p_\beta : \mathbb{R}^n \to \mathbb{R} \text{ given by}$$
$$p_\beta(x) := \frac{(x-x_0)^\beta}{\beta!} := \prod_{i=1}^n \frac{(x-x_0)^{\beta_i}}{\beta_i!}\,. \tag{7.1}$$

Then the derivatives of p_β satify [4]

$$\partial^\alpha p_\beta(x) = \begin{cases} p_{\beta-\alpha}(x) & \text{if } \alpha \leq \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{7.2}$$

[4]Definition: $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for all i.

therefore $\partial^\alpha p_\beta(x_0) = \delta_{\alpha,\beta}$. Now let $|\beta| \leq m$. Without restriction it can be assumed that $M = N = 1$. Then [5]

$$L(p_\beta) = \sum_{\alpha\leq\beta} a_\alpha \partial^\alpha p_\beta = a_\beta + \sum_{\alpha<\beta} a_\alpha p_{\beta-\alpha}\,.$$

It is $\sum_{\alpha<\beta} a_\alpha p_{\beta-\alpha}$ continuous, if a_α are continuous for all $\alpha < \beta$. Because $L(p_\beta)$ is continuous, then it follows that a_β is continuous. Thus, the continuity of a_β is inductively shown. □

Proof of uniqueness. If $L = 0$, then it follows inductively in β that $a_\beta = 0$, like in the proof above. □

7.2 Scalar operators. Let L be as above. The operator L can be written as

$$L(u) = \left(\sum_{j=1}^{N} L_{ij}(u_j)\right)_{i=1,\ldots,M},$$

where $L_{ij} : \mathcal{C}^m(\Omega) \to \mathcal{C}^0(\Omega)$ are ***scalar differential operators*** with the representation

$$L_{ij}(v)(x) = \sum_{|\alpha|\leq m} (a_\alpha)_{ij}(x)\partial^\alpha v(x),$$

$$a_\alpha(x) = ((a_\alpha)_{ij}(x))_{i=1,\ldots,M;j=1,\ldots,N}\,.$$

7.3 Transposed operator. Let $L : \mathcal{C}^m(\mathcal{U};\mathbb{R}^N) \to \mathcal{C}^0(\mathcal{U};\mathbb{R}^M)$ be a classical linear differential operator as in 7.1 of order m

$$L(u) = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha u \quad \text{with } a_\alpha \in \mathcal{C}^m(\mathcal{U};\mathbb{R}^{M\times N})$$

Then there exists a unique $L^{\mathrm{T}} : \mathcal{C}^m(\mathcal{U};\mathbb{R}^M) \to \mathcal{C}^0(\mathcal{U};\mathbb{R}^N)$, a classical linear differential operator of order m with

$$\int_\Omega L^{\mathrm{T}}(v)\bullet u \,\mathrm{dL}^n = \int_\Omega v\bullet L(u)\,\mathrm{dL}^n$$

for all $u \in \mathcal{C}_0^m(\mathcal{U};\mathbb{R}^N)$ and all $v \in \mathcal{C}_0^m(\mathcal{U};\mathbb{R}^M)$. This differential operator is given by

$$L^{\mathrm{T}}(v) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial^\alpha({a_\alpha}^{\mathrm{T}} v).$$

We call L^{T} the ***transposed operator*** of L or the ***formally adjoint operator*** of L. It is $L^{\mathrm{TT}} = L$.

Proof. For all u, v as in the assertion it holds

$$\int_\mathcal{U} v\bullet L(u)\,\mathrm{dL}^n = \sum_{|\alpha|\leq m}\int_\mathcal{U} v\bullet(a_\alpha\partial^\alpha u)\,\mathrm{dL}^n$$

$$= \sum_{|\alpha|\leq m}\int_\mathcal{U} ({a_\alpha}^{\mathrm{T}} v)\bullet\partial^\alpha u\,\mathrm{dL}^n = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}\int_\mathcal{U} \partial^\alpha({a_\alpha}^{\mathrm{T}} v)\bullet u\,\mathrm{dL}^n$$

[5] Definition: $\alpha < \beta$ means that $\alpha \leq \beta$ and $\alpha \neq \beta$.

after partial integration. Let $M : \mathcal{C}^m(\mathcal{U};\mathbb{R}^M) \to \mathcal{C}^0(\mathcal{U};\mathbb{R}^N)$ be a linear differential operator with

$$\int_{\mathcal{U}} M(v)\bullet u \,\mathrm{dL}^n = \int_{\mathcal{U}} v\bullet L(u)\,\mathrm{dL}^n$$

For all u, v as in the assertion. Then

$$\int_{\mathcal{U}} M(v)\bullet u \,\mathrm{dL}^n = \int_{\mathcal{U}} \Big(\sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha({a_\alpha}^{\mathrm{T}} v)\Big)\bullet u \,\mathrm{dL}^n\,.$$

Da dies für alle Funktionen u gilt, folgt

$$M(v) = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha({a_\alpha}^{\mathrm{T}} v)\,.$$

Dass dies ein linearer Differentialoperator wie in 7.1 ist, folgt aus der ***Leibniz-Regel***

$$\partial^\alpha(vw) = \sum_{\beta\,:\,0\le\beta\le\alpha} \binom{\alpha}{\beta}\partial^{\alpha-\beta} v\cdot\partial^\beta w$$

for functions $v, w \in \mathcal{C}^m(\mathcal{U};\mathbb{R})$, where

$$\binom{\alpha}{\beta} := \prod_{i=1}^{n}\binom{\alpha_i}{\beta_i}.$$

Applying this Leibniz rule we obtain

$$\begin{aligned} M(v) &= \sum_{\alpha:|\alpha|\le m}\ \sum_{\beta\,:\,0\le\beta\le\alpha} (-1)^{|\alpha|}\binom{\alpha}{\beta}\,\partial^{\alpha-\beta}\,{a_\alpha}^{\mathrm{T}}\cdot\partial^\beta v \\ &= \sum_{\beta:|\beta|\le m}\Bigg(\sum_{\alpha\,:\,|\alpha|\le m,\ \alpha\ge\beta} (-1)^{|\alpha|}\binom{\alpha}{\beta}\,\partial^{\alpha-\beta}\,{a_\alpha}^{\mathrm{T}}\Bigg)\,\partial^\beta v, \end{aligned}$$

which is a representation as in 7.1. □

7.4 Remark. Let $L(u) = \Big(\sum_{j=1}^{N} L_{ij}(u_j)\Big)_{i=1,\dots M}$. Then

$$(L_{ij})^{\mathrm{T}} = (L^{\mathrm{T}})_{ji}.$$

7.5 Distributional definition. If L is an operator as in 7.1 and the coefficients $a_\alpha \in \mathcal{C}^\infty(\mathcal{U};\mathbb{R}^{M\times N})$, then the operator L for a distribution $S \in \mathscr{D}'(\mathcal{U};\mathbb{R}^N)$ is defined as

$$L(S) := \sum_{|\alpha|\le m} a_\alpha\partial^\alpha S \quad \text{ in } \mathscr{D}'(\mathcal{U};\mathbb{R}^M).$$

Here we used the matrix multiplication of a vector valued distribution, that is, for $\zeta \in \mathscr{D}(\mathcal{U};\mathbb{R}^M)$

$$\boldsymbol{\langle \zeta\,, L(S)\,\rangle = \langle\, L^{\mathrm{T}}(\zeta)\,, S\,\rangle} = \sum_{|\alpha|\le m}\boldsymbol{\langle\, {a_\alpha}^{\mathrm{T}}\zeta\,, \partial^\alpha S\,\rangle}.$$

Once we have defined linear differential operators, we are able to consider fundamental solutions, especially for the general case, that means, for systems.

7.6 Fundamental solution. Let

$$L\colon\mathcal{C}^m(\mathbb{R}^n;\mathbb{R}^N)\to\mathcal{C}^0(\mathbb{R}^n;\mathbb{R}^M)$$

be a linear differential operator as in 7.1 with constant coefficients. Then [6]

$$F=(F_{jk})_{j=1,\dots,N;k=1,\dots,M}\in\mathscr{D}'(\mathbb{R}^n;\mathbb{R}^{N\times M})$$

is called ***fundamental solution*** for L, if

$$L(F)=\boldsymbol{\delta}_0\mathrm{Id}_{\mathbb{R}^M} \tag{7.3}$$

in the space of distributions $\mathscr{D}'(\mathbb{R}^n;\mathbb{R}^{M\times M})$. The definition (7.3) reads

$$\langle L^{\mathbf{T}}\zeta\,,F\rangle=\langle\zeta\,,LF\rangle=\mathrm{trace}\,\zeta(0)\qquad\text{for }\zeta\in\mathscr{D}(\mathbb{R}^n;\mathbb{R}^M\times\mathbb{R}^M). \tag{7.4}$$

Alternative: We can write

$$F=(F_{jk})_{j=1,\dots,N;k=1,\dots,M}\text{ with }F_{jk}\in\mathscr{D}'(\mathbb{R}^n;\mathbb{R})$$

and the property (7.3) is for $i,k=1,\dots,M$ in $\mathscr{D}'(\mathbb{R}^n;\mathbb{R})$

$$\sum_{j=1}^N L_{ij}(F_{jk})=\begin{cases}\boldsymbol{\delta}_0 & \text{for } i=k,\\ 0 & \text{otherwise.}\end{cases} \tag{7.5}$$

Further alternatives: The equation (7.5) can also be written for $i,k=1,\dots,M$ as

$$\sum_{j=1}^N L_{ij}(F_{jk})=\delta_{i,k}\boldsymbol{\delta}_0\,. \tag{7.6}$$

Writing this for test functions $\zeta\in\mathscr{D}(\mathbb{R}^n;\mathbb{R})$ it becomes for $i,k=1,\dots,M$

$$\sum_{j=1}^N\langle\zeta\,,L_{ij}(F_{jk})\rangle=\delta_{i,k}\zeta(0)\,. \tag{7.7}$$

Replacing ζ by ζ_{ik} and summing over i and k one obtains that for all $\zeta=(\zeta_{ik})_{i,k=1,\dots,M}\in\mathscr{D}(\mathbb{R}^n;\mathbb{R}^{M\times M})$

$$\sum_{i,k=1}^M\sum_{j=1}^N\langle\zeta_{ik}\,,L_{ij}(F_{jk})\rangle=\sum_{ik}\delta_{i,k}\zeta_{ik}(0)=\sum_k\zeta_{kk}(0)\,. \tag{7.8}$$

Now taking instead of ζ_{ik} a function ζ_i and summing over i one obtains that for all $\zeta=(\zeta_i)_{i=1,\dots,M}\in\mathscr{D}(\mathbb{R}^n;\mathbb{R}^M)$

$$\sum_{i,j}\langle\zeta_i\,,L_{ij}(F_{jk})\rangle=\sum_i\delta_{i,k}\zeta_i(0)=\zeta_k(0)\,. \tag{7.9}$$

Definition: Here the operator L applied to a matrix F is defined by matrix multiplication. F maps into $N\times M$-matrices, the coefficients of L into $M\times N$-matrices, so that finally $L(F)$ maps into $M\times M$-matrices. *Notice:* The definition of fundamental solutions for systems usually is not covered in literature.

Hence fundamental solutions are distributional solutions for the operator L.

[6] $\mathbb{R}^{N\times M}$ denotes the set of $N\times M$-matrices.

7.7 Special case of a single equation (N=M=1). Let $L : \mathcal{C}^m(\mathbb{R};\mathbb{R}) \to \mathcal{C}^0(\mathbb{R};\mathbb{R})$ be a linear differential operator with constant coefficients. Then a distribution $F \in \mathscr{D}'(\mathbb{R}^n;\mathbb{R})$ is called ***fundamental solution*** of L, if

$$L(F) = \boldsymbol{\delta}_0 \quad \text{in } \mathscr{D}'(\mathbb{R}^n;\mathbb{R})\,.$$

We mention the following special standard case that the distributional fundamental solution consists of the distribution of a function.

7.8 Special case of a function as fundamental solution. Let $F = (F_{jk})_{jk} \in \mathcal{L}^1_{loc}(\mathbb{R}^n;\mathbb{R}^{N\times M})$. If $[F] := ([F_{jk}])_{jk}$ is a (distributional) fundamental solution, then also F is called ***fundamental solution***.

References: Tréves [14], Alt [3].

We shall get to know the most important fundamental solutions. Here a list:

- $L(u) := u'$ (see 7.9(1))
- $L(u) := u''$ (see 7.9(2))
- $L(u) := u' - Au$ (see 7.13)
- $L(u) :=$ div$u$ (see 8.2)
- $L(u) := \partial_i u$ (see 7.14)
- $L(u) := \partial_{\overline{z}} u = \frac{1}{2}\big(\partial_{x_1} u + \mathrm{i}\,\partial_{x_2} u\big)$ (see 8.9)
- $L(u) := -\Delta u$ (see 8.3)
- $L(u) := \partial_t u - \Delta u$ (see 9.2)
- $L(u) := \partial_t^2 u - \Delta u$ (see 9.6)

Es gibt auch Differetialgleichungen, welche keine Fundamentallösung haben, unter ihnen der Gradientenoperator, den wir in Abschnitt 10 behandeln. Falls $M = N = 1$ ist, gibt es jedoch immer eine Fundamentallösung, wie das Theorem von Ehrenpreis sagt.

As most simple example we present the fundamental solutions of two ordinary differential operators. This fundamental solutions are functions.

7.9 Gewöhnliche Differentialgleichung. Es sei $n = 1$ und der skalare Fall $N = M = 1$ in 7.6 (genauer: 7.8) gegeben. Dann gilt:

(1) Für $L(u) := u'$ für $u \in \mathcal{C}^1(\mathbb{R})$ definiert

$$F(x) := \mathcal{X}_{[0,\infty[}(x) = \begin{cases} 1 & \text{für } x \geq 0, \\ 0 & \text{für } x < 0, \end{cases}$$

eine Fundamentallösung $F \in \mathcal{L}^1_{loc}(\mathbb{R})$. Jede andere Fundamentallösung aus $\mathcal{L}^1_{loc}(\mathbb{R})$ ist bis auf eine additive Konstante gleich F.

(2) Für $L(u) := u''$ für $u \in \mathcal{C}^2(\mathbb{R})$ definiert

$$F(x) := \frac{1}{2}|x|$$

eine Fundamentallösung $F \in \mathcal{C}^0(\mathbb{R})$. Jede andere Fundamentallösung aus $\mathcal{L}^1_{loc}(\mathbb{R})$ ist bis auf eine (affin) lineare Funktion gleich F.

