Conference on Difference Equations,

Special Functions and Applications

July 25 - July 30, 2005, Munich, Germany

Dedicated to the Memory of Bernd Aulbach

Munich University of Technology (TUM) /

Arcisstrasse, Theresienstrasse

Munich, July 2005

Dear Participants,

welcome to the International Conference on Difference Equations, Special Funktions and Applications in Munich. This is the first joint meeting of the three communities ISDE, OPSFA and SIDE.

We wish you a successful week during this meeting and hope that it will give you a lot of inspiration.

The Organizers

Local Organizing Committee:
Prof. Dr. Peter Kloeden
Prof. Dr. Rupert Lasser
Prof. Dr. Frank Nijhoff
Dr. Andreas Ruffing

Scientific Committee:
Richard Askey
Bernd Aulbach
Christian Berg
Alexander Bobenko
Saber Elaydi
Basil Grammaticos
Jarmo Hietarinta
Mourad Ismail
Nalini Joshi
Gerry Ladas
Rupert Lasser
Lance Littlejohn
Vassilis Papageorgiou
Allan Peterson
George Sell
Alexandr Sharkovsky
Sergei Suslov
Pavel Winternitz

Contents

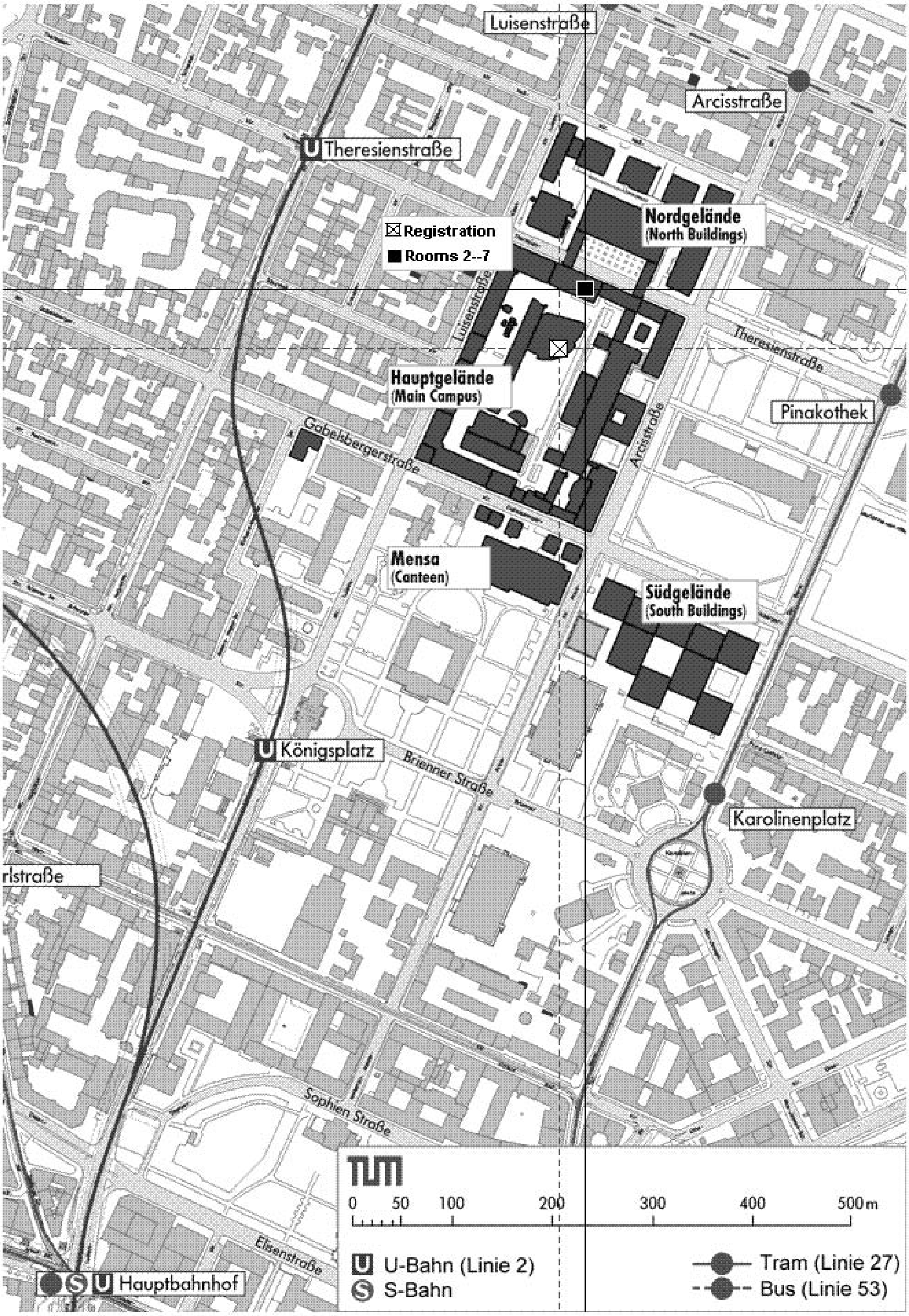
Proceedings of the Conference

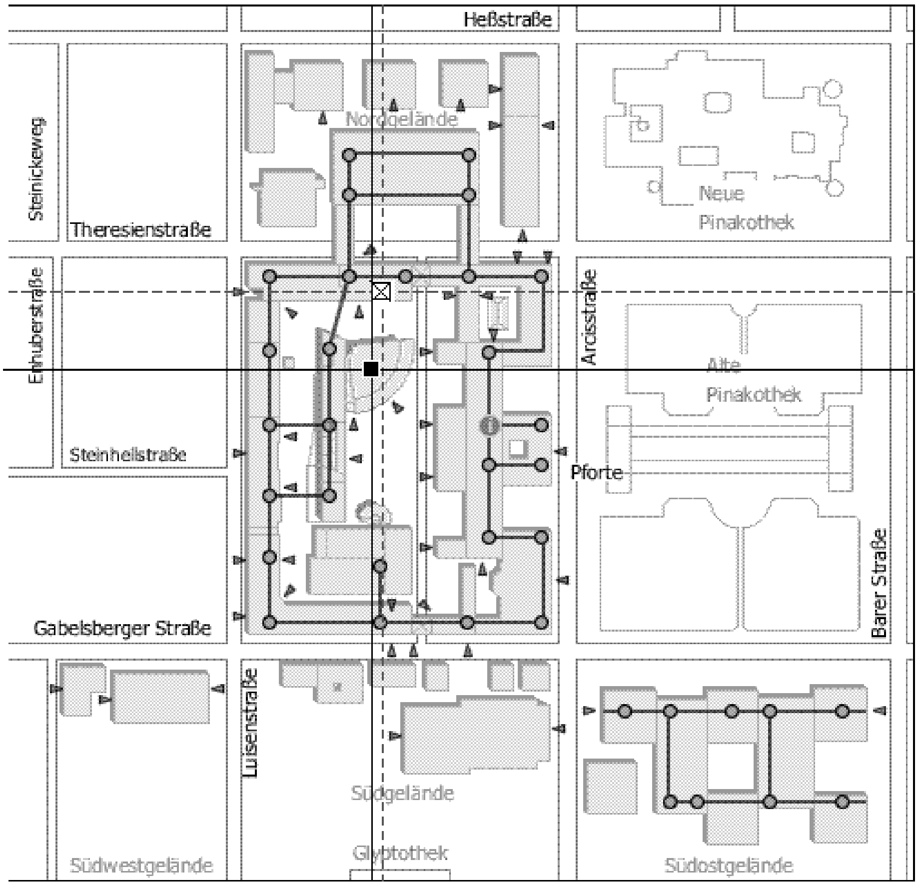
We intend to publish the proceedings of the conference by World Scientific. The deadline for submission of manuscripts is the first of November, 2005. All papers should be sent to Denise Wilson (dwilson@trinity.edu).

