On the Existence of a Solution for a Model of Stem Cell Differentiation

Gabriele Witterstein
Munich University of Technology, Center for Mathematical Sciences,
Boltzmannstr. 3, D - 85747 Garching bei München,
E-Mail: gw@ma.tum.de,
Phone: +49-89 28916832, Fax:+49-89 28916809

SUMMARY
In two dimensions we regard a model describing the biological material change in the process of stem cell differentiation. The arising system consists in the Navier-Stokes equations for viscous, compressible, isothermal flow and models a creep movement mainly caused by a mass production and affected by the change-over in the stem cell culture. These equations are coupled with an Allen-Cahn equation modeling the solidification of a liquid. As the main result, we show the existence of a weak, spherically symmetric solution in a plain domain.

KEY WORDS:
Compressible Navier-Stokes equations, Allen-Cahn equation, Modeling biomaterials.

1 Introduction
Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with a smooth boundary and $\Omega_T:=(0,T] \times \Omega$. We shall study the following system of equations in $\Omega_T$:

$$
\rho_t + \text{div} (\rho v) = \lambda,
$$

(1)

$$
(\rho v)_t + \text{div} (\rho v \otimes v + \Pi) = v \lambda,
$$

(2)
\[ \Phi_t + v \cdot \nabla \Phi = -W_0 \Phi - A \Phi - \left[ c p_1(\theta) - c p_2(\theta) \right] \rho + \delta^2 \Delta \Phi + (1 - \Phi) \frac{\lambda}{\rho}, \quad (3) \]

with \( W_0 \Phi = c_1 \ln \left( \frac{\Phi}{1 - \Phi} \right) \) and \( A \Phi = c_2 (1 - 2\Phi) \).

Hereby applies following boundary and radially symmetric, initial conditions for \((\rho, \rho v, \Phi)\):

\[ \vec{n} \cdot \nabla \Phi = 0 \quad \text{on} \quad (0,T) \times \partial \tilde{\Omega}, \quad (4) \]

and

\[ \rho(0,x) = \rho_0(\|x\|) \quad \text{in} \quad \tilde{\Omega}, \quad (5) \]

\[ (\rho v)(0,x) = \rho_0(\|x\|) v_0(\|x\|) x/\|x\| \quad \text{in} \quad \tilde{\Omega}, \quad (6) \]

\[ \Phi(0,x) = \Phi_0(\|x\|) \quad \text{in} \quad \tilde{\Omega}, \quad (7) \]

with \( \Phi_0(\|x\|) \in H^{1,2}(\tilde{\Omega}), 0 \leq \Phi_0 \leq 1 \). As usual \( \vec{n} \) denotes the outer normal vector of \( \partial \tilde{\Omega} \). The function \( \rho \) represents the total mass, \( v \) the velocity, where as \( \Phi \) distinguishes between liquid and solid sub-domains of \( \tilde{\Omega}_T \). The initial functions \( \rho_0(\|x\|) \) and \( (\rho_0 v_0)(\|x\|) \) are supposed to be in \( C^1(\tilde{\Omega}) \).

Further \( \Pi := (\Pi_k)_{k=1,2} :=(\Pi_{ki})_{k,i=1,2} \), denotes the usual stress tensor containing the pressure und the viscosity coefficients. Then \( \Pi \) is given by

\[ \Pi = +p_f I - \tau + \delta^2 \nabla \Phi \otimes \nabla \Phi \]

and

\[ \tau = \mu_1 \text{div} v I + \mu_2 \left( \frac{\partial}{\partial x_i} v_j + \frac{\partial}{\partial x_j} v_i \right)_{ij}. \]

Equation (1) describes the mass transport in a velocity field \( v \). Mass doesn’t vanish, but they is produced out by the mass production \( \lambda > 0 \). Equation (2) models the impulse conservation with additional impulse source \( \lambda v \), emerging necessarily by the objectivity principle. And equation (3) describes the phase transition from liquid to solid induced by an inorganic substance. The movement of the free boundary is independent of the temperature change, but is determined by the chemical potential of the different phases. \( c p_1 \equiv c p_1(\theta) \) and \( c p_2 \equiv c p_2(\theta) \) are constants independent on the absolute temperature \( \theta \). \( c_1, c_2 > 0 \) are specific material constants. \( \delta \) determines the thickness of the transition layer between the pure phases. \( \mu_1, \mu_2 \) are real, positive constants, stands for the viscosity coefficients. In this process, the temperature plays no role, but the compressibility of the fluids. The pressure \( p \equiv p_f \) induced by the free energy \( f \), is a given function depending on the total mass density \( \rho \) and the phase parameter \( \Phi \).

A feasible way to model the change-over from liquid to solid in the Navier-Stokes equations relies on the controlling of the variable, viscosity coefficients \( \mu_1 \) and \( \mu_2 \).
depending on $\rho$ and $\Phi$. If $\Phi=0$, that means, we regard solid material, the $\mu_1$ and $\mu_2$ have to be grown up to an huge number. In the physical view this stops the possibility of movement of a mass particle and simulate that each mass particle is incorporated in a fixed lattice structure as it is usual in solid bodies.

But this concept we don’t follow in this paper. Here we handle the problem by an effective definition of the pressure function (section 2, assumption 5). The idea of the definition is, we try to approach the pressure to a constant function at such a rate as the material solidify.

This paper is organized as follows. In section 2 we give the weak formulation of the problem and concretize the basic assumptions under that we start our examination. In section 3 we deal with an auxiliary problem of the type

$$\Phi_t + v \Phi_r = -c_1 \ln \left( \frac{\Phi}{1 - \Phi} \right) - c_2 (1 - 2\Phi) - \text{dcp}q + \delta^2 \Phi_{rr} + \delta^2 \Phi_{rr} + (1 - \Phi)A \quad (8)$$

in $\Omega_T$ for some functions $q$ and $v$. In section 4 we deal with the Navier-Stokes equations in two dimensions. We are mainly concentrated on the treatment of the production term $\lambda$ and the additional stress tensor $\delta^2 \nabla \Phi \otimes \nabla \Phi$ arising as a result of the phase transition. In section 5 we couple both solution parameter.

## 2 The Weak Formulation of the Problem and Basic Assumptions

**Assumptions:** Here we regard the case, that

1. $\lambda(\rho):=A\rho$, where $A$ a real, positive constant;
2. $A$ is large, so that $A - 2c_2 > 0$;
3. $\tilde{\Omega} := B_R(0) \setminus B_a(0)$, where $a < R$, that means, we regard the exterior sphere problem;
4. $\text{dcp} := cp_1 - cp_2 \geq 0$;
5. $p(\rho,\Phi) = \sqrt{\rho} \Phi - \delta_1 \frac{1}{\rho}$, where $\delta_1 > 0$ is an arbitrary real number.

If the mass $\rho$ vanishes then the pressure should be very small. If $\rho \to \infty$, then the pressure increase as long as cytoplasm is existent, that means, as long as $\Phi > 0$. The second summand acts as corrector term. We choose $\delta_1$ very small in order to keep its influence as small as possible.
We denote with \( ||\cdot||_p \) the norm of \( L^p(\tilde{\Omega}) \) and with \( ||.||_{m,2},(.,.)_{m,2} \) the norm or the scalar product of the Sobolev space \( H^{m,2}(\tilde{\Omega}) \), respectively. In the same manner we consider \( (H^{m,2}(\tilde{\Omega})^*, ||.||_{m,2,*}) \). Further, \( C \) denotes a positive, generic constant, and \( \epsilon_j \) stands also for a generic parameter arising only in connection with Young’s inequality.

Setting all in all, we regard the system

\[
\begin{align*}
\rho_t + \text{div} (\rho v) &= A\rho, \quad (9) \\
(\rho v_i)_t + \text{div} (\rho v_i v) + p(\rho, \Phi)|_{x_i} + \delta^2 \text{div} (\Phi|_{x_i} \nabla \Phi) &= \mu_1 \Delta v_i + \mu_2 \text{div} v|_{x_i} + v A\rho, \quad i=1,2, \quad (10) \\
\Phi_t + v \cdot \nabla \Phi &= -c_1 \ln \left( \frac{\Phi}{1-\Phi} \right) - c_2 (1 - 2\Phi) - \text{dcp} \rho + \delta^2 \Delta \Phi + (1 - \Phi) A, \quad (11)
\end{align*}
\]

holding conditions (4)–(7).