Wir werden in 7.13 noch eine Verallgemeinerung des Differentialoperators von 7.9(1) kennenlernen. Die Fundamentallösung in 7.9(1) ist die Heaviside-Funktion 2.6(2).

Proof (1). Sei $\zeta \in \mathcal{C}_0^\infty(\mathbb{R})$. Dann gilt

$$\langle \zeta, [F]' \rangle = \int_{\mathbb{R}} (-\zeta'(x))F(x)\,\mathrm{d}x = -\int_0^\infty \zeta'(x)\,\mathrm{d}x = \zeta(0) = \langle \zeta, \delta_0 \rangle.$$

Ist $\widetilde{F} \in \mathcal{L}^1_{loc}(\mathbb{R})$ eine weitere Fundamentallösung, so gilt $[F - \widetilde{F}]' = 0$. Dann folgt $F - \widetilde{F}$ ist fast überall eine konstante Funktion (ein Polynom 0-ten Grades). □

Proof (2). Es gilt

$$\langle \zeta, [F]'' \rangle = \int_{\mathbb{R}} \zeta''(x)F(x)\,\mathrm{d}x = \int_{-\infty}^0 \zeta''(x)\frac{-x}{2}\,\mathrm{d}x + \int_0^\infty \zeta''(x)\frac{x}{2}\,\mathrm{d}x.$$

$$= \int_{-\infty}^0 \frac{1}{2}\zeta'(x)\,\mathrm{d}x - \int_0^\infty \frac{1}{2}\zeta'(x)\,\mathrm{d}x \quad \text{(Partielle Integration)}$$

$$= \frac{1}{2}\zeta(0) + \frac{1}{2}\zeta(0) = \zeta(0) = \langle \zeta, \delta_0 \rangle.$$

Ist $\widetilde{F} \in \mathcal{L}^1_{loc}(\mathbb{R})$ eine andere Fundamentalösung, so folgt aus $[F - \widetilde{F}]'' = 0$, dass $F - \widetilde{F}$ fast überall eine affin lineare Funktion ist. □

Fundamentallösungen werden dazu benutzt, Integraldarstellungen von Lösungen der Differentialgleichung herzuleiten, d.h. eine Integraldarstellung von u durch g, wenn die Differentialgleichung $L(u) = g$ erfüllt ist. Wir betrachten zunächst den Spezialfall $L(u) := u'$.

7.10 Integraldarstellung für u'. Sei $I =]a, b[$, $a, b \in \mathbb{R}$, $u, g \in \mathcal{L}^1(I)$, $x_0 \in \overline{I}$. Dann sind äquivalent:

(1) $[u]' = [g]$ in $\mathscr{D}'(I)$.

(2) Es gibt ein $u_0 \in \mathbb{R}$ mit

$$u(x) = u_0 + \int_{x_0}^x g(y)\,\mathrm{d}y \qquad \text{für fast alle } x \in I.$$

Beachte: Ist $I = \mathbb{R}$ und hat g kompakten Träger, so ist das Integral gleich

$$F * g(x) = \int_{\mathbb{R}} F(x-y)g(y)\,\mathrm{d}y = \int_{-\infty}^x g(y)\,\mathrm{d}y\,, \tag{7.10}$$

wobei F die Fundamentallösung aus 7.9(1) ist. Es ist also $u := F * g$ eine partikuläre Lösung der Differentialgleichung $[u]' = [g]$.

Die rechte Seite dieser Identität ist stetig in x. Dies bedeutet, dass $u \in \mathcal{L}^1(I)$ einen stetigen Repräsentanten besitzt.

Proof (2) ⇒ (1). Eine Änderung von x_0 bewirkt nur eine Änderung von u_0, also sei ohne Einschränkung $x_0 = a$. Dann ist für $\zeta \in \mathcal{C}_0^\infty(I)$

$$\begin{aligned}\langle \zeta\,, [u]' \rangle_{\mathscr{D}'(I)} &= -\int_I \zeta'(x) u(x)\,\mathrm{d}x \\ &= -\int_I \zeta'(x) u_0 \,\mathrm{d}x - \int_a^b \zeta'(x) \int_a^x g(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Mit partieller Integration ist der erste Summand

$$= -\int_I \zeta'(x)\,\mathrm{d}x \cdot u_0 = 0,$$

und mit dem Satz von Fubini ist der zweite Summand

$$= -\int_a^b g(y) \left(\int_y^b \zeta'(x)\,\mathrm{d}x \right) \mathrm{d}y = \int_a^b g(y)\zeta(y)\,\mathrm{d}y = \langle \zeta\,, [g] \rangle_{\mathscr{D}'(I)}.$$

□

Proof (1) ⇒ (2). Definiere

$$\widetilde{u}(x) := \int_a^x g(y)\,\mathrm{d}y.$$

Es gilt $[\widetilde{u}]' = [g]$ (siehe im Beweisteil ((2) ⇒ (1))), also muss $[u - \widetilde{u}]' = 0$ sein. Somit folgt $u - \widetilde{u} = u_0$ fast überall für ein $u_0 \in \mathbb{R}$, konsequenterweise gilt für fast alle $x \in I$

$$u(x) = u_0 + \widetilde{u}(x) = u_0 + \int_a^x g(y)\,\mathrm{d}y.$$

□

We now give a generalization of (7.10) to general differential equations. Having a fundamental solution one is able to construct a solution u of $Lu = g$, at least if g is a smooth function having compact support, as we shall see in 7.12.

7.11 Motivation. Sei F eine $\mathcal{L}^1_{loc}$-Fundamentallösung des Differentialoperators L im Falle $N = M = 1$, also

$$L[F] = \delta_0.$$

Damit gilt für eine Verschiebung des Ursprungs unter Ausnutzung der Voraussetzung, dass L konstante Koeffizienten hat,

$$L[F(\sqcup - x_0)] = \delta_{x_0}. \tag{7.11}$$

Darüber hinaus gilt wegen der Linearität von L für $x_1, \dots, x_m \in \mathbb{R}^n$ und $c_1, \dots, c_m \in \mathbb{R}$

$$L\Big[\sum_{i=1}^m c_i F(\sqcup - x_i)\Big] = \sum_{i=1}^m c_i \delta_{x_i}\,,$$

was also ein ***Superpositionsprinzip*** bedeutet. Bei richtiger Belegung der c_i strebt die rechte Seite gegen $[g]$ und dann das Argument von L auf der linken Seite gegen $[F * g]$. Diese heuristische Betrachtung motiviert die Vermutung, dass $[F * g]$ die Gleichung $L[u] = [g]$ löst. Siehe dazu den folgenden Satz.

Proof. Der Beweis von (7.11) ist wie folgt:

$$\begin{aligned}
\langle \zeta , L[F(\sqcup - x_0)] \rangle &= \int_{\mathbb{R}^n} (L^{\mathrm{T}}(\zeta))(x)\, F(x - x_0)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} (L^{\mathrm{T}}(\zeta))(x + x_0)\, F(x)\, \mathrm{d}x = \langle L^{\mathrm{T}}(\zeta(\sqcup + x_0)) , [F] \rangle \\
&= \langle \zeta(\sqcup + x_0) , L[F] \rangle = \langle \zeta(\sqcup + x_0) , \delta_0 \rangle \\
&= \zeta(x + x_0)|_{x=0} = \zeta(x_0) = \langle \zeta , \delta_{x_0} \rangle .
\end{aligned}$$

Die Behauptung ist in der Tat richtig: Zerlegt man den $\mathbb{R}^n$ gleichmäßig in Quader Q_i der Kantenlänge ε und wählt $g \in \mathcal{L}^\infty_{loc}(\mathbb{R}^n)$, $c_i := \int_{Q_i} g(x)\,\mathrm{d}x$, so lässt sich die Konvergenz im Distributionssinn folgendermaßen einsehen:

$$\begin{aligned}
\left| \left\langle \zeta , \sum_i c_i \delta_{x_i} - [g] \right\rangle \right| &= \left| \sum_i c_i \zeta(x_i) - \int_{\mathbb{R}^n} \zeta(x) g(x)\, \mathrm{d}x \right| \\
&= \left| \sum_i \int_{Q_i} (\zeta(x_i) - \zeta(x)) g(x)\, \mathrm{d}x \right| \\
&\leq \sup_{|y_1 - y_2|_\infty \leq \varepsilon} |\zeta(y_1) - \zeta(y_2)| \cdot ||g||_{L^1(\mathbb{R}^n)} .
\end{aligned}$$

was im Limes für $\varepsilon \to 0$ gegen 0 geht. Im gleichen Sinne lässt sich die Konvergenz von $[\sum_{i=1}^m c_i F(\sqcup - x_i)]$ gegen $[F * f]$ einsehen. □

7.12 Theorem. Let $F = (F_{jk})_{jk} \in \mathcal{L}^1_{loc}(\mathbb{R}^n; \mathbb{R}^{N\times M})$ be a function, which is a fundamental solution of L as in 7.8, and $g \in \mathcal{L}^1(\mathbb{R}^n; \mathbb{R}^M)$ with compact support in $\mathbb{R}^n$. Then

$$u := F * g = \left(\sum_{k=1}^{M} F_{jk} * g_k \right)_{j=1,\dots,N} \in \mathcal{L}^1_{loc}(\mathbb{R}^n; \mathbb{R}^N)$$

solves $L(u) = g$, that is,

$$\sum_j L_{ij}[u_j] = [g_i] \quad \text{für } i = 1, \dots, M.$$

Proof. Es gilt für $\zeta \in \mathscr{D}(\mathbb{R}^n)$

$$\begin{aligned}
\left\langle \zeta , \sum_j L_{ij}[u_j] \right\rangle_{\mathscr{D}(\mathbb{R}^n)} &= \sum_j \left\langle (L_{ij})^{\mathrm{T}}(\zeta) , [u_j] \right\rangle_{\mathscr{D}(\mathbb{R}^n)} \\
&= \sum_j \int_{\mathbb{R}^n} \big((L_{ij})^{\mathrm{T}}(\zeta) \big)(x)\, u_j(x)\, \mathrm{d}x \\
&= \sum_{j,k} \int_{\mathbb{R}^n} \big((L_{ij})^{\mathrm{T}}(\zeta) \big)(x) \int_{\mathbb{R}^n} F_{jk}(x-y) g_k(y)\, \mathrm{d}y\, \mathrm{d}x \\
&= \sum_{j,k} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \big((L_{ij})^{\mathrm{T}}(\zeta) \big)(x) F_{jk}(x-y)\, \mathrm{d}x \right) g_k(y)\, \mathrm{d}y .
\end{aligned}$$

Durch eine Verschiebung $x \rightsquigarrow x+y$ wird dies zu

$$= \sum_{j,k} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \big((L_{ij})^{\mathrm{T}}(\zeta)\big)(x+y) F_{jk}(x)\,\mathrm{d}x \Big) g_k(y)\,\mathrm{d}y$$

$$= \sum_{k} \int_{\mathbb{R}^n} \underbrace{\Big(\sum_j \int_{\mathbb{R}^n} (L_{ij})^{\mathrm{T}}(\zeta(\sqcup + y)) F_{jk}\,\mathrm{dL}^n \Big)}_{= \sum_j \left\langle (L_{ij})^{\mathrm{T}}(\zeta(\sqcup+y))\,,[F_{jk}] \right\rangle_{\mathscr{D}(\mathbb{R}^n)}} g_k(y)\,\mathrm{d}y$$

$$= \sum_{k} \int_{\mathbb{R}^n} \Big\langle \zeta(\sqcup + y)\,, \sum_j L_{ij}[F_{jk}] \Big\rangle_{\mathscr{D}(\mathbb{R}^n)} g_k(y)\,\mathrm{d}y$$

$$= \sum_{k} \int_{\mathbb{R}^n} \delta_{i,k}\zeta(y) g_k(y)\,\mathrm{d}y \quad \text{(nach Definition der Fundamentallösung)}$$

$$= \int_{\mathbb{R}^n} \zeta(y) g_i(y)\,\mathrm{d}y = \langle \zeta\,, [g_i] \rangle_{\mathscr{D}(\mathbb{R}^n)}\,,$$

das heißt $\sum_j L_{ij}[u_j] = [g_i]$. □

We apply this now to a general system of linear first order differential equations with constant coefficients (hence $n = 1$, $m = 1$, $M = N$ in 7.6). If we denote the variable in $\mathbb{R}$ with t, they have the form, for $g \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$L(u) := u' - Au = g$$

and if we define

$$\widetilde{u}(t) := \mathrm{e}^{-tA} u(t)$$

it satisfies

$$\widetilde{u}' = \big(\mathrm{e}^{-tA}u\big)' = \mathrm{e}^{-tA}(u' - Au) = \mathrm{e}^{-tA} g =: \widetilde{g}$$

so that by 7.9(2) with a vector u_0

$$\widetilde{u}(t) = u_0 + \int_{-\infty}^t \widetilde{g}(s)\,\mathrm{d}s = u_0 + \int_{-\infty}^t \mathrm{e}^{-sA} g(s)\,\mathrm{d}s$$

hence

$$u(t) = \mathrm{e}^{tA}\Big(u_0 + \int_{-\infty}^t \mathrm{e}^{-sA} g(s)\,\mathrm{d}s\Big) = \mathrm{e}^{tA}u_0 + \int_{-\infty}^t \mathrm{e}^{(t-s)A} g(s)\,\mathrm{d}s$$
$$= \mathrm{e}^{tA}u_0 + \int_{\mathbb{R}} F(t-s) g(s)\,\mathrm{d}s = \mathrm{e}^{tA}u_0 + (F * g)(t)\,,$$

where F is defined in the following.

7.13 ODE system. Let $L(u) := u' - Au$ for $u \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^N)$, where $A \in \mathbb{R}^{N\times N}$ is a $N \times N$-matrix. Then

$$F(t) := \begin{cases} \mathrm{e}^{tA} & \text{for } t > 0, \\ 0 & \text{for } t < 0, \end{cases}$$

defines a fundamental solution $F \in \mathcal{L}^1_{loc}(\mathbb{R};\mathbb{R}^{N\times N})$ of L. Every other $\mathcal{L}^1_{loc}$-fundamental solution has the form

$$t \longmapsto F(t) + \mathrm{e}^{tA}\, C_0$$

with a constant matrix $C_0 \in \mathbb{R}^{N\times N}$.

Proof der Fundamentallösung. Es ist zunächst für $\eta \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{R}^N)$

$$L^{\mathrm{T}}\eta = -\eta' - A^{\mathrm{T}}\eta\,.$$

Dann folgt für $\zeta \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{R}^{N\times N})$

$$\begin{aligned}
&\langle \boldsymbol{\zeta}\,, \boldsymbol{L[F]}\,\rangle_{\mathscr{D}(\mathbb{R})} = \big\langle \boldsymbol{L^{\mathrm{T}}\zeta}\,, \boldsymbol{[F]}\,\big\rangle_{\mathscr{D}(\mathbb{R})} \\
&= \int_0^\infty (L^{\mathrm{T}}\zeta)(t) \bullet \mathrm{e}^{tA}\,\mathrm{d}t = -\int_0^\infty \Big(\zeta'(t) + A^{\mathrm{T}}\zeta(t)\Big) \bullet \mathrm{e}^{tA}\,\mathrm{d}t \\
&= -\int_0^\infty \underbrace{\Big(\mathrm{e}^{tA^{\mathrm{T}}}\zeta'(t) + \mathrm{e}^{tA^{\mathrm{T}}}A^{\mathrm{T}}\zeta(t)\Big)}_{=\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{tA^{\mathrm{T}}}\zeta(t)\right)} \bullet \mathrm{Id}\,\mathrm{d}t \\
&= -\int_0^\infty \mathrm{trace}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{tA^{\mathrm{T}}}\zeta(t)\big)\Big)\,\mathrm{d}t = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{trace}\,\big(\mathrm{e}^{tA^{\mathrm{T}}}\zeta(t)\big)\Big)\,\mathrm{d}t \\
&= \mathrm{trace}\,\big(\mathrm{e}^{tA^{\mathrm{T}}}\zeta(t)\big)\Big|_{t=0} = \mathrm{trace}\,\zeta(0) = \langle \boldsymbol{\zeta}\,, \boldsymbol{\delta_0}\mathbf{Id}\,\rangle_{\mathscr{D}(\mathbb{R})}\,,
\end{aligned}$$

was $L[F] = \boldsymbol{\delta}_0\mathrm{Id}$ bedeutet. □

Proof der Eindeutigkeit. Zum Beweis der Eindeutigkeit nehmen wir an, $\widetilde{F}$ sei eine weitere Fundamentallösung. Definiere $H := \widetilde{F} - F$, dann ist $L[H] = 0$. Also gilt für $\zeta \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{R}^{N\times N})$, da $(L^{\mathrm{T}}\zeta)(t) = -(\zeta'(t) + A^{\mathrm{T}}\zeta(t))$,

$$\begin{aligned}
&0 = \langle \boldsymbol{\zeta}\,, \boldsymbol{L[H]}\,\rangle_{\mathscr{D}(\mathbb{R})} = \big\langle \boldsymbol{L^{\mathrm{T}}\zeta}\,, \boldsymbol{[H]}\,\big\rangle_{\mathscr{D}(\mathbb{R})} \\
&= \int_0^\infty (L^{\mathrm{T}}\zeta)(t) \bullet H(t)\,\mathrm{d}t = -\int_0^\infty \Big(\zeta'(t) + A^{\mathrm{T}}\zeta(t)\Big) \bullet H(t)\,\mathrm{d}t \\
&= -\int_0^\infty \underbrace{\Big(\mathrm{e}^{tA^{\mathrm{T}}}\zeta'(t) + \mathrm{e}^{tA^{\mathrm{T}}}A^{\mathrm{T}}\zeta(t)\Big)}_{=\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{tA^{\mathrm{T}}}\zeta(t)\right)} \bullet \Big(\mathrm{e}^{-tA}H(t)\Big)\,\mathrm{d}t\,.
\end{aligned}$$

Definieren wir nun

$$\widetilde{H}(t) := \mathrm{e}^{-tA}H(t)$$

und für $\widetilde{\zeta} \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{R}^{N\times N})$

$$\zeta(t) := \mathrm{e}^{-tA^{\mathrm{T}}}\widetilde{\zeta}(t)\,,\ \text{ also } \mathrm{e}^{tA^{\mathrm{T}}}\zeta(t) = \widetilde{\zeta}(t)\,,$$

so haben wir gezeigt, dass für alle $\widetilde{\zeta}$

$$0 = -\int_0^\infty \widetilde{\zeta}'(t) \bullet \widetilde{H}(t)\,\mathrm{d}t = -\Big\langle \boldsymbol{\widetilde{\zeta}'}\,, \boldsymbol{[\widetilde{H}]}\,\Big\rangle_{\mathscr{D}(\mathbb{R})} = \Big\langle \boldsymbol{\widetilde{\zeta}}\,, \boldsymbol{[\widetilde{H}]'}\,\Big\rangle_{\mathscr{D}(\mathbb{R})}\,,$$

also $[\widetilde{H}]' = 0$. Es folgt, dass es eine $(N \times N)$-Matrix C_0 gibt mit $\widetilde{H}(t) = C_0$, was zu zeigen war. □

Als letztes Beispiel in diesem Abschnitt betrachten wir die Ableitung $u \mapsto \partial_i u$. Deren Fundamentallösung ist eine Distribution, und zwar ein Linienintegral.

7.14 Die Ableitung ∂_i. Betrachte die eindimensionale Halbgerade

$$\Gamma_i := \{s\mathbf{e}_i\,;\ s \geq 0\}.$$

Dann ist die Distribution

$$F = \mathrm{H}^1 \llcorner \Gamma_i$$

eine Fundamentallösung des Operators $L(u) := \partial_i u$.

Proof. Die Distribution ist für $\zeta \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$

$$\langle \zeta\,, F\rangle_{\mathscr{D}(\mathbb{R}^n)} = \int_{\Gamma_i} \zeta \,\mathrm{dH}^1 = \int_0^\infty \zeta(s\mathbf{e}_i)\,\mathrm{d}s\,.$$

Dann ist

$$\begin{aligned}&\langle \zeta\,, LF\rangle_{\mathscr{D}(\mathbb{R}^n)} = \big\langle L^{\mathrm{T}}\zeta\,, F\big\rangle_{\mathscr{D}(\mathbb{R}^n)} = \langle -\partial_i\zeta\,, F\rangle_{\mathscr{D}(\mathbb{R}^n)}\\ &= -\int_0^\infty (\partial_i\zeta)(s\mathbf{e}_i)\,\mathrm{d}s = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\big(\zeta(s\mathbf{e}_i)\big)\,\mathrm{d}s = \zeta(0) = \langle \zeta\,, \boldsymbol{\delta}_0\rangle_{\mathscr{D}(\mathbb{R}^n)}\,,\end{aligned}$$

also $LF = \boldsymbol{\delta}_0$. □

In this section we have mainly seen fundamental solutions for ODE's. Besides these we present further examples and among them the classical fundamental solutions in section 8 and 9.

8 Space dependent fundamental solutions

In this section the coordinates are given as before,

$$x \in \mathbb{R}^n\,,$$

so that we can use the previous section 7 to the space $\mathbb{R}^n$. In this section we give the fundamental solutions for the following operators:

- divergence operator (see 8.2),
- Laplace operator (see 8.3),
- Cauchy-Riemann operator (see 8.9).

The first example deals with the divergence operator.

8.1 Divergence operator. The ***Divergence operator*** is defined as mapping $L:\mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n) \to \mathcal{C}^0(\mathbb{R}^n;\mathbb{R})$, it is $N = n$ and $M = 1$ in 7.1, by

$$L(u) := \operatorname{div} u = \sum_{i=1}^{n} \partial_i u_i\,.$$

Eine Fundamentallösung des Divergenzoperators ist gegeben durch

8.2 Fundamental solution for divergence operator. Let $L(u) = \operatorname{div}(u)$ for $u \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}^n)$. Then

$$F(x) := \frac{1}{\sigma_n}\frac{x}{|x|^n}$$

defines a fundamental solution $F \in \mathcal{L}^1_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ of L. *Definition:* $\sigma_n := \mathrm{H}^{n-1}(\partial\mathrm{B}_1(0))$. *Important notice:* This is in fact only <u>one</u> fundamental solution of the divergence operator. There exist many other fundamental solutions with a different behaviour at the origin. This means, that a uniqueness statement as for example in theorem 7.13 is inpossible.

Proof. Es ist $N = n$. Da $M = 1$ werden in der Definition des Operators Matrizen in $\mathbb{R}^{1\times n}$ mit $\mathbb{R}^n$ identifiziert und in der Definition der Fundamentallösung $\mathbb{R}^{n\times 1}$ mit $\mathbb{R}^n$. Also ist $L = (L_j)_{j=1,\dots,n} = (\partial_j)_{j=1,\dots,n}$ und $F = (F_j)_{j=1,\dots,n}$. Dass F Fundamentallösung ist, heißt also

$$\sum_{j=1}^{n} \partial_j[F_j] = \boldsymbol{\delta}_0\,,$$

was also zu zeigen ist. Nun ist für $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$

$$\sum_j \langle\,\boldsymbol{\zeta}\,,\boldsymbol{\partial_j}[\boldsymbol{F_j}]\,\rangle = \sum_j \langle\,\boldsymbol{-\partial_j\zeta}\,,[\boldsymbol{F_j}]\,\rangle$$
$$= -\sum_j \int_{\mathbb{R}^n} F_j\,\partial_j\zeta\,\mathrm{dL}^n = -\int_{\mathbb{R}^n} F\bullet\nabla\zeta\,\mathrm{dL}^n\,.$$

Wir betrachten das Gebiet mit ausgestochener Kugel

$$-\int_{\mathbb{R}^n\setminus\mathrm{B}_\varepsilon(0)} F\bullet\nabla\zeta\,\mathrm{dL}^n \to -\int_{\mathbb{R}^n} F\bullet\nabla\zeta\,\mathrm{dL}^n \quad \text{for } \varepsilon\to 0$$

und benutzen den Satz von Gauß, wobei klar ist, dass im klassischen Sinne $\operatorname{div} F = 0$ in $\mathbb{R}^n \setminus \{0\}$ gilt. Partielle Integration ergibt dann

$$\begin{aligned}
&- \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(0)} F \bullet \nabla \zeta \, \mathrm{dL}^n = \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta \, F \bullet \nu_{\mathrm{B}_\varepsilon(0)} \, \mathrm{dH}^{n-1} + \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(0)} \zeta \operatorname{div} F \, \mathrm{dL}^n \\
&= \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta(x) \frac{1}{\sigma_n} \frac{x}{|x|^n} \bullet \frac{x}{|x|} \, \mathrm{dH}^{n-1}(x) = \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta(x) \frac{1}{\sigma_n \varepsilon^{n-1}} \, \mathrm{dH}^{n-1}(x) \\
&= \frac{1}{\sigma_n \varepsilon^{n-1}} \int_{\partial \mathrm{B}_\varepsilon(0)} \zeta(x) \, \mathrm{dH}^{n-1}(x) = \frac{1}{\sigma_n} \int_{\partial \mathrm{B}_1(0)} \zeta(\varepsilon y) \, \mathrm{dH}^{n-1}(y) \\
&\to \zeta(0) \quad \text{for } \varepsilon \to 0 \,,
\end{aligned}$$

da $\zeta(\varepsilon y)$ in der Variablen $y \in \partial \mathrm{B}_1(0)$ gleichmäßig gegen $\zeta(0)$ konvergiert. □

Now let us introduce the fundamental solution of the negative Laplace operator, given by $L(u) := -\Delta u = \operatorname{div}(-\nabla u)$.

8.3 Fundamental solution of $-\Delta$. The function

$$F(x) := \begin{cases} \dfrac{1}{(n-2)\sigma_n} |x|^{2-n} & \text{for } n \geq 3 \\ -\dfrac{1}{2\pi} \log |x| & \text{for } n = 2 \\ -\dfrac{1}{2} |x| & \text{for } n = 1 \end{cases}$$

defines a ***fundamental solution*** $F \in L^1_{loc}(\mathbb{R}^n; \mathbb{R})$ of the differential operator $-\Delta$. *Definition:* Here σ_n is the surface area of the sphere $\mathbb{S}^{n-1} := \partial \mathrm{B}_1(0) \subset \mathbb{R}^n$, where $\sigma_1 = 2$, $\sigma_2 = 2\pi$, and $\sigma_n = n\kappa_n$. Here κ_n is the volume of the unit ball $\mathrm{B}_1(0) \subset \mathbb{R}^n$.

Proof. Es sei für $r > 0$

$$\psi(r) := \begin{cases} \dfrac{1}{(n-2)\sigma_n} r^{2-n} & \text{for } n \geq 3 \\ -\dfrac{1}{2\pi} \log r & \text{for } n = 2 \\ -\dfrac{r}{2} & \text{for } n = 1 \end{cases}$$

dann ist

$$F(x) = \frac{1}{\sigma_n} \psi_n(|x|) \text{ und } \psi_n'(r) = -\frac{1}{r^{n-1}} \,.$$

Also gilt für $x \in \mathbb{R}^n \setminus \{0\}$

$$\partial_i F(x) = \frac{1}{\sigma_n} \psi_n'(|x|) \frac{x_i}{|x|} = -\frac{1}{\sigma_n} \frac{x_i}{|x|^n} =: -G_i(x).$$

Nun ist

$$G(x) = \frac{1}{\sigma_n} \frac{x}{|x|^n} \text{ in } \mathbb{R}^n \setminus \{0\}$$

die Fundamentallösung in $\mathcal{L}^1(\mathbb{R}^n; \mathbb{R})$ vom div-Operator, also nach 8.2

$$\sum_i \partial_i [G_i] = \boldsymbol{\delta}_0 \quad \text{in } \mathscr{D}'(\mathbb{R}^n).$$

Zu zeigen bleibt also noch

$$\partial_i[F] = -[G_i] \quad \text{in } \mathscr{D}'(\mathbb{R}^n), \tag{8.1}$$

denn dann folgt

$$-\Delta[F] = \sum_i \partial_i(-\partial_i[F]) = \sum_i \partial_i[G_i] = \boldsymbol{\delta}_0 \quad \text{in } \mathscr{D}'(\mathbb{R}^n),$$

was zu zeigen war. Die Behauptung (8.1) kann mit eine der folgenden Beweisansätzen verifiziert werden:

- Durch Herausstechen einer ε-Kugel um 0 wie im Beweis von 8.2.
- Zeige, dass $F \in W^{1,1}_{loc}(\mathbb{R}^n)$ liegt, also $\partial_i[F] = [\partial_i F]$.

□

Es gilt die folgende Eindeutigkeitsaussage.