We say the triplet \((\rho, v, \Phi) \in L^2(0,T; H_0^{1,2}(\tilde{\Omega})) \times L^2(0,T; H_0^{1,2}(\tilde{\Omega}, \mathbb{R}^2)) \times L^2(0,T; H^{1,2}(\tilde{\Omega}))\) is a weak solution of the system (9)–(11) provided that \( \rho \geq 0 \) a.e., iff

\[
- \int_{\Omega_T} \xi_1(t) (\rho - \rho_0) - \int_{\Omega_T} \nabla \xi_1 \cdot \rho v - \int_{\Omega_T} \xi_1 A\rho = 0
\]

for all \( \xi_1 \in L^2(0,T; H_0^{1,2}(\tilde{\Omega})) \) with \( \xi_1|_t \in L^2(\tilde{\Omega}_T), \xi_1(t)=0 \) for \( t \) near by \( T \),

\[
- \int_{\Omega_T} \xi_i(t) (\rho v_i - \rho_0 v_0) - \int_{\Omega_T} \left[ \nabla \xi_i \cdot \rho v_i + \xi_i|_{x_i} p(\rho, \Phi) + \delta^2 \nabla \Phi \cdot \Phi|_{x_i} \nabla \Phi \right] = - \int_{\Omega_T} \left[ \mu_1 \nabla \xi_i \cdot \nabla v_i + \mu_2 \nabla \xi_i \cdot v|_{x_i} \right] + \int_{\Omega_T} \xi_i v_i A\rho,
\]

for all \( \xi_{2,3} \in L^2(0,T; H_0^{1,2}(\tilde{\Omega}, \mathbb{R}^2)) \) with \( \xi_{2,3}|_t \in L^2(\tilde{\Omega}_T), \xi_{2,3}(t)=0 \) for \( t \) near by \( T \).

\[
- \int_{\Omega_T} \xi_4(t) (\Phi - \Phi_0) + \int_{\Omega_T} \delta^2 \nabla \xi_4 \cdot \nabla \Phi \\
+ \int_{\Omega_T} \xi_4 \left[ v \cdot \nabla \Phi + c_1 \ln \left( \frac{\Phi}{1-\Phi} \right) + c_2 (1 - 2\Phi) - (1 - \Phi) A \right] = - \int_{\Omega_T} \xi_4 \text{dcp} \rho
\]

for all \( \xi_4 \in L^2(0,T; H^{1,2}(\tilde{\Omega})) \) with \( \xi_4|_t \in L^2(\tilde{\Omega}_T), \xi_4(t)=0 \) for \( t \) near by \( T \).

Here we concentrate on the weak, spherically symmetric solution of the system (9)–(11).
Theorem 1 Assume that \( p(\rho, \Phi) = \sqrt{\rho} \Phi - \delta \frac{1}{\rho} \), where \( C \) is a constant, and that the given initial data \( \rho_0, v_0, \Phi_0 \) satisfy
\[
\int_\Omega \left[ \frac{1}{2} \rho_0(r) v_0(r)^2 + \rho_0(r)^2 \right] r dr \leq C_0,
\]
\[
0 \leq \Phi_0 \leq 1,
\]
for some constant \( C_0 \), where \( \rho_0 \leq C_1 \) for some constant \( C_1 \).

Then the problem (9)–(11), (4)–(7) has a weak solution in the sense described above, for which \( \rho = \rho(t, \|x\|), v = v(t, \|x\|) \frac{x}{\|x\|} \) and \( \Phi = \Phi(t, \|x\|) \).

Let \( \Omega \) be the interval \( \Omega := (a, R) \). If we let \( r = \|x\| \) and take
\[
\rho = \rho(t, r), \quad v = v(t, r) \frac{x}{r} \quad \text{and} \quad \Phi = \Phi(t, r)
\]
in (9)–(11), then we obtain the corresponding system for spherically symmetric solutions in \((0, T) \times (a, R)):
\[
\rho_t + (\rho v)_r + \frac{\rho v}{r} = A \rho, \tag{15}
\]
\[
(\rho v)_t + (\rho v^2 + p(\rho, \Phi) + \delta^2 \Phi^2)_r + \frac{\rho v^2}{r} + \delta^2 \frac{\Phi^2}{r} = \mu \left( v_r + \frac{v}{r} \right)_r + A \rho, \tag{16}
\]
\[
\Phi_t + v \Phi_r = -c_1 \ln \left( \frac{\Phi}{1 - \Phi} \right) - c_2 (1 - 2\Phi) - dcp \rho + \delta^2 \Phi_{rr} + \delta^2 \frac{\Phi_r}{r} + (1 - \Phi) A, \tag{17}
\]
where \( \mu := \mu_1 + \mu_2 \) and the initial-boundary conditions (5)–(7) and (4) are valid. In the first we concentrate on the weak solution of equation (17).

3 Existence of a Weak Solution for the Problem (17), (4), (7)

In this section we deal with an auxiliary problem of the type
\[
\Phi_t + v \Phi_r = -c_1 \ln \left( \frac{\Phi}{1 - \Phi} \right) - c_2 (1 - 2\Phi) - dcpq + \delta^2 \Phi_{rr} + \delta^2 \frac{\Phi_r}{r} + (1 - \Phi) A \tag{18}
\]
in \( \Omega_T \) for some function \( q \) and \( v \).

We regard \( \Phi_1 := \Phi \) and \( \Phi_2 := 1 - \Phi \) and introduce \( w := \Phi_2 - \Phi_1 \). Then \( 1 + w = 2\Phi_2 \) and \( 1 - w = 2\Phi_1 \). With that we can problem (18) transform to
\[
-\frac{1}{2} w_t - \frac{1}{2} vv_r = c_1 \ln \left( \frac{1+w}{1-w} \right) - c_2 w - dcpw - \frac{1}{2} \delta^2 w_{rr} - \frac{1}{2} \delta^2 \frac{w_r}{r} + \frac{1}{2} (1+w) A.
\]
Now the task is to evaluate the solution from the following initial - Neumann boundary value problem

\[ w_t - \delta^2 w_{rr} + \left( v - \frac{\delta^2}{r} \right) w_r + \left[ 2c_1 \ln \left( \frac{1+w}{1-w} \right) + (A - 2c_2)w \right] = 2 dc pq - A. \] (19)

\[ \vec{n} \cdot w_r |_{\partial \Omega} = 0 \quad \text{and} \quad w(0,.) |_{\Omega} = w_0 \] (20)

Here the most arising problem consists in the treatment of the non-linear term of the logarithmic free energy \( 2c_1 \ln \left( \frac{1+w}{1-w} \right) \). This term we want to treat by the series development. We introduce \( g_N \)

\[ g_N(w) := 4c_1 \sum_{k=0}^{N} \frac{w^{2k+1}}{2k+1}. \]

Now, we fix \( N \in \mathbb{N} \). With that we regard the corresponding \( N \)--problem on \((0,T) \times \Omega \)

\[ w_t - \delta^2 w_{rr} + \left( v - \frac{\delta^2}{r} \right) w_r + \left[ 4c_1 \sum_{k=0}^{N} \frac{w^{2k+1}}{2k+1} + (A - 2c_2)w \right] = 2 dc pq - A. \] (21)

\[ \vec{n} \cdot w_r |_{\partial \Omega} = 0 \quad \text{and} \quad w(0,.) |_{\Omega} = w_0 \] (22)

Writing in divergence form, we consider on \( \Omega_T \) and on the boundary \((0,T) \times \partial \Omega \).

\[ w_t - \left( \delta^2 w_r - (v - \delta^2 \frac{1}{r})w \right)_r - \left( v_r + \frac{\delta^2}{r^2} \right) w + 4c_1 \sum_{k=0}^{N} \frac{w^{2k+1}}{2k+1} + (A - 2c_2)w = 2 dc pq - A \]

\[ =: a(w_{rr}, w) = a(w) \quad \text{=: e}_N(w) \quad \text{=: f(q)} \] (23)

\[ \vec{n} \cdot w_r |_{\partial \Omega} = 0 \quad \text{and} \quad w(0,.) |_{\Omega} = w_0. \] (24)

We set

\[ \langle \xi, A(w_{rr}, w) \rangle = \int_{\Omega} \xi_r a(w_{rr}, w), \quad A(w) \equiv A(w_{rr}, w), \quad \langle \xi, e_N(w) \rangle = \int_{\Omega} \xi e_N(w). \]

Now for each \( N \in \mathbb{N} \) we are looking for a solution \( w_N \).

**Theorem 2** For each fixed \( N \in \mathbb{N} \), let \( v \in C^0((0,T]; H^{1,2}(\Omega)) \) and \( q \in L^2(0,T;L^2(\Omega)) \) be given. Furthermore, we have initial values \( w_0 = 1 - 2\Phi_0 \in H^{1,2}(\Omega) \). Then, there is a weak solution \( w_N \in L^2(0,T;H^{1,2}(\Omega)) \) of the problem (23)–(24).

First we consider the non-linear elliptic problem. We show that for each fixed \( N \) the operator \( A + E_N \) is bounded. Particular attention we give for the non-linear, polynomial term \( g_N \). For each fixed \( N \) we can show that this operator fulfills the growth condition, especially in a space with only one dimension. Then, we can apply the existence theorem for non-linear operators (see [1]).
Lemma 3 Let \( v \in H^{1,2}(\Omega) \) and \( q \in L^2(\Omega) \) be given. The non-linear elliptic equation
\[
A(w) + E_N(w) = f \quad \text{in} \quad \Omega, \\
\overrightarrow{n} \cdot w|_r = 0 \quad \text{on} \quad \partial \Omega
\]
has an unique weak solution \( w \in H^{1,2}(\Omega) \).

Proof. We use the existence theorem in [1] (Theorem 2.1) for non-linear equations.
\[
F_N := A + E_N, \quad F_N : H^{1,2}(\Omega) \rightarrow H^{1,2}(\Omega)^*.
\]

[A]: For each fixed \( N \in \mathbb{N} \): \( F_N - f \) is bounded on bounded subsets of \( H^{1,2}(\Omega) \).