8.4 Eindeutigkeit der Fundamentallösung von $-\Delta$. Ist $n \geq 3$, so ist die Fundamentallösung in 8.3 die einzige Fundamentallösung $F \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ von $-\Delta$, die

$$F(x) \to 0 \text{ for } |x| \to \infty$$

erfüllt, d.h. die im Unendlichen verschwindet.

Proof. .... □

Using theorem 7.12 we conclude:

8.5 Integral representation for the Laplace operator. Let $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ with compact support. Then

$$u(x) := \int_{\mathbb{R}^n} F(x-y)f(y)\,\mathrm{d}y = \begin{cases} \dfrac{1}{(n-2)\sigma_n} \displaystyle\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\,\mathrm{d}y & \text{for } n \geq 3 \\ -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log\frac{1}{|x-y|} \cdot f(y)\,\mathrm{d}y & \text{for } n = 2 \\ -\dfrac{1}{2} \displaystyle\int_{\mathbb{R}} |x-y| f(y)\,\mathrm{d}y & \text{for } n = 1 \end{cases}$$

defines a function in $\mathcal{L}^1_{loc}(\mathbb{R}^n)$ solving the differential equation

$$-\Delta[u] = [f] \quad \text{in } \mathscr{D}'(\mathbb{R}^n).$$

Before we study differential equations for complex functions let us make some essential comments.

8.6 Remark on complex numbers. We identify $\mathbb{C} = \mathbb{R}^2$.

(1) Die komplexe Multiplikation $z \to wz$ hat die Matrixdarstellung

$$wz = \mathbf{C}_w z\,, \quad \mathbf{C}_w := \begin{bmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{bmatrix}$$

mit $w = w_1 + \mathrm{i}w_2 \in \mathbb{C}$, $w_1, w_2 \in \mathbb{R}$.

(2) Für $w, z \in \mathbb{C}$ gilt $w \bullet z = \mathrm{Re}\,(w\overline{z})$.

(3) Die ***Wirtinger-Ableitungen*** sind für komplexwertige Funktionen u

$$\partial_{\overline{z}} u := \frac{1}{2}(\partial_1 u + \mathrm{i}\partial_2 u)\,, \quad \partial_z u := \frac{1}{2}(\partial_1 u - \mathrm{i}\partial_2 u)$$

Es gilt $\overline{\partial_{\overline{z}} u} = \partial_z \overline{u}$ und $\overline{\partial_z u} = \partial_{\overline{z}} \overline{u}$.

(4) For $k \in \mathbb{Z}$ we have the identities $\partial_{\overline{z}}(z^k) = 0$ und $\partial_z(z^k) = kz^{k-1}$.

Proof (1). Es gilt

$$\mathbf{C}_w z = \mathbf{C}_w \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} w_1 z_1 - w_2 z_2 \\ w_2 z_1 + w_1 z_2 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\,(wz) \\ \mathrm{Im}\,(wz) \end{bmatrix} = wz\,.$$

□

Proof (2). .... □

Proof (3). .... □

Proof (4). Es ist mit $z = (z_1, z_2)$

$$\partial_1 \left((z_1 + \mathrm{i}z_2)^k\right) = k(z_1 + \mathrm{i}z_2)^{k-1}\,,$$
$$\partial_2 \left((z_1 + \mathrm{i}z_2)^k\right) = k(z_1 + \mathrm{i}z_2)^{k-1} \cdot \mathrm{i}\,.$$

Daraus folgt die Behauptung. □

8.7 Cauchy-Riemann operator. The ***Cauchy-Riemann operator*** is defined as mapping $L : \mathcal{C}^1(\mathbb{R}^2; \mathbb{R}^2) \to \mathcal{C}^0(\mathbb{R}^2; \mathbb{R}^2)$ by

$$\begin{aligned} L(u) &:= \frac{1}{2} \begin{bmatrix} \partial_1 u_1 - \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 \end{bmatrix} = \partial_{\overline{z}} u \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \partial_1 u + \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \partial_2 u = \frac{1}{2}\partial_1 u + \frac{\mathrm{i}}{2}\partial_2 u \end{aligned}$$

Hence $m = 1$, $n = M = N = 2$ in 7.1. In complex notation the operator and its transposed is given by

$$L(u) = \partial_{\overline{z}} u = \frac{1}{2}(\partial_1 u + \mathrm{i}\partial_2 u)\,,$$
$$L^{\mathrm{T}}(v) = -\partial_z v = -\frac{1}{2}(\partial_1 v - \mathrm{i}\partial_2 v)\,.$$

Bemerkung: Die Darstellung mit Matrizen zeigt, dass der Operator von der Gestalt 7.1 ist.

Proof representations. Es ist

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C}_{\mathrm{i}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.2}$$

Also

$$L(u) := \frac{1}{2}\begin{bmatrix}\partial_1 u_1 - \partial_2 u_2\\ \partial_2 u_1 + \partial_1 u_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\partial_1 u + \frac{1}{2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\partial_2 u = \frac{1}{2}(\partial_1 u + \mathrm{i}\partial_2 u) = \partial_{\overline{z}} u.$$

□

Proof of L^{T}. Es ist für $\zeta \in \mathcal{C}_0^1(\mathbb{R}^2;\mathbb{R}^2)$

$$\zeta \bullet L(u) = \mathrm{Re}\,\big(\zeta\,\overline{L(u)}\big) = \mathrm{Re}\,\big(\zeta\,\overline{\partial_{\overline{z}} u}\big) = \mathrm{Re}\,\big(\zeta\,\partial_z \overline{u}\big) = \mathrm{Re}\,\partial_z(\zeta\,\overline{u}) - \mathrm{Re}\,\big(\partial_z\zeta\,\overline{u}\big)\,.$$

Da

$$\int_{\mathbb{R}^2} \mathrm{Re}\,\partial_z(\zeta\,\overline{u})\,\mathrm{dL}^2 = \mathrm{Re}\Big(\int_{\mathbb{R}^2} \partial_z(\zeta\,\overline{u})\,\mathrm{dL}^2\Big) = 0\,,$$

folgt

$$\langle\, \boldsymbol{\zeta}\,, \boldsymbol{L(u)}\,\rangle = \int_{\mathbb{R}^2} \zeta \bullet L(u)\,\mathrm{dL}^2 = \int_{\mathbb{R}^2} \Big(\mathrm{Re}\,\partial_z(\zeta\,\overline{u}) - \mathrm{Re}\,\big(\partial_z\zeta\,\overline{u}\big)\Big)\,\mathrm{dL}^2 = \int_{\mathbb{R}^2} \mathrm{Re}\,\big(-\partial_z\zeta\,\overline{u}\big)\,\mathrm{dL}^2 = \int_{\mathbb{R}^2} (-\partial_z\zeta)\bullet u\,\mathrm{dL}^2 = \langle\, \boldsymbol{-\partial_z\zeta}\,, \boldsymbol{u}\,\rangle,$$

das heißt $\langle\, \boldsymbol{\zeta}\,, \boldsymbol{L(u)}\,\rangle = \big\langle\, \boldsymbol{L^{\mathrm{T}}\zeta}\,, \boldsymbol{u}\,\big\rangle$, where $L^{\mathrm{T}} = -\partial_z$. □

Before we present the fundamental solution for this operator, this general remark on fundamental solutions:

8.8 Representation of fundamental solutions. Let $F = (F_{jk})_{j=1,\dots,N;k=1,\dots,M}$ be a distributional fundamental solution in $\mathscr{D}'(\mathbb{R}^n;\mathbb{R}^{N\times M})$ as in 7.6 of the operator $L\colon\mathcal{C}^m(\mathbb{R}^n;\mathbb{R}^N) \to \mathcal{C}^0(\mathbb{R}^n;\mathbb{R}^M)$. This is equivalent to, for $k = 1,\dots,M$,

$$\sum_{j=1}^{N} \big\langle\, \boldsymbol{(L^{\mathrm{T}}\zeta)_j}\,, \boldsymbol{F_{jk}}\,\big\rangle = \zeta_k(0) \quad \text{for } \zeta \in \mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R}^M)\,. \tag{8.3}$$

If F is a $\mathcal{L}^1_{loc}$-fundamental solution in $\mathcal{L}^1_{loc}(\mathbb{R}^n;\mathbb{R}^{N\times M})$ then this is equivalent to

$$\int_{\mathbb{R}^n} F^{\mathrm{T}} L^{\mathrm{T}}\zeta\,\mathrm{dL}^n = \zeta(0) \quad \text{for } \zeta \in \mathcal{C}_0^\infty(\mathbb{R}^n;\mathbb{R}^M)\,. \tag{8.4}$$

Proof of (8.3). Es wurde in Abschnitt 7 in (7.9) gezeigt, dass die Eigenschaft von F Fundamentallösung zu sein äquivalent ist zu

$$\sum_{i,j} \langle\, \boldsymbol{\zeta_i}\,, \boldsymbol{L_{ij}F_{jk}}\,\rangle = \zeta_k(0)\,.$$

Nun ist

$$\sum_i \langle\, \boldsymbol{\zeta_i}\,, \boldsymbol{L_{ij}F_{jk}}\,\rangle = \sum_i \Big\langle\, \boldsymbol{(L_{ij})^{\mathrm{T}}\zeta_i}\,, \boldsymbol{F_{jk}}\,\Big\rangle = \sum_i \big\langle\, \boldsymbol{L^{\mathrm{T}}{}_{ji}\zeta_i}\,, \boldsymbol{F_{jk}}\,\big\rangle = \Big\langle\, \boldsymbol{\sum_i L^{\mathrm{T}}{}_{ji}\zeta_i}\,, \boldsymbol{F_{jk}}\,\Big\rangle = \big\langle\, \boldsymbol{(L^{\mathrm{T}}\zeta)_j}\,, \boldsymbol{F_{jk}}\,\big\rangle$$

woraus (8.3) folgt. □

Proof of (8.4). Ist F eine lokal integrierbare Fundamentallösung, so besagt (8.3)

$$\zeta_k(0) = \sum_j \langle \boldsymbol{(L^{\mathrm{T}}\zeta)_j} \,, \boldsymbol{[F_{jk}]} \rangle = \sum_j \int_{\mathbb{R}^n} (L^{\mathrm{T}}\zeta)_j F_{jk} \, \mathrm{dL}^n$$
$$= \int_{\mathbb{R}^n} \sum_j {F^{\mathrm{T}}}_{kj} (L^{\mathrm{T}}\zeta)_j \, \mathrm{dL}^n = \int_{\mathbb{R}^n} (F^{\mathrm{T}} L^{\mathrm{T}} \zeta)_k \, \mathrm{dL}^n ,$$

was (8.4) impliziert. □

Wir kommen nun zur Fundamentallösung des Cauchy-Riemann Operators.

8.9 Fundamental solution of the Cauchy-Riemann operator. Let L as in 8.7. Then in complex notation

$$F(z) := \frac{1}{\pi z} \tag{8.5}$$

defines a fundamental solution $F \in \mathcal{L}^1_{loc}(\mathbb{C};\mathbb{C})$ for the Cauchy-Riemann-Operator, i.e. for the $\partial_{\overline{z}}$-operator. In real notation this fundamental solution is

$$F(z) := \frac{1}{\pi |z|^2} \begin{bmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{bmatrix} = \frac{1}{\pi |z|^2} \mathbf{C}_{\overline{z}} \,,$$

now defined as function $F \in \mathcal{L}^1_{loc}(\mathbb{R}^2;\mathbb{R}^{2\times 2})$. *Remark:* In the complex notation the fact that this function is a fundamental solution is equivalent to the property

$$- \int_{\mathbb{R}^2} \overline{F} \partial_z \zeta \, \mathrm{dL}^2 = \zeta(0) \quad \text{ for all } \zeta \in \mathcal{C}_0^\infty(\mathbb{C};\mathbb{C}) \,. \tag{8.6}$$

Proof. Nach 8.8 ist zu zeigen, dass

$$\int_{\mathbb{R}^2} F^{\mathrm{T}} L^{\mathrm{T}} \zeta \, \mathrm{dL}^2 = \zeta(0)$$

ist für alle

$$\zeta = \zeta_1 + \mathrm{i}\zeta_2 = (\zeta_1, \zeta_2) \in \mathcal{C}_0^\infty(\mathbb{R}^2;\mathbb{R}^2) \,.$$

Nun ist $L^{\mathrm{T}}\zeta = -\partial_z \zeta$ nach 8.7 und

$$F^{\mathrm{T}}(z) = \frac{1}{\pi |z|^2} \mathbf{C}_z$$

die komplexe Multiplikation mit

$$\frac{1}{\pi |z|^2} z = \frac{1}{\pi \overline{z}} = \overline{F(z)} \,,$$

woraus also folgt, dass die Behauptung (8.6) in der Bemerkung zu zeigen ist. Nun gilt für $\varepsilon \searrow 0$

$$- \int_{\mathbb{R}^2} \overline{F} \partial_z \zeta \, \mathrm{dL}^2 \leftarrow - \int_{\mathbb{R}^2 \setminus \mathrm{B}_\varepsilon(0)} \overline{F} \partial_z \zeta \, \mathrm{dL}^2$$
$$= - \int_{\mathbb{R}^2 \setminus \mathrm{B}_\varepsilon(0)} \partial_z (\overline{F} \zeta) \, \mathrm{dL}^2 \quad (\text{wegen } \partial_z \overline{F} = 0 \text{ in } \mathbb{C} \setminus \{0\})$$
$$= - \frac{1}{2} \int_{\partial \mathrm{B}_\varepsilon(0)} \overline{F} \zeta (\nu_1 - \mathrm{i}\nu_2) \, \mathrm{dH}^1 \quad (\text{es ist } \overline{F}(z) = \frac{1}{\pi \overline{z}})$$
$$= - \frac{1}{2\pi\varepsilon} \int_{\partial \mathrm{B}_1(0)} \zeta \, \mathrm{dH}^1 \to \zeta(0),$$

also ist F eine Fundamentallösung. □

Using theorem 7.12 we conclude:

8.10 Integral representation for the Cauchy-Riemann operator. Let $f \in \mathcal{L}^1(\mathbb{C};\mathbb{C})$ be a function with compact support. Then

$$u(z) := \int_{\mathbb{R}^2} \frac{f(y)}{\pi(z-y)} \,\mathrm{dL}^2(y)$$

defines a solution $u \in \mathcal{L}^1_{loc}(\mathbb{C};\mathbb{C})$ of the differential equation

$$\partial_{\overline{z}}[u] = [f] \quad \text{in } \mathscr{D}'(\mathbb{C}).$$

This is equivalent to

$$-\int_{\mathbb{R}^2} u\,\partial_{\overline{z}}\zeta \,\mathrm{dL}^2 = \int_{\mathbb{R}^2} f\zeta \,\mathrm{dL}^2 \quad \text{for } \zeta \in \mathcal{C}^\infty_0(\mathbb{C};\mathbb{C}).$$

9 Time dependent fundamental solutions

In this section we give the main fundamental solutions for differential operators involving time t. So the coordinates are

$$(t,x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{1+n} = \mathbb{R}^{\underline{n}} \text{ with } \underline{n} = n+1\,,$$

and we apply the definitions of section 7 to the space $\mathbb{R}^{\underline{n}} = \mathbb{R} \times \mathbb{R}^n$. In this section we give the fundamental solutions for the following operators:

- Heat operator,
- Schrödinger operator,
- Wave operator in 1D and 2D,
- Wave operator in 3D.