We have \( \xi \in H^{1,2}(\Omega) \).
\[
|\langle \xi, A(w) \rangle| \leq \int_\Omega |\xi| |\delta^2 w|_r - (v - \delta^2 \frac{1}{r})w| \leq C(\delta^2, ||v||_\infty, a) ||\xi||_{1,2} ||w||_{1,2} \quad (25)
\]

Further, with the general Hölder inequality
\[
|\langle \xi, E_N(w) - f \rangle| \leq \int_\Omega |\xi| \left( |v|_r + \frac{\delta^2}{r^2} |w| + 4c_1 \sum_{k=0}^N \frac{1}{2k+1} |w|^{2k+1} + |A - 2c_2| |w| \right) + \int_\Omega |\xi| |f|
\]
\[
\leq C(\delta^2/a^2, A, c_2) ||\xi||_2 ||w||_2 + ||\xi||_4 ||v||_r ||w||_4 + 4c_1 \sum_{k=0}^N \int_\Omega |\xi| |w|^{2k+1} + ||\xi||_2 ||f||_2
\]

Regard the finite sum
\[
\sum_{k=0}^N \int_\Omega |\xi| |w|^{2k+1} \leq \sum_{k=0}^N \left( \int_\Omega |w|^{(2k+1)q'} \right)^{1/q'} \left( \int_\Omega |\xi|^q \right)^{1/q}
\]
\[
\leq \sum_{k=0}^N ||w||^{2k+1}_{(2k+1)q'} ||\xi||_q \leq \sum_{k=0}^N ||w||^{2k+1}_{1,2} ||\xi||_{1,2}
\]

Using the embedding theorem, because it is \( \frac{1}{2} > -\frac{1}{q} \), \( \forall 1 \leq q < \infty \), then it holds \( H^{1,2}(\Omega) \hookrightarrow L^q(\Omega), \forall 1 \leq q < \infty \). That holds, in fact, for \( q := (2k+1)q' \) for all \( k \). Totaling, we have
\[
||(F_N - f)(w)||_{1,2,*} \leq C( ||v||_\infty, ||v||_r, ||\delta^2, a, A, c_2|| ||w||_{1,2} + ||f||_2 + 4c_1 \sum_{k=0}^N ||w||^{2k+1}_{1,2}.
\]

[B]: \( F_N - f \) is coercive.
\[
\langle w, A(w) + E_N(w) - f \rangle
\]

7
\[
\int_{\Omega} \delta^2 |w| r^2 + \int_{\Omega} (v - \frac{\delta^2}{r}) w w_r + \int_{\Omega} \left(4c_1 \sum_{k=0}^{N} \frac{w_{2k+2}}{2k+1} \right) + (A - 2c_2) w^2 \right) - \int_{\Omega} f w.
\]

With Young's inequality for \(\epsilon, \epsilon_1\)
\[
\geq \delta^2 \|w\|_2^2 - \int_{\Omega} |w_r| \left(\frac{\epsilon}{2} |v|^2 + \frac{1}{2} |v - \frac{\delta^2}{r}|^2 \right) + \int_{\Omega} 4c_1 \sum_{k=0}^{N} \frac{w_{2k+2}}{2k+1} \left(\frac{\epsilon}{4\epsilon_1} \|v\|_4^4 + \delta^8/a^4 \right) + \int_{\Omega} \frac{C}{4\epsilon_1} \left(\frac{\|v\|_4^4 + \delta^8/a^4 \right)
\]

We choose \(\epsilon, \epsilon_1\) so small, so that \(\delta^2 - \epsilon > 0, \frac{1}{3} \epsilon_1 > 0\) and \(4c_1 - \frac{\epsilon}{2} > 0\), then
\[
\langle w, A(w) + E_N(w) - f \rangle \geq C \|w\|_{i,2}^2 - C. \tag{27}
\]

[C]: \(F_N\) is weak lower semicontinuous and weak continuous. \(A(., w)\) is monotone and fulfills the continuity condition for each \(w \in H^{1,2}\). Let \(w_m \rightarrow w\) weak in \(H^{1,2}(\Omega)\), then it converges \(w_m \rightarrow w\) f.a. in \(\Omega\) for a subsequence. With (25) \(A(w_m) \rightarrow A(w) \in H^{1,2}(\Omega)\).

[C.a]: Let \(w_m\) weak to \(w\) in \(H^{1,2}(\Omega)\). Then, we have \(w_m e_N(., w_m) + C \geq 0\). With (26) and the Lemma of Fatou we conclude \(\int_{\Omega} w e_N(., w) \leq \liminf_{m \rightarrow \infty} \int_{\Omega} w_m e_N(., w_m)\)

[C.b]: Let \(w_m\) weak to \(w\) in \(H^{1,2}(\Omega)\). Then it converge strongly in \(L^2(\Omega)\) and for a subsequence pointwise almost everywhere. Then \(e_N(x, w_m) \rightarrow e_N(x, w)\) almost pointwise. With equation (26) and because \(w_m\) bounded in \(H^{1,2}(\Omega)\), we have \(e_N(., w_m)\) is bounded in \(L^2(\Omega)\). Because \(L^2(\Omega)\) reflexive, there is a weak convergent subsequence in \(L^2(\Omega)\). Because \(H^{1,2}(\Omega) \subset L^2(\Omega)\), it is \(\int_{\Omega} \xi e_N(., w_m) \rightarrow \int_{\Omega} \xi e_N(., w)\) for all \(\xi \in H^{1,2}(\Omega)\) with \(m \rightarrow \infty\). Then \(E_N(., w_m) \rightarrow E_N(., w)\) weak* in \(H^{1,2}(\Omega)^*\).

Now, we regard the time discrete problem. For \(h \in \mathbb{R}, h > 0\), we define the difference quotient by
\[
\partial_t^{-h} w_N(t) := \frac{1}{h} (w_N(t) - w_N(t-h)) \quad \text{and} \quad \partial_t^h w_N := \frac{1}{h} (w_N(t+h) - w_N(t)).
\]

8
$T_h(\mathbb{R};L^2(\Omega))$ denotes the vector space of step functions with step length $h$ from $\mathbb{R}$ into the Hilbert space $L^2(\Omega)$.

**Lemma 4** Let $h \in \mathbb{R}, h > 0$. There is a unique time discrete solution $w_{Nh} \in T_h(\mathbb{R};L^2(\Omega))$ with

$$w_{Nh}(t) = w_0 \quad \text{for} \quad t < 0 \quad \text{and} \quad w_{Nh}(t) \in H^{1,2}(\Omega) \quad \text{for} \quad t > 0.$$  

which fulfills

$$\int_0^\infty \left[ \int_\Omega -\partial_t^h \xi(t)(w_{Nh}(t) - w_0) + \int_\Omega \xi(t)(a + e_N)(w_{Nh}(t)) \right] dt = \int_0^\infty \int_\Omega \xi(t)f(t)dt \quad (28)$$

for all $\xi \in L^\infty(\mathbb{R};H^{1,2}_0(\Omega))$ with bounded support.

**Proof.** We consider $F_{Na}: H^{1,2}(\Omega) \to H^{1,2}(\Omega)^*$ defined by $F_{Na} := \alpha I + (A + E_N)$. That means

$$\langle \xi, F_{Na}(u) \rangle_{1,2} = \alpha \langle \xi, u \rangle_2 + \langle \xi, (A + E_N)(u) \rangle_{1,2}.$$  

For the Operator $F_{Na}$ holds: [A] $F_{Na}$ maps bounded sets onto bounded sets, [B] $F_{Na}$ is coercive for $\alpha \geq \alpha_0$, and [C] $F_{Na}$ fulfills the continuity conditions. We prove [A], [B] and [C] in the same manner like Lemma 3.

Now we put $\alpha = \frac{1}{h}$ with $0 < h \leq \frac{1}{\alpha_0}$. And we set $w_{Nh}^0 := w_0 \in H^{1,2}(\Omega)$ and it is

$$\langle \xi, f_h^i \rangle_{1,2} + \frac{1}{h} \int_\Omega \xi w_{Nh}^{i-1} \in (H^{1,2}(\Omega))^* \quad \text{for} \quad i \in \mathbb{N}.$$  

Hereby, it is

$$f_h^i := \frac{1}{h} \int_{(i-1)h}^{ih} f(t)dt.$$  

Then, with the existence theorem of elliptic non-linear operator type, lemma 3, there exists an unique $w_{Nh} \in H^{1,2}(\Omega)$ with

$$\langle \xi, F_{Na}(w_{Nh}) \rangle = \langle \xi, f_h^i \rangle_{1,2} + \frac{1}{h} \int_\Omega \xi w_{Nh}^{i-1} \quad \text{for all} \quad \xi \in H^{1,2}_0(\Omega).$$  

Then we plug these in the definitions, we get

$$\frac{1}{h} \int_\Omega \xi (w_{Nh}^i - w_{Nh}^{i-1}) + \langle \xi, F_{Na}(w_{Nh}) - f_h^i \rangle_{1,2} = 0 \quad \text{for all} \quad i \in \mathbb{N}, \xi \in H^{1,2}_0(\Omega). \quad (29)$$

We set

$$w_{Nh} := \begin{cases} w_{Nh}^i & \text{for} \quad (i - 1)h < t \leq ih, i \in \mathbb{N}, \\ w_0 & \text{for} \quad i = 0. \end{cases}$$

and further
\[ f_h := \begin{cases} f_i^h & \text{for } (i-1)h < t \leq ih, i \in \mathbb{N}, \\ 0 & \text{for } t \leq 0 \end{cases}. \]

With equation (29) we can write
\[ \int_{\Omega} \xi \partial_t^{-h} w_{Nh}(t) + \langle \xi, F_N(w_{Nh}(t)) \rangle_{1,2} = \langle \xi, f_h(t) \rangle_{1,2} \quad \text{for all } t > 0 \text{ and } \xi \in H_0^{1,2}(\Omega). \]