We begin with the heat operator.

9.1 Heat operator. Let $a > 0$. The ***heat operator*** is defined as mapping $L : \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}) \to \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ via

$$L(u) := \partial_t u - \Delta u\,.$$

Thus n is arbitrary, and $M = N = 1$ in 7.1. The transposed operator is

$$L^{\mathrm{T}}(v) = -\partial_t v - \Delta v.$$

Proof. It is $L^{\mathrm{T}}(v) = (-1)^1 \partial_t v - \sum_{i=1}^n (-1)^2 \partial_i^2 v = -\partial_t v - \Delta v$. □

Let us consider the corresponding fundamental solution:

9.2 Fundamental solution of the heat operator. By

$$F(t,x) := \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} \exp\left(-\dfrac{|x|^2}{4t}\right) & \text{for } t > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

a fundamental solution $F \in \mathcal{L}^1_{loc}(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ of L is defined. *Remark:* F is a real-analytic function in $]0,\infty[\times\mathbb{R}^n$ and ∞-many differentiable in $(\mathbb{R}\times\mathbb{R}^n)\setminus\{0\}$. Moreover F has the representation

$$F(t,x) = \psi_{\sqrt{t}}(x) \text{ for } t > 0$$

with

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^n}\psi\left(\frac{x}{\varepsilon}\right)\,, \quad \psi(x) := \frac{1}{(4\pi)^{n/2}} \mathrm{e}^{-\frac{|x|^2}{4}}\,.$$

Here $(\psi_\varepsilon)_{\varepsilon>0}$ is a Dirac sequence, that is ψ has integral 1.

Proof. Let $t > 0$. We compute the partial derivatives of F:

$$\partial_{x_i} F(t,x) = -\frac{x_i}{2t} F(t,x)\,,$$
$$\partial_{x_j}\partial_{x_i} F(t,x) = \left(-\frac{\delta_{i,j}}{2t} + \frac{x_i x_j}{4t^2}\right) F(t,x)\,,$$

hence

$$\Delta F(t,x) = \left(-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right) F(t,x) = \partial_t F(t,x)\,,$$

thtat is $L(F) = 0$ in $]0,\infty[\times\mathbb{R}^n$. We have to show that

$$L[F] = \boldsymbol{\delta}_{(0,0)} \quad \text{in } \mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)\,.$$

Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{R}^n)$. Then

$$\begin{aligned}
&\langle \boldsymbol{\zeta}\,, \boldsymbol{L[F]} \rangle = \langle \boldsymbol{L^{\mathrm{T}}\zeta}\,, \boldsymbol{[F]} \rangle \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} (-\partial_t\zeta - \Delta\zeta) F \,\mathrm{dL}^{n+1} \\
&\leftarrow \int_\varepsilon^\infty \int_{\mathbb{R}^n} (-\partial_t\zeta - \Delta\zeta) F \,\mathrm{dL}^n\,\mathrm{dL}^1 \quad \text{as } \varepsilon \to 0 \\
&= \int_{\mathbb{R}^n} (\zeta F)(\varepsilon, x)\,\mathrm{d}x + \int_\varepsilon^\infty \int_{\mathbb{R}^n} \zeta \underbrace{(\partial_t F - \Delta F)}_{=0} \,\mathrm{dL}^n\,\mathrm{dL}^1 \\
&= \int_{\mathbb{R}^n} \zeta(\varepsilon, x)\psi_{\sqrt{\varepsilon}}(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \zeta(0, x)\psi_{\sqrt{\varepsilon}}(x)\,\mathrm{d}x + \int_{\mathbb{R}^n} \underbrace{(\zeta(\varepsilon,x) - \zeta(0,x))}_{\to 0 \text{ unif. in } x} \psi_{\sqrt{\varepsilon}}(x)\,\mathrm{d}x \\
&\to \zeta(0,0) = \langle \boldsymbol{\zeta}\,, \boldsymbol{\delta_{(0,0)}} \rangle \quad \text{as } \varepsilon \to 0
\end{aligned}$$

where we have used the properties of a Dirac sequence, that is for $\varepsilon \to 0$

$$\psi_{\sqrt{\varepsilon}} * \zeta(0, -\sqcup) \to \zeta(0,0)\,,$$
$$\left|\int_{\mathbb{R}^n} (\zeta(\varepsilon,x) - \zeta(0,x))\psi_{\sqrt{\varepsilon}}(x)\,\mathrm{d}x\right| \le \sup_{x\in\mathbb{R}^n} |\zeta(\varepsilon,x) - \zeta(0,x)| \to 0\,.$$

□

The second example is the

9.3 Schrödinger operator. The ***Schrödinger operator*** with constant coefficients is defined as mapping $L:\mathcal{C}^2(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^2)\to\mathcal{C}^0(\mathbb{R}\times\mathbb{R}^2;\mathbb{R}^2)$ by

$$L(u) := \frac{h}{\mathrm{i}}\partial_t u - \frac{h^2}{m}\Delta u\,.$$

If we write $u=(u_1,u_2)=u_1+\mathrm{i}u_2$ this definition is equivalent to

$$L(u) = \begin{bmatrix} h\partial_t u_2 - \frac{h^2}{m}\Delta u_1 \\ -h\partial_t u_1 - \frac{h^2}{m}\Delta u_2 \end{bmatrix} = h\partial_t \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix} - \frac{h^2}{m}\Delta \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and it is $n=M=N=2$ in 7.1. *Constants:* The constants are m, the mass of the particle, and h, the Planck constant.

References: The Schrödinger equation you find in [**?**, –] and in [15, Sect. 1.5], where it is said: “It has the effect of not being Lorentz-invariant and therefore of not fitting in the relativistic formulation of quantum mechanics. It is still used as an approximation, but in a more rigorous setup, it has been replaced by Dirac’s equations.” See [15, Example 15.1].

It’s fundamental solution is given by

9.4 Fundamental solution of Schrödinger’s operator. ...... in Bearbeitung .....

Proof outside $(0,0)$. □

Proof in a neighbourhood of $(0,0)$. □

9.5 Wave operator. ...... in Bearbeitung .....

9.6 Fundamental solution of the wave operator. ...... in Bearbeitung .....

In Bearbeitung

10 Gradient

The following is true:

10.1 Theorem. If $U \in \mathscr{D}'(\mathcal{U};\mathbb{R})$ with

$$\partial_i U = F_i \text{ in } \mathscr{D}'(\mathcal{U};\mathbb{R}) \text{ for } i = 1,\dots,n\,, \tag{10.1}$$

where $F_i \in \mathscr{D}'(\mathcal{U};\mathbb{R})$, then

$$\partial_j F_i = \partial_i F_j \text{ in } \mathscr{D}'(\mathcal{U};\mathbb{R}) \text{ for } i,j = 1,\dots,n\,. \tag{10.2}$$

Proof. $\partial_j F_i = \partial_j \partial_i U = \partial_i \partial_j U = \partial_i F_j$, which is based on the basic fact, that the differential operators ∂_i and ∂_j commute on distributions, because they commute on test functions. □

Therefore, if $L_i = \partial_i$, that is

$$L = \nabla : \mathcal{C}^1(\mathbb{R}^n;\mathbb{R}) \to \mathcal{C}^0(\mathbb{R}^n;\mathbb{R}^n)$$

is the ***gradient operator***, this operator has no fundamental solution. Because, if it would have one, by Theorem 7.12 it would have a solution $U \in \mathscr{D}'(\mathbb{R}^n;\mathbb{R})$ of the equation (10.1) for any right side, therefore also for one which not satisfies (10.2). This is a contradiction.

11 Cauchy's principal value

The concept of distribution is particularly suitable for considering limits. To these limits belongs the Cauchy principal value.

We show the following statement about distributions that have a pointwise limit. This theorem is also called "sequential completeness of the space $\mathscr{D}'$", see Walter [16, §4 II], Jäger [9, 2.2 Satz], Gelfand & Schilow [6, Kapitel I §5 6].

11.1 Theorem. Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set and $(T_m)_{m\in\mathbb{N}}$ a sequence in $\mathscr{D}'(\mathcal{U};Y)$. If for all $\zeta \in \mathscr{D}(\mathcal{U};Y)$ the limit

$$\langle \zeta , T \rangle := \lim_{m\to\infty} \langle \zeta , T_m \rangle \tag{11.1}$$

exists, then $T \in \mathscr{D}'(\mathcal{U};Y)$. Moreover, for $U \subset\subset \mathcal{U}$ there exist $C_U \geq 0$ and $k_U \in \mathbb{N}_0$ such that for all $m \in \mathbb{N}$

$$|\langle \zeta , T_m \rangle| \leq C_U \|\zeta\|_{\mathcal{C}^{k_U}(\overline{U};Y)} \text{ for all } \zeta \in \mathcal{C}_0^\infty(U;Y)\,.$$

The assumption of the theorem says that $\lim_{m\to\infty} \langle \zeta , T_m \rangle$ exists for each ζ. Let l_ζ denote this limit. Then $\zeta \mapsto l_\zeta$ is a linear mapping and the theorem says that this map is indeed again an element in $\mathscr{D}'(\mathcal{U};Y)$, which is called T. The last statement of the theorem holds for this sequences, so that also the constants depend on this sequence.

Proof that $T \in \mathscr{D}'(\mathcal{U};Y)$ [16, §4 II]. We have to show that

$$\begin{aligned} &\forall\, U \subset\subset \mathcal{U} :\ \exists\, C \geq 0 :\ \exists\, k \in \mathbb{N}_0 : \\ &\forall\, \zeta \in \mathcal{C}_0^\infty(U) :\ |\langle \zeta , T \rangle| \leq C\|\zeta\|_{\mathcal{C}^k(\overline{U};Y)}\,. \end{aligned} \tag{11.2}$$

Assume this is not true, that is,

$$\begin{aligned} &\exists\, U \subset\subset \mathcal{U} :\ \forall\, C \geq 0 :\ \forall\, k \in \mathbb{N}_0 : \\ &\exists\, \zeta \in \mathcal{C}_0^\infty(U) :\ |\langle \zeta , T \rangle| > C\|\zeta\|_{\mathcal{C}^k(\overline{U};Y)}\,. \end{aligned} \tag{11.3}$$

Choose a sequence $(C_k)_{k\in\mathbb{N}}$ with $C_k \to \infty$ as $k \to \infty$. Then (11.3) implies

$$\begin{aligned} &\exists\, U \subset\subset \mathcal{U} :\ \forall\, k \in \mathbb{N} : \\ &\exists\, \zeta \in \mathcal{C}_0^\infty(U) :\ |\langle \zeta , T \rangle| > C_k\|\zeta\|_{\mathcal{C}^k(\overline{U};Y)}\,. \end{aligned} \tag{11.4}$$

Now let $U \subset\subset \mathcal{U}$ according to (11.4) and for $k \in \mathbb{N}$ let $\zeta_k \in \mathcal{C}_0^\infty(U)$ as in (11.4), that is,

$$|\langle \zeta_k , T \rangle| > C_k\|\zeta_k\|_{\mathcal{C}^k(\overline{U})}\,.$$

For definiteness let $C_k = 2^{2k}$. Then the modified functions (it is $\zeta_k \neq 0$)

$$\widetilde{\psi}_k := \frac{\zeta_k}{2^k\|\zeta_k\|_{\mathcal{C}^k(\overline{U})}}$$

satisfy, by the linearity of T (which follows trivially), $\left|\left\langle \widetilde{\psi}_k , T \right\rangle\right| > 2^k$. If we multiply $\widetilde{\psi}_k$ by a certain number σ_k with $|\sigma_k| = 1$ we obtain $\psi_k := \sigma_k\widetilde{\psi}_k$ satisfying

$$\operatorname{Re}\langle \psi_k , T \rangle > 2^k\,, \tag{11.5}$$

we take for example $\sigma_k = |\Big\langle\, \widetilde{\boldsymbol{\psi}}_{\boldsymbol{k}} \,,\boldsymbol{T}\,\Big\rangle|^{-1}\overline{\Big\langle\, \widetilde{\boldsymbol{\psi}}_{\boldsymbol{k}} \,,\boldsymbol{T}\,\Big\rangle}$. Obviously it is for all k

$$\operatorname{supp}\psi_k \subset U \quad \text{and} \quad \|\psi_k\|_{\mathcal{C}^k(\overline{U})} \le 2^{-k}\,, \tag{11.6}$$

therefore $\|\psi_k\|_{\mathcal{C}^l(\overline{U})} \to 0$ as $k \to \infty$ for all $l \in \mathbb{N}$. Since $T_m \in \mathscr{D}'(\mathcal{U};Y)$, there exist $C(T_m,U)$ and $k(T_m,U)$ with

$$|\,\langle\, \boldsymbol{\zeta} \,,\boldsymbol{T_m}\,\rangle\,| \le C(T_m,U)\|\zeta\|_{\mathcal{C}^{k(T_m,U)}(\overline{U})} \text{ for all } \zeta \in \mathscr{D}(U;Y)\,.$$

We conclude that $|\,\langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T_m}\,\rangle\,| \le C(T_m,U)\|\psi_k\|_{\mathcal{C}^{k(T_m,U)}(\overline{U})} \to 0$ as $k \to \infty$. Hence for each m

$$\lim_{k\to\infty} |\,\langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T_m}\,\rangle\,| = 0\,. \tag{11.7}$$

On the other hand, for fixed k, it follows $\langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T_m}\,\rangle \to \langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T}\,\rangle$ as $m \to \infty$, that is, by (11.5) for each k

$$\lim_{m\to\infty} \operatorname{Re}\langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T_m}\,\rangle > 2^k\,. \tag{11.8}$$

We now use (11.7) and (11.8) to construct subsequences $(R_i,\varphi_j) = (T_{m_i},\psi_{k_j})$ for $i,j \in \mathbb{N}$, so that

$$\begin{aligned} &\big|\,\langle\, \boldsymbol{\varphi_j} \,,\boldsymbol{R_i}\,\rangle\,\big| < 2^{-j} \text{ for } i < j,\\ &\operatorname{Re}\langle\, \boldsymbol{\varphi_j} \,,\boldsymbol{R_i}\,\rangle \ge 2^j \text{ for } i \ge j. \end{aligned} \tag{11.9}$$

This subsequences are constructed inductively in $l \in \mathbb{N}$, where for given l the inequalities are considered as

$$\begin{aligned} &\big|\,\langle\, \boldsymbol{\varphi_j} \,,\boldsymbol{R_i}\,\rangle\,\big| < 2^{-j} \text{ for } 1 \le i < j \le l,\\ &\operatorname{Re}\langle\, \boldsymbol{\varphi_j} \,,\boldsymbol{R_i}\,\rangle \ge 2^j \text{ for } 1 \le j \le i \le l. \end{aligned} \tag{11.10}$$

The base of induction is $l = 1$. Then the only relevant term is

$$\operatorname{Re}\langle\, \boldsymbol{\varphi_j} \,,\boldsymbol{R_i}\,\rangle \ge 2^j \text{ for } 1 = j = i = l = 1\,.$$

We set $\varphi_1 := \psi_1$. By (11.8) we have that the limit of $\operatorname{Re}\langle\, \boldsymbol{\psi_1} \,,\boldsymbol{T_m}\,\rangle$ as $m \to \infty$ is bigger than 2. Hence for some $m_1 \in \mathbb{N}$ we have

$$\operatorname{Re}\langle\, \boldsymbol{\psi_1} \,,\boldsymbol{T_m}\,\rangle \ge 2 \text{ for } m \ge m_1.$$

Set $R_1 := T_{m_1}$ which implies $\operatorname{Re}\langle\, \boldsymbol{\varphi_1} \,,\boldsymbol{R_1}\,\rangle \ge 2$.