Now, let be the function \( \xi \in L^\infty([0, \infty), H_0^{1,2}(\Omega)) \) with bounded support. Then with integration over \( t \), we get
\[ \int_{0}^{\infty} \int_{\Omega} \xi(t) \partial_t^{-h} w_{Nh}(t) dt + \int_{0}^{\infty} \langle \xi(t), F_N(w_{Nh}(t)) \rangle_{1,2} dt = \int_{0}^{\infty} \langle \xi(t), f_h \rangle_{1,2} dt. \]
It is because \( \partial_t^{-h} w_{Nh} = \partial_t^{-h} (w_{Nh} - w_0) \) and \( w_{Nh}(t) - w_0 = 0 \) for \( t < 0 \). Further \( \xi(t) = 0 \) for large positive \( t \), then we can conclude
\[ \int_{0}^{\infty} \int_{\Omega} \xi(t) \partial_t^{-h} w_{Nh}(t) dt = \frac{1}{h} \int_{\mathbb{R}} \int_{\Omega} \xi(t) w_{Nh}(t) dt - \frac{1}{h} \int_{\mathbb{R}} \int_{\Omega} \xi(t) w_{Nh}(t-h) dt \]
\[ = - \int_{0}^{\infty} \int_{\Omega} \partial_t^h \xi(t) (w_{Nh}(t) - w_0) dt. \]
Then we have proved the assertion. \( \square \)

**Lemma 5** We have for the time discrete solution an uniform estimate and we can take \( h \to 0 \). It holds following energy estimation
\[ \sup_{0 < t < t_0} \int_{\Omega} w_{Nh}(t)^2 + \int_{0}^{t_0} \|w_{Nh}\|_{1,2}^2 \leq C \quad \text{f.a. } t_0 \in (0, T). \]

**Proof first part.** We set in equation (29) \( \xi(t) := w_{Nh}(t) \) for \( t \in \mathbb{R} \) and integrate over \( [0, ih] \) for \( i \in \mathbb{N} \). Then it is made a similar computation as it is done below in the proof of theorem 6.

**Proof second part.** From the energy estimate we know, \( w_{Nh} \) is bounded in \( L^2(0, T; H^{1,2}(\Omega)) \). Because the weak compactness of bounded sets in Hilbert spaces, there is a subsequence \( h \searrow 0 \) and a \( w_N \) in \( L^2(0, T; H^{1,2}(\Omega)) \), so that
\[ \langle w_{Nh} - w_N, \xi \rangle \longrightarrow 0 \quad \text{for } h \to 0 \text{ for all } \xi \in L^2(0, T; H^{1,2}(\Omega)). \quad (30) \]

We start with equation (28). The convergence of the time derivative term is standard. We don’t explicitly describe here. The only interesting term is the second summand on the left-hand side. For \( \varphi \in C([0, T]; H_0^{1,2}(\Omega)) \) we define \( \tilde{\xi} \) by
\[ \langle w, \tilde{\xi} \rangle := \int_{0}^{T} \langle \varphi(t), (a + e_N)(w(t)) \rangle_{1,2} dt \quad \text{for } w \in L^2(0, T; H^{1,2}(\Omega)). \]
From lemma 3 part [A] it follows \( \tilde{\xi} \in L^2(0,T;H^{1,2}(\Omega)^*) \) and from that directly the assertion.

\[ \square \]

REMARK. In the same manner follows the uniqueness of the problem.

Now we have for each \( N \in \mathbb{N} \) a weak solution of the \( N \)-problem (21)–(22). For this solution we can further derive an uniform bound in \( N \).

**Theorem 6** Let \( q \in L^2(0,T;L^2(\Omega)) \) and \( v \in L^4(0,T;L^4(\Omega)) \). For all \( N \in \mathbb{N} \), there is an uniform estimation for the weak solution of (21)–(22) in \( N \). That means, there is a constant \( C \) independent of \( N \) with

\[
\| w_N(t_0) \|^2 + \int_0^{t_0} \| w_N \|^2_{1,2} \leq C \quad \text{f.a. } t_0 \in (0,T).
\]

**Proof.** We take equation (21), thereby using \( w_N \) as test function. Then

\[
\int_{\Omega_0} w_N(t) w_N + \int_{\Omega_0} \delta^2 |w_N| r^2 + \int_{\Omega_0} \left( v - \frac{\delta^2}{r} \right) w_N |r| w_N
\]

\[
+ \int_{\Omega_0} \left[ 4c_1 \sum_{k=0}^N \frac{w_{N,2k+1}}{2k+1} + (A - 2c_2) w_N \right] w_N = \int_{\Omega_0} \left[ 2dcpq - A \right] w_N.
\]

From that we have

\[
\int_0^{t_0} \frac{1}{2} \frac{d}{dt} \| w_N \|^2 + \delta^2 \int_{\Omega_0} |w_N| r^2 + \int_{\Omega_0} \left( v - \frac{\delta^2}{r} \right) w_N |r| w_N
\]

\[
+ \int_{\Omega_0} \left[ 4c_1 \sum_{k=0}^N \frac{w_{N,2k+1}}{2k+1} - 2c_2 w_N^2 \right] + A \int_{\Omega_0} w_N^2 = \int_{\Omega_0} f w_N.
\]

From the calculation with Young’s inequality (see also above lemma 3, proof, part [B]) we have

\[
\int_{\Omega_0} \left( v - \frac{\delta^2}{r} \right) w_N |r| w_N \geq \int_0^{t_0} -\epsilon_1 \| w_N |r| \|^2 - \frac{\epsilon}{4\epsilon_1} \| w_N \|^4 - \frac{C}{4\epsilon_1} (\| v \|^4 + \delta^8 / a^4).
\]

Choose \( \epsilon_1 \) so small that \( \delta^2 - \epsilon_1 > 0 \). After that, choose \( \epsilon \) so small, so that \( \frac{4\epsilon_1}{3} - \frac{\epsilon}{4\epsilon_1} > 0 \) holds. We define

\[
\tilde{E}_N := 4c_1 \sum_{k=0}^N \frac{w_{N,2k+1}}{2k+1} - 2c_2 w_N^2 - \frac{\epsilon}{4\epsilon_1} w_N^4.
\]
We want to find a lower bound for the operator $\widetilde{E}_N$. We have $c_1 > 0$, then
\begin{align*}
\widetilde{E}_N &= \int_0^{t_0} \int_\Omega \left\{ 4c_1 \frac{w_N^2}{1} + \left( \frac{4c_1}{3} - \frac{\epsilon}{4c_1} \right) w_N^4 + 4c_1 \left( \frac{w_N^6}{5} + \frac{w_N^8}{7} + ... \right) - 2c_2 w_N^2 \right\} dxdt \\
&= 4c_1 \int_0^{t_0} \int_\Omega w_N^2 dxdt + \int_0^{t_0} \int_\Omega \left\{ \tilde{c}_1 w_N^4 - 2c_2 w_N^2 \right\} dxdt,
\end{align*}
\begin{align*}
\widetilde{E}_N &\geq 4c_1 \int_0^{t_0} \|w_N\|_2^2 - \frac{c_2^2}{\tilde{c}_1} |\Omega| t_0.
\end{align*}
(33)

The last inequality holds, because it is valid
\[(\tilde{c}_1 w_N^2 - c_2)^2 \geq 0 \iff \tilde{c}_1^2 w_N^4 - 2 \cdot \tilde{c}_1 c_2 w_N^2 + c_2^2 \geq 0 \iff \tilde{c}_1 w_N^4 - 2c_2 w_N^2 \geq -\frac{c_2^2}{\tilde{c}_1}.
\]

Using (32),(33) in equation (31) and applying the Young inequality, we get
\begin{align*}
\frac{1}{2} (\|w_N(t_0)\|_2^2 - \|w_N(0)\|_2^2) + \min(\delta^2 - \epsilon_1, 4c_1, A) \int_0^{t_0} \|w_N\|_{1,2}^2 \\
&\quad - \frac{c_2^2}{\tilde{c}_1} |\Omega| t_0 - \int_0^{t_0} C \left( \|v\|^4_4 + \delta^8 / a^4 \right) \leq \int_0^{t_0} \frac{\epsilon}{2} \|w_N\|_2^2 + \int_0^{t_0} \frac{1}{2\epsilon} \|f\|_2^2.
\end{align*}
(34)

We choose $\epsilon > 0$ so small, that $C_3 := \min(\delta^2 - \epsilon_1, 4c_1, A) - \epsilon > 0$. Then with equation (35) we have
\begin{align*}
\|w_N(t_0)\|_2^2 + C \int_0^{t_0} \|w_N\|_{1,2}^2 \\
&\quad \leq \|w_0\|_2^2 + \frac{2c_2^2}{\tilde{c}_1} |\Omega| t_0 + C \int_0^{t_0} \|v\|^4_4 + 2C \delta^8 / a^4 |\Omega| t_0 + C \int_0^{t_0} \|f\|_2^2.
\end{align*}
(35)

With the assumptions, it follows the assertion. □

**Lemma 7** It holds with $\|w_0\|_\infty \leq 1, v \in L^4(0,T;H^{1,4}(\Omega)), q \in L^2(0,T;L^2(\Omega))$,
\[\int_0^{t_0} \|g_N(w_N)\|_2^2 \leq C \quad C \text{ independent on} \ N.\]

**Proof.** From equation (23), multiply with $g_N(w_N)$ and integrate over $\Omega$:
\begin{align*}
\int_\Omega w_N|g_N(w_N)| + \int_\Omega \delta^2 w_N|g_N(w_N)| - \int_\Omega \left( v - \frac{\delta^2}{r} \right) w_N g_N(w_N)
\end{align*}
(36)