We have to do the induction step from $l-1 \ge 1$ to l. First we fulfil

$$\big|\,\langle\, \boldsymbol{\varphi_j} \,,\boldsymbol{R_i}\,\rangle\,\big| < 2^{-j} \text{ for } 1 \le i < j \le l\,. \tag{11.11}$$

For $1 \le i < j < l$ (that is $1 \le i < j \le l-1$) the inequality has been shown before. Therefore let $1 \le i < j = l$. Since $i < l$ (that is $i \le l-1$) the R_i are already chosen as $R_i = T_{m_i}$. Thus it remains to show

$$|\,\langle\, \boldsymbol{\varphi_l} \,,\boldsymbol{T_{m_i}}\,\rangle\,| < 2^{-l} \text{ for } 1 \le i < l\,,$$

where we want to set $\varphi_l = \psi_{k_l}$. Now $1 \le i < l$ defines finitly many i and for each such i we know from (11.7) that $|\,\langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T_{m_i}}\,\rangle\,| \to 0$ as $k \to \infty$. Therefore we can choose a number $k_l \in \mathbb{N}$ ($k_l \ge l$) so that for all these i

$$|\,\langle\, \boldsymbol{\psi_k} \,,\boldsymbol{T_{m_i}}\,\rangle\,| < 2^{-l} \text{ for } k \ge k_l\,.$$

We set $\varphi_l := \psi_{k_l}$. Thus the first inequalities (11.11) are shown. Next we look for the second inequalities

$$\mathrm{Re}\langle \boldsymbol{\varphi_j} , \boldsymbol{R_i} \rangle \geq 2^j \text{ for } 1 \leq j \leq i \leq l\,. \tag{11.12}$$

The case $1 \leq j \leq i \leq l-1$ are already done, and in these inequalities the $\varphi_j = \psi_{k_j}$ for $1 \leq j \leq l$ are already chosen, in particular $\varphi_l = \psi_{k_l}$ before in this induction step. Thus the new terms are for $1 \leq j \leq i = l$ and we have to choose m_l and $R_l = T_{m_l}$. The relevant new terms are for $1 \leq j \leq l$

$$\mathrm{Re}\Big\langle \boldsymbol{\psi_{k_j}} , \boldsymbol{T_{m_l}} \Big\rangle$$

and they have to be estimated from below. We know from (11.8) that as $m \to \infty$ the term $\mathrm{Re}\Big\langle \boldsymbol{\psi_{k_j}} , \boldsymbol{T_m} \Big\rangle$ converges to a value bigger that 2^{k_j}. Therefore we can choose a number $m_l \in \mathbb{N}$ ($m_l \geq l$) so that for $1 \leq j \leq l$

$$\mathrm{Re}\Big\langle \boldsymbol{\psi_{k_j}} , \boldsymbol{T_m} \Big\rangle \geq 2^{k_j} \geq 2^j \text{ for } m \geq m_l\,.$$

We set $R_l := T_{m_l}$. Thus the second inequalities (11.12) are shown.

Therefore we have proved the inequalities in (11.9). If we define for $m \in \mathbb{N}$

$$\eta_m := \sum_{j=1}^m \varphi_j\,,$$

this implies for $m > i$

$$\begin{aligned}
&\mathrm{Re}\langle \boldsymbol{\eta_m} , \boldsymbol{R_i} \rangle = \sum_{j=1}^m \mathrm{Re}\langle \boldsymbol{\varphi_j} , \boldsymbol{R_i} \rangle \\
&\geq \sum_{j=1}^i \mathrm{Re}\langle \boldsymbol{\varphi_j} , \boldsymbol{R_i} \rangle + \sum_{j=i+1}^m \mathrm{Re}\langle \boldsymbol{\varphi_j} , \boldsymbol{R_i} \rangle \\
&\geq \sum_{j=1}^i 2^j - \sum_{j=i+1}^m 2^{-j} \geq \sum_{j=1}^i 2^j - \sum_{j=i+1}^\infty 2^{-j} \geq \sum_{j=1}^i 2^j - 2^{-i} \\
&\geq 2^i\,,
\end{aligned}$$

hence

$$\mathrm{Re}\langle \boldsymbol{\eta_m} , \boldsymbol{R_i} \rangle \to \infty \qquad \text{for } m > i \text{ with } i \to \infty. \tag{11.13}$$

This is in contradiction to what we show now. It exists

$$\eta := \lim_{m\to\infty} \eta_m = \sum_{j=1}^\infty \varphi_j\,, \quad \eta \in \mathscr{D}(\mathcal{U};Y)\,. \tag{11.14}$$

To show (11.14), we conclude from (11.6) that for every $k \in \mathbb{N}$, since $k_i \geq i \geq k$ for large i,

$$\begin{gathered}
\operatorname{supp}\varphi_i = \operatorname{supp}\psi_{k_i} \subset U\,, \\
\|\varphi_i\|_{\mathcal{C}^k(U)} = \|\psi_{k_i}\|_{\mathcal{C}^k(U)} \leq \|\psi_{k_i}\|_{\mathcal{C}^{k_i}(U)} \leq 2^{-k_i} \leq 2^{-i}\,,
\end{gathered}$$

which shows that η is pointwise defined and in addition η is in $\mathscr{D}(\mathcal{U};Y)$. Thus (11.14) is shown. Letting $\zeta_m := \eta - \eta_m \in \mathscr{D}(\mathcal{U};Y)$ it follows from 2.3 that

$$\langle \boldsymbol{\zeta_m} , \boldsymbol{R_i} \rangle \to 0 \text{ as } m \to \infty\,,$$

hence

$$\langle \boldsymbol{\eta} , \boldsymbol{R_i} \rangle = \lim_{m\to\infty} \langle \boldsymbol{\eta} - \boldsymbol{\zeta_m} , \boldsymbol{R_i} \rangle = \lim_{m\to\infty} \langle \boldsymbol{\eta_m} , \boldsymbol{R_i} \rangle\,. \tag{11.15}$$

Obviously, (11.15) contradicts (11.13). □

Proof of additional property [9, 2.2 Beweis des Satzes]. We assume that the first part is already proved, that is $T \in \mathscr{D}'(\mathcal{U};Y)$. We have to show that

$$\begin{aligned}&\forall\, U \subset\subset \mathcal{U} :\ \exists\, C \geq 0 :\ \exists\, k \in \mathbb{N}_0 :\\ &\forall\, m \in \mathbb{N} :\ \forall\, \zeta \in \mathcal{C}_0^\infty(U) :\ |\langle \boldsymbol{\zeta} , \boldsymbol{T_m} \rangle| \leq C\|\zeta\|_{\mathcal{C}^k(\overline{U};Y)} .\end{aligned} \tag{11.16}$$

Assume this is not true, that is,

$$\begin{aligned}&\exists\, U \subset\subset \mathcal{U} :\ \forall\, C \geq 0 :\ \forall\, k \in \mathbb{N}_0 :\\ &\exists\, m \in \mathbb{N} :\ \exists\, \zeta \in \mathcal{C}_0^\infty(U) :\ |\langle \boldsymbol{\zeta} , \boldsymbol{T_m} \rangle| > C\|\zeta\|_{\mathcal{C}^k(\overline{U};Y)} .\end{aligned} \tag{11.17}$$

Choose in particular $C = 2^{2k}$.Then (11.17) implies

$$\begin{aligned}&\exists\, U \subset\subset \mathcal{U} :\ \forall\, k \in \mathbb{N}_0 :\\ &\exists\, m \in \mathbb{N} :\ \exists\, \zeta \in \mathcal{C}_0^\infty(U) :\ |\langle \boldsymbol{\zeta} , \boldsymbol{T_m} \rangle| > 2^{2k}\|\zeta\|_{\mathcal{C}^k(\overline{U};Y)} .\end{aligned} \tag{11.18}$$

Now let $U \subset\subset \mathcal{U}$ according to (11.18) and for $k \in \mathbb{N}$ let T_{m_k} and $\zeta_k \in \mathcal{C}_0^\infty(U)$ as in (11.18), that is,

$$|\langle \boldsymbol{\zeta_k} , \boldsymbol{T_{m_k}} \rangle| > 2^{2k}\|\zeta_k\|_{\mathcal{C}^k(\overline{U})} .$$

Then

$$\widetilde{\psi}_k := \frac{\zeta_k}{2^k\|\zeta_k\|_{\mathcal{C}^k(\overline{U})}}$$

satisfy $\left|\left\langle \widetilde{\boldsymbol{\psi}}_{\boldsymbol{k}} , \boldsymbol{T_{m_k}} \right\rangle\right| > 2^k$ by the linearity of T_{m_k}. If we multiply $\widetilde{\psi}_k$ by the number

$$\sigma_k := \frac{\overline{\left\langle \widetilde{\boldsymbol{\psi}}_{\boldsymbol{k}} , \boldsymbol{T_{m_k}} \right\rangle}}{|\left\langle \widetilde{\boldsymbol{\psi}}_{\boldsymbol{k}} , \boldsymbol{T_{m_k}} \right\rangle|}$$

satisfying $|\sigma_k| = 1$ we obtain that $\psi_k := \sigma_k\widetilde{\psi}_k$ satisfies

$$\mathrm{Re}\,\langle \boldsymbol{\psi_k} , \boldsymbol{T_{m_k}} \rangle > 2^k \text{ for all } k \ , \tag{11.19}$$

and obviously it is for all k

$$\operatorname{supp}\psi_k \subset U \quad \text{ and } \quad \|\psi_k\|_{\mathcal{C}^k(\overline{U})} \leq 2^{-k} , \tag{11.20}$$

therefore $\|\psi_k\|_{\mathcal{C}^l(\overline{U})} \to 0$ as $k \to \infty$ for all $l \in \mathbb{N}$. Since $T_m \in \mathscr{D}'(\mathcal{U};Y)$, there exist $C(T_m,U)$ and $k(T_m,U)$ with

$$|\langle \boldsymbol{\zeta} , \boldsymbol{T_m} \rangle| \leq C(T_m,U)\|\zeta\|_{\mathcal{C}^{k(T_m,U)}(\overline{U})} \text{ for all } \zeta \in \mathscr{D}(U;Y) .$$

We conclude that $|\langle \boldsymbol{\psi_k} , \boldsymbol{T_m} \rangle| \leq C(T_m,U)\|\psi_k\|_{\mathcal{C}^{k(T_m,U)}(\overline{U})} \to 0$ as $k \to \infty$, that is for each m

$$|\langle \boldsymbol{\psi_k} , \boldsymbol{T_m} \rangle| \to 0 \text{ as } k \to \infty . \tag{11.21}$$

We now use (11.19) and (11.21) to construct subsequences $(R_i,\varphi_j) = (T_{m_{k_i}},\psi_{k_j})$ for $i,j \in \mathbb{N}$, which satisfy

$$\begin{aligned}&|\langle \boldsymbol{\varphi_j} , \boldsymbol{R_i} \rangle - \langle \boldsymbol{\varphi_j} , \boldsymbol{T} \rangle| < 2^{-j} \text{ for } j < i,\\ &|\langle \boldsymbol{\varphi_j} , \boldsymbol{R_i} \rangle| < 2^{-j} \text{ for } i < j,\\ &\mathrm{Re}\,\langle \boldsymbol{\varphi_i} , \boldsymbol{R_i} \rangle \geq 2^i \text{ for all } i.\end{aligned} \tag{11.22}$$

This subsequences are constructed inductively in $l \in \mathbb{N}$, where for given l the inequalities are considered as

$$\begin{aligned} &\big|\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle - \langle \boldsymbol{\varphi_j}, \boldsymbol{T}\rangle\big| < 2^{-j} \text{ for } 1 \le j < i \le l, \\ &\big|\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle\big| < 2^{-j} \text{ for } 1 \le i < j \le l, \\ &\mathrm{Re}\langle \boldsymbol{\varphi_i}, \boldsymbol{R_i}\rangle \ge 2^i \text{ for } 1 \le i \le l. \end{aligned} \tag{11.23}$$

The base of induction is $l = 1$. Then the only relevant term is

$$\mathrm{Re}\langle \boldsymbol{\varphi_i}, \boldsymbol{R_i}\rangle \ge 2^i \text{ for } i = l = 1.$$

We set $\varphi_1 := \psi_1$ and $R_1 := T_{m_1}$ and use (11.19).