12
Now, we treat the third summand on the left-hand side

$$ - \int_{\Omega} \left( v_r + \frac{\delta^2}{r^2} \right) w_N g_N(w_N) + \| g_N(w_N) \|^2 + \int_{\Omega} (A - 2c_2) w_N g_N(w_N) = \int_{\Omega} f g_N(w_N). $$

It is

$$ \int_{\Omega} w_N g_N(w_N) = \frac{d}{dt} \int_{\Omega} G_N(w_N) \quad \text{with} \quad G_N(w) := 4c_1 \sum_{k=0}^{N} \frac{w^{2k+2}}{(2k+2)(2k+1)}, $$

$$ \int_{\Omega} \delta^2 w_N g_N(w_N)|w_N|^2 \geq 0 \quad \text{with} \quad g_N'(w) = 4c_1 \sum_{k=0}^{N} w^{2k}. \quad (37) $$

And with Young's inequality, it holds

$$ - \int_{\Omega} \left( v - \frac{\delta^2}{r} \right) w_N g_N(w_N) \geq - \int_{\Omega} g_N'(w_N) \left\{ \| v \|_\infty + \frac{\delta^2}{a} \right\} \frac{\epsilon}{2} |w_N|^2 \geq - \int_{\Omega} g_N'(w_N) \left( \| v \|_\infty + \frac{\delta^2}{a} \right) \left\{ \frac{\epsilon}{2} |w_N|^2 + \frac{1}{2\epsilon} |w_N|^2 \right\}. $$

Now, choose $\epsilon$ so small, so that

$$ \int_{\Omega} g_N'(w_N) \left\{ \delta^2 - \left( \| v \|_\infty + \frac{\delta^2}{a} \right) \frac{\epsilon}{2} \right\} |w_N|^2 \geq 0. $$

Further

$$ - \int_{\Omega} \left( v_r + \frac{\delta^2}{r^2} \right) w_N g_N(w_N) \geq - \int_{\Omega} \left[ \epsilon_1 |g_N(w_N)|^2 + \frac{\epsilon_2}{4\epsilon_1} |w_N|^4 + \frac{C}{4\epsilon_2 \epsilon_1} (|v_r|^4 + \delta^8/a^8) \right], $$

and

$$ \int_{\Omega} f g_N(w_N) \leq \frac{\epsilon_3}{2} \| g_N(w_N) \|^2 + \frac{1}{2\epsilon_3} \| f \|^2. $$

Plugging these in equation (36) and choose $\epsilon_1, \epsilon_2, \epsilon_3$ small, we get

$$ \frac{d}{dt} \int_{\Omega} G_N(w_N) + C \int_{\Omega} g_N'(w_N) |w_N|^2 - \int_{\Omega} g_N'(w_N) \left( \| v \|_\infty + \frac{\delta^2}{a} \right) |w_N|^2 \quad (38) $$

$$ + C \| g_N(w_N) \|^2 - C \| w_N \|^4 + \int_{\Omega} (A - 2c_2) w_N g_N(w_N) \leq C \| f \|^2 + C \int_{\Omega} (|v_r|^4 + \delta^8/a^8). $$

Now, we treat the third summand on the left-hand side

$$ - \int_{\Omega} g_N'(w_N) \left( \| v \|_\infty + \frac{\delta^2}{a} \right) |w_N|^2 \geq - C(\| v \|_\infty, \delta^2, a) \left[ \frac{\epsilon_4}{2} \| g_N'(w_N) \|^2 + \frac{1}{2\epsilon_4} \| w_N \|^4 \right]. $$

(39)
It is
\[
\frac{C}{2} \| g_N(w_N) \|_2^2 = \frac{C}{2} \int_{\Omega} (4c_1)^2 \left( \sum_{k=0}^{N} \frac{w_N^{2k+1}}{2k+1} \right)^2 \\
= \frac{C}{2} \int_{\Omega} (4c_1)^2 \left\{ \sum_{k=0}^{N} \frac{w_N^{(2k+1)^2}}{(2k+1)^2} + \sum_{k_1 < k_2 \leq N} \frac{2w_N^{2(k_1+k_2)+2}}{(2k_1+1)(2k_2+1)} \right\},
\]
\[
-C \frac{\epsilon_4}{2} \| g'_N(w_N) \|_2^2 = -C \frac{\epsilon_4}{2} \int_{\Omega} (4c_1)^2 \left\{ \sum_{k=0}^{N} \frac{w_N^{2k-2}}{1} + \sum_{k_1 < k_2 \leq N} \frac{2w_N^{2(k_1+k_2)}}{1} \right\}.
\]

By comparison of the coefficients in (40) and (41), it is easy to see, that we can choose \( \epsilon_4 \) so small so that:
\[
\frac{1}{2} \| g_N(w_N) \|_2^2 - C \frac{\epsilon_4}{2} \| g'_N(w_N) \|_2^2 \geq -C \frac{\epsilon_4}{2} (4c_1)^2.
\] (42)

In the same manner as above, because \((A-2c_2) > 0\)
\[
\int_{\Omega} (A-2c_2)w_N g_N(w_N) - C \| w_N \|_4^4 \geq -C.
\] (43)

Plugging (39)–(43) in (38) and integrate over \((0,t_0)\), it holds
\[
\int_{\Omega} G_N(w_N(t_0,\cdot)) \geq 0 - \int_{\Omega} G_N(w_0) + \int_{0}^{t_0} \| g_N(w_N) \|_2^2 \leq C(\| f \|_2, \| v \|_\infty, \| v \|_1^4, \delta, a, t_0).
\]

whereby \( C \) is independent on \( N \). For \( n \to \infty \) and \( \| w_0 \| < 1 \) we have
\[
G_N(w_0) \to (1+w_0) \ln(1+w_0) + (1-w_0) \ln(1-w_0) \leq C.
\]

**Lemma 8** Because \( w_0 \in H^{1,2}(\Omega) \), f.a. \( t_0 \), it is valid
\[
\int_{0}^{t_0} \| w_N \|_2^2 + C \| w_N \|_1^2 \leq C \quad C \text{ independent on } N.
\]

**Proof.** For the detailed proof regard the time discretized case and go to the limit for \( h \downarrow 0 \). Here we only present the bound. In equation (36), we use \( w_N |_t \) instead of \( g_N(w_N) \):
\[
\int_{\Omega} w_N |_t w_N |_t + \int_{\Omega} \delta^2 w_N |_t w_N |_t + \int_{\Omega} \left( v - \frac{\delta^2}{r} \right) w_N |_t w_N |_t + \int_{\Omega} g_N(w_N) w_N |_t + \int_{\Omega} (A-2c_2)w_N w_N |_t = \int_{\Omega} f w_N |_t.
\] (44)
Integrate over \((0,t_0)\) and use Young’s inequality, we get
\[
\int_0^{t_0} \|w_{N,t}\|^2 + \frac{\delta^2}{2} \|w_{N,t_0}(t_0,.)\|^2 + \int_0^{t_0} G_N(w_N)(t_0,.) + \int_0^{t_0} (A - 2c_2)w_N^2(t_0,.) \geq 0
\]
\[
\leq \frac{\delta^2}{2} \|w_{0,t}\|^2 + \int_\Omega G_N(w_0) + (A - 2c_2)\|w_0\|^2 + \int_0^{t_0} \frac{\epsilon}{2} \|w_{N,t}\|^2 + \int_0^{t_0} \frac{1}{2\epsilon} \|f\|^2
\]
\[
+ \int_0^{t_0} \frac{\epsilon}{2} \|w_{N,t}\|^2 + \int_0^{t_0} \frac{1}{2\epsilon} \left(\|v\|_\infty + \frac{\delta^2}{a}\right) \|w_{N,t}\|^2.
\]
Because \(w_0 \in H^{1,2}(\Omega)\), it is \(\frac{\delta^2}{2} \|w_{0,t}\|^2 + (A - 2c_2)\|w_0\|^2 \leq C\). Similar to the end of the proof of theorem 7, it is \(\int_\Omega G_N(w_0) \leq C\). Choose \(\epsilon\), so that \(1 - \epsilon > 0\) and apply the Gronwall lemma, we get the claim. □

**Theorem 9** Let \(\|w_0\|_\infty \leq 1, v \in H^{1,4}(0,T;L^4(\Omega)) \cap C^0(0,T;H^{1,4}(\Omega))\) and \(q \in H^{1,2}(0,T;L^2(\Omega))\) as above. There is a weak solution of the problem (21)-(22).

**Proof.** First we have to show that the limit exists.
For an extensive proof take the succeeding inequalities in a time discrete scheme and pass \(h \searrow 0\). Similarly, we have to the coerciveness considerations and under the usage of the bound results of lemma 8 and theorem 6
\[
\langle w_{N,t}, \frac{d}{dt}(A + E_N - f)(w_N) \rangle \geq C\|w_{N,t}\|^2 - C_1\|w_{N,t}\|^2 - C_2\|q|\|^2
\]
and
\[
\langle -(A + E_N - f)(w_N), \frac{d}{dt}(A + E_N - f)(w_N) \rangle = -\frac{1}{2}\|A + E_N - f)(w_N)\|^2.
\]
From (45) it follows
\[
\frac{1}{2}\frac{d}{dt}\|A + E_N - f)(w_N)\|^2 + C\|w_{N,t}\|^2 \underbrace{\leq C_1\|A + E_N - f)(w_N)\|^2}_{\geq 0}
\]
With the Gronwall lemma follows \(\|(A + E_N - f)(w_N)\|^2 \leq C\). From that \(\|w_{N,t}\|^2 \leq C\). With the plugging of (39)–(43) in (38), it holds
\[
C\|g_N(w_N)\|^2 \leq C(\|f\|^2, \|v\|_\infty, \|v_t\|^4, \delta, a) + \frac{1}{2\epsilon}\|w_{N,t}\|^2,
\]
we get \(\|g_N(w_N)\|^2 \leq C\). All in all, from above, we infer \(w_N\) is uniform bounded in \(L^2(0,T;H^{1,2}(\Omega))\). Then from the compactness theorem, we deduce \(w_N \rightharpoonup w\) weak in \(L^2(0,T;H^{1,2}(\Omega))\) for \(N_m\). Furthermore \(w_N \rightharpoonup w\) weak in \(H^{1,2}(0,T;H^{1,2}(\Omega^*)\). Further
$w_N \to w$ weak* in $L^\infty(0,T;L^2(\Omega))$. Moreover, by above estimate there exists a $g$ such that

$$g_N(w_N(t,x)) \longrightarrow g \quad \text{in } L^\infty(0,T;L^2(\Omega)) \text{ weak*}.$$ 

Choose $\{t_k\}$ dense in $(0,T)$. Then $w_N(t_k,.) \to w(t,.)$ weak in $L^2(\Omega)^*$. For an arbitrary small $\epsilon \in (0,1)$ and for every $t \in (0,T)$, we denote

$$\Omega_N^\epsilon(t) := \{x \in \Omega : |w_N(t,x)| > 1 - \epsilon\} \quad \text{and} \quad \Omega^\epsilon(t) := \{x \in \Omega : |w(t,x)| > 1 - \epsilon\}.$$

Then we have

$$C \geq \left[ \int_{\Omega_N^\epsilon(t)} g_N^2(w_N(t,.)) \right]^{1/2} \geq |\Omega_N^\epsilon(t)|^{1/2} \left[ \inf_{x \in \Omega_N^\epsilon(t)} \left( 4c_1 \sum_{k=0}^{N} \frac{w_N(t,x)^{2k+1}}{2k+1} \right) \right]^{1/2} \geq 2c_1 |\Omega_N^\epsilon(t)|^{1/2} \sum_{k=0}^{N} \frac{(1-\epsilon)^{2k+1}}{2k+1} \geq 2c_1 |\Omega_N^\epsilon(t)|^{1/2} \ln \left( \frac{2-\epsilon}{\epsilon} \right)$$

and from that

$$|\Omega_N^\epsilon(t)|^{1/2} \leq \frac{C}{2c_1 \ln \left( \frac{2-\epsilon}{\epsilon} \right)} \quad \text{and} \quad \frac{C}{2c_1 \ln \left( \frac{2-\epsilon}{\epsilon} \right)} \epsilon \to 0.$$ 

With the lemma of Fatou we conclude

$$|\Omega^\epsilon(t)| = \int_{\Omega} \chi_{\Omega^\epsilon(t)} \leq \liminf_{N \to \infty} \int_{\Omega} \chi_{\Omega_N^\epsilon(t)} \leq \liminf_{N \to \infty} |\Omega_N^\epsilon(t)|.$$

Regard $\epsilon \to 0$, it follows

$$|\{x \in \Omega : |w(t,x)| \geq 1\}| = 0.$$

Now, we have to prove that $g = 2c_1 \ln \left( \frac{1+w}{1-w} \right)$. We regard

$$|g_N(w_N(t,x)) - 2c_1 \ln \left( \frac{1+w_N}{1-w_N} \right)(t,x)| \leq 4c_1 \sum_{k=N+1}^{\infty} \frac{|w_N(t,x)^{2k+1}}{2k+1}$$

$$\leq 4c_1 \frac{1}{2N+3} \sum_{k=N+1}^{\infty} (1-\epsilon)^{2k+1}$$

$$\leq 4c_1 \frac{1}{2N+3} \frac{1}{\epsilon} \to 0 \quad \text{as } N \to \infty.$$

Therefore

$$|g_N(w_N) - 2c_1 \ln \left( \frac{1+w}{1-w} \right)| \leq |g_N(w_N) - 2c_1 \ln \left( \frac{1+w_N}{1-w_N} \right)| + |2c_1 \ln \left( \frac{1+w_N}{1-w_N} \right) - 2c_1 \ln \left( \frac{1+w}{1-w} \right)|$$

16
With the theorem of Lebesgue
\[ \int_{\Omega} g_N(w_N) \xrightarrow{N \to \infty} \int_{\Omega} 2c_1 \ln \left( \frac{1+w}{1-w} \right) . \]
Because the above convergence is unique, it holds \( g = 2c_1 \ln \left( \frac{1+w}{1-w} \right) . \)

**Lemma 10** Let \( v \in C^0(0,T; H^{1,4}(\Omega)) \). The solution of the problem (21)-(22) is unique.

**Proof.** Let \( w_1 \) and \( w_2 \) are two solutions of the equivalent problem (19)–(20). We set \( u := w_1 - w_2 \). Then we have
\[
 u_t - \delta^2 u_{rr} + \left( v - \frac{\delta^2}{r} \right) u_r + (W_0|w(w_1) - W_0|w(w_2)) + (A - 2c_2) u = 0 .
\]
Taking the scalar product with \( u \), it follows with \( u_0 \equiv 0 \):
\[
 \frac{1}{2} \| u(t) \|^2_2 + C \int_0^t \| u \|^2_{1,2} + \int_{\Omega_t} \left( 2c_1 \ln \left( \frac{1+w_1}{1-w_1} \right) - 2c_1 \ln \left( \frac{1+w_2}{1-w_2} \right) \right) (w_1 - w_2) \]
\[
 - \int_{\Omega_t} (A - 2c_2) u^2 \leq C \int_0^t \| u \|^2_2 .
\]
We have
\[
 \int_{\Omega_t} \left( 2c_1 \ln \left( \frac{1+w_1}{1-w_1} \right) - 2c_1 \ln \left( \frac{1+w_2}{1-w_2} \right) \right) (w_1 - w_2) \geq 0 ,
\]
because \( \frac{1+w}{1-w} \) is monotone and \( \ln \) is monotone. Therefore, \( \ln \left( \frac{1+w}{1-w} \right) \) is a monotone function in \( w \) and the inequality is valid.

All in all
\[
 \frac{1}{2} \| u(t) \|^2_2 + C \int_0^t \| u \|^2_{1,2} \leq C \int_0^t \| u \|^2_2 .
\]
By Gronwall’s inequality follows the assertion. \( \square \)

### 4 Existence of the Solution from Equation (15) and (16)

The considerations of this section are motivated by the work of [2] and [6], who proved the existence of a global weak solution of the Navier-Stokes equations for
viscous, compressible, isothermal flow, where the mass production don’t arise ($\lambda \equiv 0$).

There is given radially symmetric initial data. That means, for $x \in \mathbb{R}^2$ is $\rho(0,x) = \rho_0(\|x\|)$ and $v(0,x) = v_0(\|x\|)x/\|x\|$, with $\rho_0(\|x\|), v_0(\|x\|) \in \mathbb{R}$. Further, $\rho_0, v_0$ satisfy
\[
\int_0^R \left[ \frac{1}{2} \rho_0(r) v_0(r)^2 + \rho_0(r)^2 \right] r dr \leq C_0
\]  
for some constant $C_0$, where $\rho_0, \rho_0^{-1} \leq C_1$ for some constant $C_1$.

Here, we regard equation (15) and (16):
\[
\rho_t + (\rho v)_r + \frac{\rho v}{r} = A \rho,
\]
\[
(\rho v)_t + (\rho v^2 + p(\rho, \Phi) + \delta^2 \Phi^2 r^2)_r + \frac{\rho v}{r} + \delta^2 \frac{\Phi^2 r^2}{r} = \mu \left( \frac{v}{r} \right)_r + v A \rho.
\]

We introduce Lagrangian coordinates $z$ for the radially symmetric system above as follow:
\[
r_{zt}(t,z) = v(t,r(t,z)).
\]

Then we get with $\rho(t,z) = \rho(t,r(t,z)), v(t,z) = v(t,r(t,z))$, that succeeding equations are equivalent to the equations (15)–(16)
\[
\rho_{zt} + \rho \frac{1}{r_{zt}} (rv)_z = \lambda(\rho),
\]
\[
v_{zt} + p(\rho, \Phi) + \delta^2 \Phi^2 r^2 + \frac{\rho v}{r} + \delta^2 \frac{\Phi^2 r^2}{r} = \mu \left( \frac{v}{r} \right)_r + v A \rho.
\]

From equation (47), (49) and the fact that $(rr_{zt})_z = (rr_{zt})_t$ we can conclude
\[
\rho_{zt} \cdot r_{zt} + \rho (rr_{zt})_z = \lambda(\rho) \cdot r_{zt}
\]
\[
\rho_{zt} \cdot r_{zt} + \rho (rr_{zt})_t = \lambda(\rho) \cdot r_{zt}
\]
\[
\frac{(rr_{zt})_t}{r_{zt}} = \frac{\lambda(\rho)}{\rho}.
\]

Therefore, for $r_{zt}$ has to be valid
\[
r_{zt} = \frac{1}{\rho r} \exp \left( \int_0^t \frac{\lambda(\rho)}{\rho} d\tau \right).
\]