We have to do the induction step from $l-1 \ge 1$ to l. First we consider the inequalities

$$\begin{aligned} \big|\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle - \langle \boldsymbol{\varphi_j}, \boldsymbol{T}\rangle\big| < 2^{-j} \text{ for } 1 \le j < i = l, \\ \big|\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle\big| < 2^{-j} \text{ for } 1 \le i < j = l. \end{aligned} \tag{11.24}$$

For the first inequality, that is for $1 \le j \le l-1$, we use that

$$\big|\langle \boldsymbol{\varphi_j}, \boldsymbol{T_{m_k}}\rangle - \langle \boldsymbol{\varphi_j}, \boldsymbol{T}\rangle\big| < 2^{-j}$$

for k large enough, since the sequence $(T_m)_m$ converges to T. Further by (11.21) we know that for $1 \le i \le l-1$

$$|\langle \boldsymbol{\psi_k}, \boldsymbol{R_i}\rangle| < 2^{-l}$$

for k large enough. Thus we can choose in both inequalities $k = k_l \ge l$ and obtain that

$$\begin{aligned} &\Big|\Big\langle \boldsymbol{\varphi_j}, \boldsymbol{T_{m_{k_l}}}\Big\rangle - \langle \boldsymbol{\varphi_j}, \boldsymbol{T}\rangle\Big| < 2^{-j} \text{ for } 1 \le j < l, \\ &\big|\langle \boldsymbol{\psi_{k_l}}, \boldsymbol{R_i}\rangle\big| < 2^{-l} \text{ for } 1 \le i < l. \end{aligned}$$

We now define $\varphi_l := \psi_{k_l}$ and $R_l := T_{m_{k_l}}$ to satisfy (11.24). It follows using (11.19)

$$\mathrm{Re}\langle \boldsymbol{\varphi_l}, \boldsymbol{R_l}\rangle = \mathrm{Re}\Big\langle \boldsymbol{\psi_{k_l}}, \boldsymbol{T_{m_{k_l}}}\Big\rangle \ge 2^{k_l} \ge 2^l\,,$$

that is the inequality

$$\mathrm{Re}\langle \boldsymbol{\varphi_l}, \boldsymbol{R_l}\rangle \ge 2^l. \tag{11.25}$$

Thus (11.24) and (11.25) show the induction step. Therefore we have proved the inequalities in (11.22). If we define for $m \in \mathbb{N}$

$$\eta_m := \sum_{j=1}^{m} \varphi_j\,,$$

this implies for $m > i$

$$\begin{aligned}
&\operatorname{Re}\langle \boldsymbol{\eta_m}, \boldsymbol{R_i}\rangle = \sum_{j=1}^{m} \operatorname{Re}\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle \\
&= \operatorname{Re}\langle \boldsymbol{\varphi_i}, \boldsymbol{R_i}\rangle + \sum_{j=i+1}^{m} \operatorname{Re}\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle \\
&+ \sum_{j=1}^{i-1} \operatorname{Re}\big(\langle \boldsymbol{\varphi_j}, \boldsymbol{R_i}\rangle - \langle \boldsymbol{\varphi_j}, \boldsymbol{T}\rangle\big) + \operatorname{Re}\Big\langle \sum_{j=1}^{i-1} \boldsymbol{\varphi_j}, \boldsymbol{T}\Big\rangle \\
&\geq 2^i - \sum_{j=i+1}^{\infty} 2^{-j} - \sum_{j=1}^{i-1} 2^{-j} + \operatorname{Re}\langle \boldsymbol{\eta_{i-1}}, \boldsymbol{T}\rangle \\
&\geq 2^i - 2 + \operatorname{Re}\langle \boldsymbol{\eta_{i-1}}, \boldsymbol{T}\rangle\,.
\end{aligned}$$

Since it is assumed that (the first part is already proved) we know $T \in \mathscr{D}'(\mathcal{U}; Y)$, it follows that

$$\sup_i |\langle \boldsymbol{\eta_{i-1}}, \boldsymbol{T}\rangle| \leq \text{const},$$

that is because $\operatorname{supp}\eta_{i-1} \subset U$ and $\|\eta_{i-1}\|_{\mathcal{C}^k(\overline{U})} \leq 1$ (see the end of the first part of the proof). Hence it follows that

$$\operatorname{Re}\langle \boldsymbol{\eta_m}, \boldsymbol{R_i}\rangle \to \infty \quad \text{for } m > i \text{ with } i \to \infty. \tag{11.26}$$

This is in contradiction to what has been shown at the end of the first part of the proof. □

References: Jäger [9, §2], Walter [16, §4 I,II,IX Der Cauchysche Hauptwert], [Wikipedia: Cauchy principal value], [Wikipedia: Cauchyscher Hauptwert].

Wir nehmen nun an, dass $T_\varepsilon \in \mathscr{D}'(\mathcal{U}; \mathbb{R})$, $\varepsilon > 0$, und dass

$$\langle \boldsymbol{\zeta}, \boldsymbol{T}\rangle := \lim_{\varepsilon \searrow 0} \langle \boldsymbol{\zeta}, \boldsymbol{T_\varepsilon}\rangle \tag{11.27}$$

für alle $\zeta \in \mathscr{D}(\mathcal{U}; \mathbb{R})$ existiert. Dann ist $T \in \mathscr{D}'(\mathcal{U}; \mathbb{R})$ nach 11.1, denn es gilt für jede Folge $(\varepsilon_m)_{m\in\mathbb{N}}$ von gegen Null konvergenten positiven Zahlen, dass

$$\lim_{m\to\infty} \langle \boldsymbol{\zeta}, \boldsymbol{T_{\varepsilon_m}}\rangle$$

existiert, und da der Limes gleich $\langle \boldsymbol{\zeta}, \boldsymbol{T}\rangle$ ist, folgt $T \in \mathscr{D}'(\mathcal{U}; \mathbb{R})$. Das wird nun ausgenutzt für uneigentliche Integrale, bei denen sich positive und negative Terme gegenseitig wegheben.

11.2 Cauchy'scher Haupwert. Es sei $\mathcal{U} \subset \mathbb{R}^n$ offen und beschränkt, sowie $\Sigma \subset \mathbb{R}^n$ abgeschlossen. Sind dann für $\varepsilon > 0$

$$g_\varepsilon \in \mathcal{C}^0(\overline{\mathcal{U}} \setminus \Sigma)$$

und existiert für alle $\zeta \in \mathscr{D}(\mathcal{U})$

$$\langle \boldsymbol{\zeta}, \boldsymbol{S}\rangle_{\mathscr{D}(\mathcal{U})} := \lim_{\varepsilon \searrow 0} \int_{\mathcal{U}\setminus \mathrm{B}_\varepsilon(\Sigma)} \zeta g_\varepsilon \,\mathrm{dL}^n\,,$$

so ist $S \in \mathscr{D}'(\mathcal{U})$. Der Integrallimes wird ***Cauchy'scher Haupwert*** genannt.

Dieser Wert wird auch mit "CH" (*de:* Cauchy'scher Haupwert), "V.P." (*fr:* valeur principale), "P.V." (*en:* principal value) bezeichnet.

Proof of $S \in \mathscr{D}'(\mathcal{U})$. Es sind durch

$$\langle \zeta, T_\varepsilon \rangle_{\mathscr{D}(\mathcal{U})} := \int_{\mathcal{U} \setminus \mathrm{B}_\varepsilon(\Sigma)} \zeta g_\varepsilon \, \mathrm{dL}^n$$

Distributionen $T_\varepsilon \in \mathscr{D}'(\mathcal{U})$ definiert, für die

$$\langle \zeta, S \rangle_{\mathscr{D}(\mathcal{U})} := \lim_{\varepsilon \searrow 0} \langle \zeta, T_\varepsilon \rangle_{\mathscr{D}(\mathcal{U})}$$

nach Voraussetzung existiert. Also ist 11.1 anwendbar. □

11.3 Example. Let $\Omega = \,]-1,1[\, \in \mathbb{R}$. The definition

$$\langle \zeta, S \rangle := \lim_{\varepsilon \searrow 0} \int_{\Omega \setminus]-\varepsilon,\varepsilon[} \frac{\zeta(x)}{x} \, \mathrm{d}x$$

defines a distribution.

Proof. We compute for $x \neq 0$

$$\frac{\zeta(x)}{x} = \zeta(x) \frac{\mathrm{d}}{\mathrm{d}x} \log(|x|) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\zeta(x)\log(|x|)\Big) - \zeta'(x)\log(|x|),$$

and therefore

$$\int_{\Omega \setminus]-\varepsilon,\varepsilon[} \frac{\zeta(x)}{x} \, \mathrm{d}x = -\Big[\, \zeta(x)\log(|x|) \,\Big]_{x=-\varepsilon}^{x=+\varepsilon} - \int_{\Omega \setminus]-\varepsilon,\varepsilon[} \zeta'(x)\log(|x|)\, \mathrm{d}x$$

with

$$\Big[\, \zeta(x)\log(|x|) \,\Big]_{x=-\varepsilon}^{x=+\varepsilon} = \log(\varepsilon)(\zeta(\varepsilon) - \zeta(-\varepsilon)) \to 0 \text{ as } \varepsilon \to 0$$

and

$$\int_{\Omega \setminus]-\varepsilon,\varepsilon[} \zeta'(x)\log(|x|)\, \mathrm{d}x \to \int_\Omega \zeta'(x)\log(|x|)\, \mathrm{d}x \text{ as } \varepsilon \to 0\,.$$

Hence 11.2, that is 11.1, can be applied with $n = 1$ and $\Sigma = \{0\}$. □

12 Topology

This section is independent of all other sections. We introduce a topology in $\mathscr{D}(\mathcal{U};Y) = \mathcal{C}_0^\infty(\mathcal{U};Y)$ and consider the dual space $\mathscr{D}(\mathcal{U};Y)^*$. The outcome will be that [7]

$$\mathscr{D}'(\mathcal{U};Y) = \mathscr{D}(\mathcal{U};Y)^* .$$

It is the result of the following procedure: The vector space $\mathcal{C}_0^\infty(\mathcal{U};Y)$ can be equipped with a topology $\mathcal{T}$ in such a way that T is a distribution if and only if T lies in the corresponding dual space, i.e. if $T : \mathcal{C}_0^\infty(\mathcal{U};Y) \to \mathbb{K}$ is linear and continuous with respect to the topology $\mathcal{T}$. We denote $\mathcal{C}_0^\infty(\mathcal{U};Y)$, equipped with the topology $\mathcal{T}$, by $\mathscr{D}(\mathcal{U};Y)$. The dual space is denoted by $\mathscr{D}(\mathcal{U};Y)^*$.

References: Alt [1, Section 3: Distributions], Jäger [9, Kapitel II §1: Die Topologie des Grundraums] and all other mathematical publications concerning the subject.

12.1 Topology on $\mathcal{C}_0^\infty(\mathcal{U};Y)$. Let $\mathcal{U} \subset \mathbb{R}^n$ be open. Define

$$p(\zeta) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|\zeta\|_{\mathcal{C}^k(\overline{D})}}{1 + \|\zeta\|_{\mathcal{C}^k(\overline{D})}} \quad \text{for } \zeta \in \mathcal{C}_0^\infty(\mathcal{U};Y) \text{ with } \mathrm{supp}(\zeta) \subset D \subset\subset \mathcal{U} ,$$

where the right-hand side is independent of the choice of D. Choose an open cover $(D_j)_{j\in\mathbb{N}}$ of $\mathcal{U}$ with sets $D_j \subset\subset D_{j+1} \subset \mathcal{U}$ for all $j \in \mathbb{N}$. For every sequence $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ with $\varepsilon_j > 0$ for $j \in \mathbb{N}$ define

$$V_\varepsilon := \mathrm{conv}\Big(\bigcup_{j\in\mathbb{N}} \{ \zeta \in \mathcal{C}_0^\infty(\mathcal{U};Y) \ ; \ \mathrm{supp}(\zeta) \subset D_j \text{ and } p(\zeta) < \varepsilon_j \} \Big) .$$

Finally, define

$$\mathcal{T} := \big\{ V \subset \mathcal{C}_0^\infty(\mathcal{U};Y) \ ; \ \text{For } \zeta \in V \text{ there exists an } \varepsilon \text{ with } \zeta + V_\varepsilon \subset V \big\} .$$

Then we can show that $\mathcal{T}$ defines a topology.

12.2 Lemma. The set $\mathcal{T}$ satisfies:

(1) p is a Fréchet metric with $p(r\zeta) \le rp(\zeta)$ for $r \ge 1$.

(2) For all ε it holds that $V_\varepsilon \in \mathcal{T}$.

(3) $\mathcal{T}$ is a topology. Hence the sets V_ε form a basis of neighbourhoods of 0 with respect to $\mathcal{T}$.

(4) $\mathcal{T}$ is independent of the choice of the cover $(D_j)_{j\in\mathbb{N}}$.

We remark that $\mathcal{T}$ is stronger than the topology induced by p. This follows from the fact that the p-ball $\mathrm{B}_\varrho(0) \subset \mathcal{C}_0^\infty(\mathcal{U})$ is a neighbourhood in the $\mathcal{T}$-topology, namely, $\mathrm{B}_\varrho(0) = V_\varepsilon$ with $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ and $\varepsilon_j = \varrho$.

[7] Now the prime is justified.