Define $A(t,z) := \exp \left( \int_0^t \frac{\lambda(\rho(t,z))}{\rho(t,z)} d\tau \right)$. Then (47)-(49) can be written by
\[
\rho_{zt} + \frac{\rho^2}{A} (rv)_z = \lambda(\rho),
\]
\[ v|_t \frac{1}{r} \Lambda + (p(\rho, \Phi) + \delta^2 \Phi^2_{|r})|_z + \delta^2 \frac{\Phi^2_{|r}}{r} \frac{\Lambda}{\rho r} = \mu \left( \left( r v|_z \frac{\rho}{\Lambda} \right) \right), \quad z > 0, t > 0, \]
\[ r|_t = v(t, r(t, z)). \]

We can equation (49) transform to
\[
2rr|_t = 2 \int_0^z (rv)|_z dz' = 2 \int_0^z (rr|_z)|_t dz',
\]
\[
\frac{d}{dt} (r^2) = \frac{d}{dt} \left( 2 \int_0^z \frac{A}{\rho} dz' \right).
\]

Then
\[ r^2 = a^2 + 2 \int_0^z \frac{A}{\rho} dz'. \]

and we can write our problem in the following form
\[
\rho|_t + \rho^2 \frac{A}{\Lambda} (rv)|_z = \lambda(\rho),
\]
\[
v|_t \frac{1}{r} \Lambda + (p(\rho, \Phi) + \delta^2 \Phi^2_{|r})|_z + \delta^2 \frac{\Phi^2_{|r}}{r} \frac{\Lambda}{\rho r} = \mu \left( \left( r v|_z \frac{\rho}{\Lambda} \right) \right), \quad z > 0, t > 0, \]
\[ r^2 = a^2 + 2 \int_0^z \frac{A}{\rho} dz'. \]

With the transformation \( u := \Lambda \rho^{-1} \), we finally get
\[
u|_t - (rv)|_z = 0, \quad (50)
\]
\[ v|_t A + r(p(\rho, \Phi) + \delta^2 \Phi^2_{|r})|_z + \delta^2 \frac{\Phi^2_{|r}}{r} u = \mu r \left( \left( r v|_z \frac{\rho}{\Lambda} \right) \right), \quad (51)
\]
\[ r^2 = a^2 + 2 \int_0^z u dz'. \quad (52)
\]

In the general form, the equations (50)-(52) are integral-differential equations. We use therefore from here up the assumption that
\[ \lambda(\rho) = A \rho \]

and that
\[ p(\rho, \Phi) = \sqrt{\rho} \Phi - \delta_1 \frac{1}{\rho}. \]

where \( A, C \) are special, positive constants. Then \( \Lambda(t, z) = \Lambda(t) = e^{At} \) and
\[ p(u, \Phi) = \sqrt{A} \Phi - \delta_1 \frac{u}{A}. \]
For the equations (50)-(52) we have a system of partial differential equations

\[ u_t - (rv)_z = 0, \]  
\[ v_t + r \left( \frac{1}{u} \frac{1}{e^{At/2}} \Phi - \delta_1 \frac{u}{e^{2At}} + \delta^2 \frac{\Phi^2}{e^{At}} \right)_z + \delta^2 \frac{\Phi^2}{r} \frac{1}{u} \frac{1}{e^{At}} = \mu \frac{r}{u} \frac{1}{e^{At}} \left| \frac{(rv)_z}{u} \right|, \]  
\[ r^2 = a^2 + 2 \int_0^z u \, dz', \]

(53) \hspace{1cm} (54) \hspace{1cm} (55)

Now, we construct approximate solutions for the exterior sphere problem in Lagrangian coordinates. These approximate solutions are generated by a semidiscrete difference scheme. Let \( z_k = kh \) for \( k = 0, 1, 2, \ldots \), and \( z_j = jh \) for \( j = \frac{1}{2}, \frac{3}{2}, \ldots \), and \( h > 0 \). We define the difference operator by

\[ \partial^h_{\ell,k} u(t) = \frac{u_{\ell+1/2}(t) - u_{\ell-1/2}(t)}{h}, \]

where \( u_{\ell}(t) = u(t, z_{\ell}) \). Then we get

\[ \dot{u}_j - \partial^h_{\ell,k} (v_j r_j) = 0, \]
\[ \dot{v}_k + r_k \partial^h_z \left( \frac{1}{\sqrt{u_k}} \frac{1}{e^{At/2}} \Phi - \delta_1 \frac{u_k}{e^{2At}} + g_k \frac{1}{e^{At}} \right) + g_k \frac{r_k}{u_k} \frac{1}{e^{At}} = \mu r_k \partial^h_z \left( \frac{\frac{\Phi^2}{r}}{u_k} \right) \frac{1}{e^{At}} = \mu r_k \partial^h_z \left( \frac{\Phi^2}{r} \right), \]
\[ r_k^2 = a^2 + 2 \sum_{j<k} u_j h, \]
\[ v_0(t) = 0, \]

(56) \hspace{1cm} (57) \hspace{1cm} (58) \hspace{1cm} (59)

where

\[ g_k = \delta^2 \frac{\Phi^2}{r}. \]

First, we consider equation (56)-(59).

Lemma 11 There is a weak solution \( \{u_j, v_k, r_k\} \) continuous in \([0, T)\) for the system of ODE’s (56)-(57) with the initial data (59).

Proof. We can take \( R = z_K \). We know with lemma 8 that f.a. \( x_k \), sup \( t \left| g_k \right| < \infty \). Therefore the whole differential operator is bounded in \( t \) and continuous in \( (u_j, v_k) \). Then with the theorem of Peano, for a finite space interval \([a, z_K]\) holds, there is a local weak solution, \( u_{1/2}, u_{3/2}, \ldots, u_{K-1/2} \) and \( v_0, \ldots, v_K \), for the system of ODE’s (56)-(57), defined up to a certain timepoint \( t' \). Now, we look for a bound in order to proceed the solution on \([0, T]\). Further regard from equation (57). We evaluate the energy estimate in the discrete form:

\[ \dot{v}_k u_k + r_k v_k \partial^h_z \left( \frac{1}{\sqrt{u_k}} \frac{1}{e^{At/2}} \Phi - \delta_1 \frac{u_k}{e^{2At}} + g_k \frac{1}{e^{At}} \right) + g_k \frac{r_k}{u_k} \frac{1}{e^{At}} = \mu r_k v_k \partial^h_z \left( \frac{\frac{\Phi^2}{r}}{u_k} \right) \frac{1}{e^{At}}, \]

where \( g_k = \delta^2 \frac{\Phi^2}{r}. \)
\[
\frac{1}{2} \sum_{k=0}^{K} v_k(t)^2 h - \int_0^t \sum_{0 < j < K} \partial_z^h (r_j v_j) \left( \frac{1}{\sqrt{u_j}} \frac{1}{e^{As/2}} \phi - \delta_1 \frac{u_j}{e^{2As}} + g_j \frac{1}{e^{As}} \right) h
\]
\[
+ \int_0^t \sum_{k=0}^{K} g_k r_k u_k v_k \frac{1}{e^{As}} h = - \int_0^t \sum_{0 < j < K} \mu \partial_z^h (r_j v_j) \left( \frac{\partial_z^h (v_j r_j)}{u_j} \right) \frac{1}{e^{As}} h,
\]
\[
\frac{1}{2} \sum_{k=0}^{K} v_k(t)^2 h - \int_0^t \sum_{0 < j < K} \dot{u}_j \left( \frac{1}{\sqrt{u_j}} \frac{1}{e^{As/2}} \phi - \delta_1 \frac{u_j}{e^{2As}} \right) h - \int_0^t \sum_{0 < j < K} \partial_z^h (r_j v_j) g_j \frac{1}{e^{As}} h
\]
\[
+ \int_0^t \sum_{0 < j < K} \mu \partial_z^h (v_j r_j) \frac{2}{u_j} h = - \int_0^t \sum_{k=0}^{K} g_k r_k u_k v_k \frac{1}{e^{As}} h.
\]

We can further transform
\[
- \int_0^t \sum_{0 < j < K} \dot{u}_j \left( \frac{1}{\sqrt{u_j}} \frac{1}{e^{As/2}} \phi - \delta_1 \frac{u_j}{e^{2As}} \right) h
\]
\[
= - \left\{ \sum_{0 < j < K} \left[ 2 \sqrt{u_j} \frac{\phi}{e^{At/2}} - \frac{1}{2} \delta_1 u_j^2 \frac{1}{e^{2At}} \right]
\]
\[
- \int_0^t \sum_{0 < j < K} 2 \sqrt{u_j} \left( \frac{\phi}{e^{As/2}} \right)_{\partial s} - \frac{1}{2} \delta_1 u_j^2 \left( \frac{1}{e^{2As}} \right)_{\partial s} \right\} h
\]
\[
= \sum_{0 < j < K} \left[ \frac{1}{2} \delta_1 u_j^2 \frac{1}{e^{2At}} - 2 \sqrt{u_j} \frac{\phi}{e^{At/2}} \right] h
\]
\[
+ \int_0^t \sum_{0 < j < K} A \delta_1 u_j^2 \frac{1}{e^{2As}} h - \int_0^t \sum_{0 < j < K} 2 \sqrt{u_j} \left( \frac{A \Phi}{2 e^{As/2}} - \frac{1}{e^{2As/2}} \phi s \right) h.
\]

It holds \( \frac{1}{4} \delta_1 u_j^2 \frac{1}{e^{2At}} - 2 \sqrt{u_j} \frac{\phi}{e^{At/2}} > -C \) for \( u_j > 0 \) on \([0, T] \). Then from equation (60) with Young’s inequality
\[
\frac{1}{2} \sum_{k=0}^{K} v_k(t)^2 h + \sum_{0 < j < K} \left[ \frac{1}{2} \delta_1 u_j^2 \frac{1}{e^{2At}} - 2 \sqrt{u_j} \frac{\phi}{e^{At/2}} \right] h
\]
\[
+ \int_0^t \sum_{0 < j < K} A \delta_1 u_j^2 \frac{1}{e^{2As}} h + \int_0^t \sum_{0 < j < K} \mu \frac{\partial_z^h (v_j r_j) \frac{2}{u_j}}{e^{As}} h
\]
\[
\leq \int_0^t \sum_{0 < j < K} \sqrt{u_j} \frac{A \Phi}{e^{As/2}} h + \int_0^t \sum_{0 < j < K} 2 u_j \frac{1}{e^{2As}} h + \int_0^t \sum_{0 < j < K} \frac{1}{2} \frac{e^{As} |\phi s|^2}{2} h
\]
\[
+ \int_0^t \sum_{k=0}^{K} \frac{g_k r_k}{2} \frac{1}{e^{2As}} h + \int_0^t \sum_{k=0}^{K} \frac{1}{2} \frac{r_k v_k^2}{2} h + \int_0^t \sum_{0 < j < K} \partial_z^h (r_j v_j) g_j \frac{1}{e^{As}} h.
\]
We have further
\[
\int_0^t \sum_{0<j<K} \frac{\partial_t^h (r_j v_j)}{e^{A_s}} g_j \frac{1}{e^{A_s}} h = \int_0^t \sum_{0<j<K} \frac{\partial_t^h (r_j v_j)}{\sqrt{u_j e^{A_s}}} \sqrt{u_j e^{A_s}} g_j h \\
\leq \int_0^t \sum_{0<j<K} \frac{\epsilon \partial_t^h (r_j v_j)^2}{u_j e^{A_s}} g_j h + \int_0^t \sum_{0<j<K} \frac{1}{2} \frac{u_j}{e^{A_s}} g_j h.
\]

Because \(0 \leq \frac{\rho_k}{r_k} \leq \frac{\rho_k}{a} \leq C\) for almost \(k\), we apply the Gronwall lemma on the functions \(u_j \frac{1}{e^{A_s}}, u_j^2 \frac{1}{e^{A_s}}\) and \(v_k^2\), one after another,

\[
\frac{1}{2} \sum_{k=0}^K v_k(t)^2 h + \sum_{0<j<K} \left[ \frac{1}{2} \delta_1 u_j^2 \frac{1}{e^{A_t}} - 2 \sqrt{u_j e^{A_t/2}} \right] h \\
+ \int_0^t \sum_{0<j<K} \left( \mu - \frac{\epsilon}{2} g_j \right) \frac{\partial_t^h (v_j r_j)^2}{u_j} \frac{1}{e^{A_s}} h \\
\leq C + \int_0^t \sum_{0<j<K} \frac{1}{2} e^{A_s} A^2 \Phi^2 h + \int_0^t \sum_{0<j<K} \frac{1}{2} e^{A_s} |\Phi|^2 h.
\]

With lemma 8 and \(0 \leq \Phi \leq 1\), it holds
\[
\int_0^t \sum_{0<j<K} \frac{1}{2} e^{A_s} A^2 \Phi^2 h + \int_0^t \sum_{0<j<K} \frac{1}{2} e^{A_s} |\Phi|^2 h \leq C.
\]

If we regard \(u_j > 0\), we get the bound
\[
\frac{1}{2} \sum_{k=0}^K v_k(t)^2 h + \sum_{0<j<K} \left[ \frac{1}{2} \delta_1 u_j^2 \frac{1}{e^{A_t}} - 2 \sqrt{u_j e^{A_t/2}} \right] h \\
+ \int_0^t \sum_{0<j<K} \left( \mu - \frac{\epsilon}{2} g_j \right) \frac{\partial_t^h (v_j r_j)^2}{u_j} \frac{1}{e^{A_s}} h \leq C.
\]

It holds,
\[
u_j \to \infty \quad \Rightarrow \quad \psi(u_j) \to \infty.
\]

From that it follows, as long as \(u_j(t) > 0\), there are constants \(C_1, C_2\) with
\[
|v_k| \leq C_1(C_0, h) \quad \text{and} \quad u_j \leq C_2(C_0, h)
\]
and
\[
a^2 \leq (r_k(t))^2 \leq a^2 + C(C_0, h) z_k.
\]

The local solution exists for all time. \(\square\)
Theorem 12 There is a weak solution \( \{u,v,r\} \) of system (50)–(52) satisfying the appropriate initial conditions, and \( u \in C([0,T];L^2(\Omega)) \cap B([0,T];H^{1,2}_{\text{loc}}(\Omega)) \), \( v \in C([0,T];L^2(\Omega)) \cap C([0,T];H_0^{1,2}(\Omega)) \), and \( r \in C([0,T] \times \Omega) \).

Proof. We construct approximate solutions \( \{u^h,v^h,r^h\} \) as follows:

- \( u^h(t,z_j) = u_j(t) \) for all \( j = \frac{1}{2}, \frac{3}{2}, \ldots \) and \( u^h(t,) \) has to be a linear function of \( z \) on each interval \([z_j, z_{j+1}]\) with the exception that \( u^h(t,) \) is constant in \( z \) on \([0, z_{1/2}]\) and on intervals \([z_j, z_{k_i}]\) and \([z_{k_i}, z_{j+1}]\) when \( k_i = j + \frac{1}{2} \);
- \( v^h(t,z_k) = v_k(t) \) and we take \( v^h(t,) \) to be linear functions of \( z \) on each interval \([z_k, z_{k+1}]\);
- finally, we define \( r^h(t,z)^2 = a^2 + 2 \int_0^z u^h(t,z') dz' \).

Then from the inequality (62) the approximate solution satisfies

\[
\int_a^R \left[ \frac{1}{2} (v^h)^2 + \overline{\psi}(u^h) \right] (t,z) dz + \int_0^t \int_a^R \frac{\mu - \epsilon}{\alpha u^h} (r^h v^h)_z^2 dz' ds \leq C \tag{65}
\]

From (63), it holds \( 0 \leq u^h(t,z) \leq C \). With a compactness argument, we can take \( h \downarrow 0 \) and conclude the existence of the solution \( \{u,v,r\} \) satisfying the equations (50), (51) and (52).

We transform this solution back to the Euclidean coordinates and get a weak solution \( \{\rho,v\} \) of system (15)–(16) satisfying (5), (6) with appropriate regularity.

5 Existence of the Solution for the Problem (1)–(7)

Theorem 13 Under assumptions made above, there is a weak solution of the problem (1)–(7).

Proof. We shall use the Leray-Schauder fix point theorem in the Banach space

\[
B := \{(\rho,v,\Phi) : (\rho,v) \in H^{1,2}(0,T;L^2(\Omega)) \times [H^{1,4}(0,T;L^4(\Omega)) \cap C^0(0,T;H^{1,4}(\Omega))], \quad \Phi \in H^{1,2}(\Omega_T) \}. \]

For this reason we regard the operator \( T_\alpha \):

\[
((\rho,v,\Phi) = T_\alpha((\rho_1,v_1,\Phi_1)) \quad \text{for all} \quad ((\rho_1,v_1,\Phi_1)) \in B, \quad 0 \leq \alpha \leq 1,
\]

23
which is defined by the solution of the problem:

\[ \rho_t + \text{div} (\rho v) - A\rho = 0, \quad (66) \]

\[ (\rho v)_t + \text{div} (\rho v v) + p(\rho, \alpha \Phi)|_{x_i} - \mu_1 \Delta v_i - \mu_2 \text{div} v|_{x_i} - v A\rho = \alpha \left( -\delta^2 \text{div} (\Phi|_x \nabla \Phi) \right), \quad (67) \]

\[ \Phi_t + \alpha v \cdot \nabla \Phi + c_1 \ln \left( \frac{\Phi}{1-\Phi} \right) + c_2 (1 - 2\Phi) - (1 - \Phi) A - \delta^2 \Delta \Phi = \alpha \left( -\text{dcp} \rho \right), \quad (68) \]

where

\[ \vec{n} \cdot \nabla \Phi = 0, \quad v = 0 \quad \text{on} \quad (0, T] \times \partial \Omega \quad (69) \]

and

\[ \rho(0, x) = \rho_0(\|x\|), \quad (\rho v)(0, x) = \rho_0(\|x\|)v_0(\|x\|)x/r, \quad \Phi(0, x) = \Phi_0(\|x\|) \quad \text{in} \quad \Omega, \quad (70) \]

for \( i = 1, 2, \ldots \).

The existence of a solution was shown in section 3 and 4. Therefore the operator \( T_\alpha \) is well defined from \( B \) into \( B \). The continuity of the Navier-Stokes equation is proved in [3] and proceeds here in a analogical way. The continuity of the Allen-Cahn equation follows with similar arguments as the uniqueness in lemma 10. Therefore \( T_\alpha \) fulfills the continuity condition needed in the Leray-Schauder theorem. In this case a fix point exists which constitute the radially symmetric solution of (1)–(7). \( \square \)

References