Proof (2). Let $\zeta \in V_\varepsilon$. Consider a finite convex combination

$$\zeta = \sum_{k=1}^{k_0} \alpha_k \zeta_k \in V_\varepsilon \quad \text{with } k_0 \in \mathbb{N},\ \alpha_k > 0,\ \sum_{k=1}^{k_0} \alpha_k = 1\ , \tag{12.1}$$

where $\zeta_k \in \mathcal{C}_0^\infty(D_{j_k})$ with $p(\zeta_k) < \varepsilon_{j_k}$. Choose $0 < \theta < 1$ such that $p(\zeta_k) < \theta \varepsilon_{j_k}$ for all $k = 1, \dots, k_0$, and set $\delta = (\delta_j)_{j \in \mathbb{N}}$ with $\delta_j := (1-\theta)\varepsilon_j$. We claim that $\zeta + V_\delta \subset V_\varepsilon$. To see this, let

$$\eta = \sum_{l=1}^{l_0} \beta_l \eta_l \in V_\delta \quad \text{with } l_0 \in \mathbb{N},\ \beta_l > 0,\ \sum_{l=1}^{l_0} \beta_l = 1\ ,$$

where $\eta_l \in \mathcal{C}_0^\infty(D_{m_l})$ with $p(\eta_l) < \delta_{m_l}$. Then, on noting (1),

$$p\big(\tfrac{1}{\theta}\zeta_k\big) \le \tfrac{1}{\theta} p(\zeta_k) < \varepsilon_{j_k} \quad \text{and} \quad p\big(\tfrac{1}{1-\theta}\eta_l\big) \le \tfrac{1}{1-\theta} p(\eta_l) < \varepsilon_{m_l}\ ,$$

i.e. $\frac{1}{\theta}\zeta_k$ and $\frac{1}{1-\theta}\eta_l$ are elements of V_ε. Hence the convexity of V_ε yields that

$$\zeta + \eta = \theta \sum_{k=1}^{k_0} \alpha_k \cdot \tfrac{1}{\theta}\zeta_k + (1-\theta) \sum_{l=1}^{l_0} \beta_l \cdot \tfrac{1}{1-\theta}\eta_l \in V_\varepsilon\ .$$

This shows that $V_\varepsilon \in \mathcal{T}$. ☐

Proof (3). We need to show that $U^1 \cap U^2 \in \mathcal{T}$, if $U^1, U^2 \in \mathcal{T}$. But this follows from $V_\varepsilon \subset V_{\varepsilon^1} \cap V_{\varepsilon^2}$, where $\varepsilon_j := \min(\varepsilon_j^1, \varepsilon_j^2)$ for $j \in \mathbb{N}$. ☐

Proof (4). Let $\big(\widetilde{D}_j\big)_{j \in \mathbb{N}}$ be another cover and let $\widetilde{V}_{\widetilde{\varepsilon}}$ with $\widetilde{\varepsilon} = (\widetilde{\varepsilon}_j)_{j \in \mathbb{N}}$ be a set defined as above, now with respect to this cover. Since $\overline{D}_j$ is compact with $\overline{D}_j \subset \mathcal{U}$, for each $j \in \mathbb{N}$ there exists an $m_j \in \mathbb{N}$ with $D_j \subset \widetilde{D}_{m_j}$. Setting $\varepsilon_j := \widetilde{\varepsilon}_{m_j}$ for $j \in \mathbb{N}$ and $\varepsilon = (\varepsilon_j)_{j \in \mathbb{N}}$ then yields that $V_\varepsilon \subset \widetilde{V}_{\widetilde{\varepsilon}}$. ☐

12.3 The space $\mathscr{D}(\mathcal{U})$. We denote the vector space $\mathcal{C}_0^\infty(\mathcal{U})$, equipped with the topology $\mathcal{T}$ from 12.1, by $\mathscr{D}(\mathcal{U})$. Then $\mathscr{D}(\mathcal{U})$ is a ***locally convex topological vector space***, i.e. it holds that:

(1) $\mathscr{D}(\mathcal{U})$ with $\mathcal{T}$ is a Hausdorff space.

(2) $\mathscr{D}(\mathcal{U})$ is a vector space and addition and scalar multiplication are continuous (as maps from $\mathscr{D}(\mathcal{U}) \times \mathscr{D}(\mathcal{U})$ to $\mathscr{D}(\mathcal{U})$ and from $\mathbb{K} \times \mathscr{D}(\mathcal{U})$ to $\mathscr{D}(\mathcal{U})$, respectively).

(3) For $\zeta \in U$ with $U \in \mathcal{T}$ there exists a convex set $V \in \mathcal{T}$ with $\zeta \in V \subset U$.

Proof (3). By their definition, the sets V_ε in 12.1 are convex. ☐

Proof (2). We claim for every V_ε that $V_\delta + V_\delta \subset V_\varepsilon$, where $\delta = (\delta_j)_{j \in \mathbb{N}}$ with $\delta_j := \frac{1}{2}\varepsilon_j$, which implies the continuity of the addition. For the proof let

$$\zeta_l \in \mathcal{C}_0^\infty\big(D_{j_l}\big) \quad \text{with } p\big(\zeta_l\big) < \delta_{j_l} \text{ for } l = 1, 2.$$

We have that $\zeta_1 + \zeta_2 = \frac{1}{2}(2\zeta_1 + 2\zeta_2)$ with $p(2\zeta_l) \le 2p(\zeta_l) \le 2\delta_{j_l} = \varepsilon_{j_l}$, and so $\zeta_1 + \zeta_2 \in V_\varepsilon$, as V_ε is convex. Then the same also holds for arbitrary elements $\zeta_1, \zeta_2 \in V_\delta$.

In order to show the continuity of the scalar multiplication at the point $(\alpha_0, \zeta_0) \in \mathbb{K} \times \mathscr{D}(\mathcal{U})$, let V_ε be given. Let $\zeta_0 \in \mathcal{C}_0^\infty(D_{j_0})$ and write

$$\alpha\zeta - \alpha_0\zeta_0 = \frac{1}{2}\big(2(\alpha-\alpha_0)\zeta_0 + 2\alpha(\zeta-\zeta_0)\big) \ .$$

Let $|\,\alpha - \alpha_0\,| < \gamma \leq \frac{1}{2}$ and let $\zeta - \zeta_0 \in \mathcal{C}_0^\infty(D_j)$ with $p(\zeta-\zeta_0) < \delta_j$, where γ, δ_j need to be chosen. Now it holds that $\|2\gamma\zeta_0\|_{\mathcal{C}^k(\overline{D}_{j_0})} \to 0$ as $\gamma \to 0$ for all $k \in \mathbb{N}$, and so it follows

$$p\big(2(\alpha-\alpha_0)\zeta_0\big) \leq p(2\gamma\zeta_0) \to 0 \qquad \text{as } \gamma \to 0.$$

If we now choose $\gamma \leq \frac{1}{2}$ with $p(2\gamma\zeta_0) < \varepsilon_{j_0}$, then $2(\alpha-\alpha_0)\zeta_0 \in V_\varepsilon$. In addition, since $|\,2\alpha\,| \leq 2(|\,\alpha_0\,| + \gamma) \leq 2\,|\,\alpha_0\,| + 1$,

$$p\big(2\alpha(\zeta-\zeta_0)\big) \leq (1 + 2\,|\,\alpha_0\,|)p(\zeta-\zeta_0) < \varepsilon_j \ ,$$

if we set $\delta_j := (1+2\,|\,\alpha_0\,|)^{-1}\varepsilon_j$. This implies that also $2\alpha(\zeta-\zeta_0) \in V_\varepsilon$, and hence $\alpha\zeta \in \alpha_0\zeta_0 + V_\varepsilon$. Then the same also follows for all $\zeta \in \zeta_0 + V_\delta$, where $\delta := (\delta_j)_{j\in\mathbb{N}}$. □

Proof (1). Let $\zeta^1, \zeta^2 \in \mathscr{D}(\mathcal{U})$ with $\zeta^1 \neq \zeta^2$ and $\zeta := \zeta^1 - \zeta^2$. We claim that

$$\big(\zeta^1 + V_\varepsilon\big) \cap \big(\zeta^2 + V_\varepsilon\big) = \emptyset \ ,$$

if $\varepsilon = (\varrho)_{j\in\mathbb{N}}$ and $\varrho > 0$ is sufficiently small. Indeed, if $\eta^1, \eta^2 \in V_\varepsilon$ with $\zeta^1 + \eta^1 = \zeta^2 + \eta^2$, then also $-\eta^1 \in V_\varepsilon$, and so

$$\zeta = \zeta^1 - \zeta^2 = (-\eta^1) + \eta^2 \in V_\varepsilon + V_\varepsilon \subset V_{2\varepsilon} \ ,$$

on recalling the proof of (2). Now write ζ as a convex combination as in (12.1), so that

$$\frac{\|\zeta_k\|_{\mathcal{C}^0}}{1 + \|\zeta_k\|_{\mathcal{C}^0}} \leq p(\zeta_k) < 2\varrho \ .$$

This implies, if $\varrho < \frac{1}{2}$, that

$$0 \neq \|\zeta\|_{\mathcal{C}^0} \leq \textstyle\sum_{k=1}^{k_0} \alpha_k \|\zeta_k\|_{\mathcal{C}^0} \leq \max_{k=1,\dots,k_0} \|\zeta_k\|_{\mathcal{C}^0} < \frac{2\varrho}{1-2\varrho} \ ,$$

which is not possible, if ϱ depending on ζ was chosen sufficiently small. □

12.4 Lemma. For every sequence $(\zeta_m)_{m\in\mathbb{N}}$ in $\mathscr{D}(\mathcal{U})$ it holds that

$$\zeta_m \to 0 \qquad \text{as } m \to \infty \text{ in } \mathscr{D}(\mathcal{U})$$

if and only if:

(1) There exists an open $D \subset\subset \mathcal{U}$, such that $\zeta_m \in \mathcal{C}_0^\infty(D)$ for all m.

(2) For all $D \subset\subset \mathcal{U}$ and all $k \in \mathbb{N}$ it holds that $\|\zeta_m\|_{\mathcal{C}^k(\overline{D})} \to 0$ as $m \to \infty$.

Proof $\Leftarrow$. On noting that $\overline{D}$ is compact and $\overline{D} \subset \mathcal{U}$, the cover in 12.1 contains a D_j such that $D \subset D_j$. Then for a given ε it follows that for ζ with $\mathrm{supp}\,(\zeta) \subset D$ and all $\bar{k}$

$$p(\zeta) \le \sum_{k=0}^{\bar{k}} 2^{-k} \|\zeta\|_{\mathcal{C}^k(\overline{D})} + \sum_{k=\bar{k}+1}^{\infty} 2^{-k} \le \|\zeta\|_{\mathcal{C}^{\bar{k}}(\overline{D})} + 2^{-\bar{k}} ,$$

which gives $p(\zeta_m) < \varepsilon_j$ for large m, if $\bar{k}$ is chosen with $2^{-\bar{k}} \le \frac{\varepsilon_j}{2}$ and m by (2) is chosen large enough that $\|\zeta_m\|_{\mathcal{C}^{\bar{k}}(\overline{D})} \le \frac{\varepsilon_j}{2}$. So $\zeta_m \in V_\varepsilon$. ☐

Proof $\Rightarrow$. If we assume that (1) is not satisfied, then there exist an open cover $(D_j)_{j\in\mathbb{N}}$ of $\mathcal{U}$ with $D_j \subset\subset \mathcal{U}$ and $D_{j-1} \subset D_j$, as well as $x_j \in D_j \setminus \overline{D_{j-1}}$ and a subsequence $m_j \to \infty$, such that $\zeta_{m_j}(x_j) \ne 0$. Then

$$U := \Big\{ \zeta \in \mathscr{D}(\mathcal{U})\ ;\ \sum_{j\in\mathbb{N}} \frac{2}{|\,\zeta_{m_j}(x_j)\,|} \|\zeta\|_{\mathcal{C}^0(\overline{D_j}\setminus D_{j-1})} \le 1 \Big\}$$

is a convex subset of $\mathscr{D}(\mathcal{U})$. On noting that for all j

$$\big\{ \zeta \in \mathcal{C}_0^\infty(D_j)\ ;\ p(\zeta) < \varepsilon_j \big\} \subset U\ , \quad \text{where} \quad \varepsilon_j := \Big(1 + \sum_{i\le j} \frac{2}{|\,\zeta_{m_i}(x_i)\,|} \Big)^{-1} ,$$

we have that $V_\varepsilon \subset U$, if $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ and V_ε is defined with respect to the cover $(D_j)_{j\in\mathbb{N}}$. The definition of the topology and the fact that $\zeta_m \to 0$ in $\mathscr{D}(\mathcal{U})$ as $m \to \infty$ yield that $\zeta_m \in V_\varepsilon$ for large m. But it follows from the construction of U that the ζ_{m_j} do not lie in U, a contradiction. This shows (1).
Now for $k \in \mathbb{N}$ and $\delta > 0$ choose $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ with $2^k \varepsilon_j = \big(1 + \frac{1}{\delta}\big)^{-1} > 0$ for all j, which yields that

$$V_\varepsilon \subset \big\{ \zeta \in \mathcal{C}_0^\infty(\mathcal{U})\ ;\ \|\zeta\|_{\mathcal{C}^k} \le \delta \big\}\ .$$

For large m we have that $\zeta_m \in V_\varepsilon$, and so $\|\zeta_m\|_{\mathcal{C}^k} \le \delta$. This shows (2). ☐

12.5 The dual space of $\mathscr{D}(\mathcal{U})$. Consider the dual space

$$\mathscr{D}(\mathcal{U})^* = \{T : \mathscr{D}(\mathcal{U}) \to \mathbb{K}\,;\ T \text{ is linear and continuous}\}$$

of $\mathcal{D}(\mathcal{U})$. Then

$$\mathscr{D}(\mathcal{U})^* = \mathscr{D}'(\mathcal{U})\ .$$

Proof $\subset$. Let $T \in \mathscr{D}(\mathcal{U})^*$. If $T \notin \mathscr{D}'(\mathcal{U})$, then there exist a $D \subset\subset \mathcal{U}$ and $\zeta_m \in \mathcal{C}_0^\infty(D)$ with

$$1 = |\,T\zeta_m\,| > m\|\zeta_m\|_{\mathcal{C}^m(\overline{D})} \quad \text{for } m \in \mathbb{N}.$$

For all $k \in \mathbb{N}$ it then follows that $\|\zeta_m\|_{\mathcal{C}^k(\overline{D})} \to 0$ as $m \to \infty$, and so 12.4 yields $\zeta_m \to 0$ as $m \to \infty$ in $\mathscr{D}(\mathcal{U})$. Now the continuity of T implies that $T\zeta_m \to 0$ as $m \to \infty$, which is a contradiction. ☐

Proof ⊃. Let $T \in \mathscr{D}'(\mathcal{U})$, let $(D_j)_{j\in\mathbb{N}}$ be the exhaustion from 12.1 and let

$$|\,T\zeta\,| \le C_j \|\zeta\|_{\mathcal{C}^{k_j}(\overline{D_j})} \quad \text{for } \zeta \in \mathcal{C}_0^\infty(D_j).$$

For $\delta > 0$ let $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ be defined by $\varepsilon_j := 2^{-k_j} \frac{\delta}{C_j+\delta}$. Then it holds that

$$\zeta \in \mathcal{C}_0^\infty(D_j) \text{ with } p(\zeta) < \varepsilon_j \quad \Longrightarrow \quad |\,T\zeta\,| \le C_j \|\zeta\|_{\mathcal{C}^{k_j}(\overline{D_j})} \le \delta\ .$$

As T is linear, it follows that $|\,T\zeta\,| \le \delta$ for all $\zeta \in V_\varepsilon$ (with V_ε as in 12.1). This proves the continuity of T. □

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