The following persons have agreed to be editors of the proceedings: Jim Cushing, Saber Elaydi, Rupert Lasser, Andreas Ruffing, Walter van Assche.

All papers will be refereed.

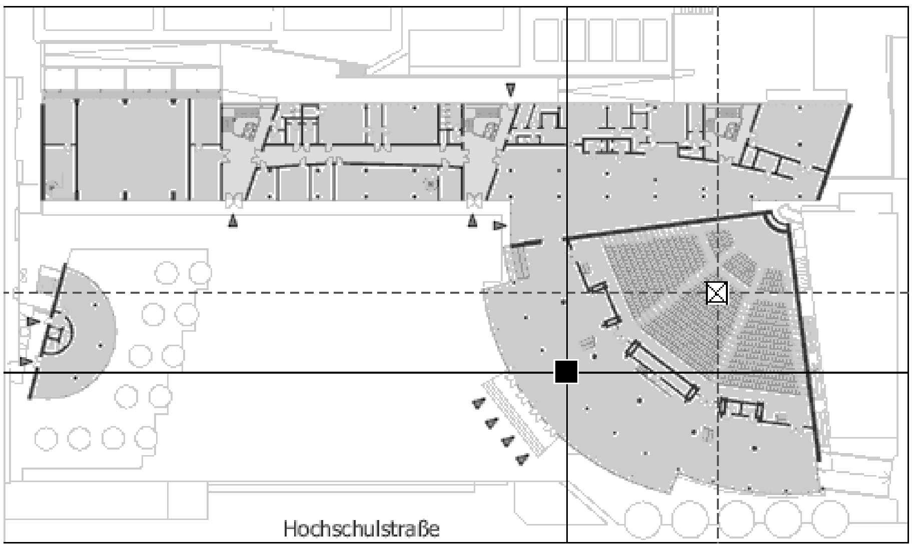
Map of the Conference Site





■ **Room 1 (Audimax)**

☒ **Rooms 2 to 7: Contributed Talks / Main Talks**



☒ **Room 1: Main Talks**

■ **Room 1: Registration, Coffee Break**

International Conference on Difference Equations, Special Functions and Applications

July 25 – July 30, 2005, TU Munich

Time	Monday, July 25	Tuesday, July 26	Wednesday, July 27	Thursday, July 28	Friday, July 29	Saturday, July 30
9 - 10	**Opening**	Sharkovsky	Saff	Joshi	Smith	Ismail
10 - 11	Simon	Geronimo / Trigiante	Devaney / Stahl	Noumi / Suslov	Bohner / Rains	Hayman / Ricci
11 - 11:^{30}	**Coffee break**	**Coffee break**	Zhedanov / van Doorn	**Coffee break**	**Coffee break**	**Coffee break**
11^{30} - 12^{00} 12^{00} - 12^{30}	Elaydi / Berg	Cushing / van Assche		Sergeyev / Marcellan	Valent / Everitt	Berezansky Koziakin
12^{30} - 14^{00}	**Lunch**	**Lunch**		**Lunch**	**Lunch**	**Closing**
14 - 15	contributed	contributed	***Excursion***	contributed	Lutz / Koepf	
15 - 16	contributed	contributed		contributed	contributed	
16 - 16^{30}	**Coffee break**	**Coffee break**		**Coffee break**	**Coffee break**	
16^{30} - 17^{30}	contributed	contributed		contributed	Workshop computer algebra	
17^{30} - 18^{30}	contributed	meetings		contributed		
	Problem session			**Dinner**		

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Munich University of Technology (TUM) / Arcisstrasse, Theresienstrasse

Timetable

Monday, July 25, 2005

Time	Room 1 (Audimax)	Room 2 (0602)	Room 3 (0606)	Room 4 (0601)	Room 5 (0670)	Room 6 (1601)	Room 7 (2601)
08:00 - 09:00	Registration						
09:00 - 10:00	**Opening**						
10:00 - 11:00	Simon (Chair: Littlejohn)						
11:00 - 11:30				**Coffee break**			
11.30 - 12.30		Elaydi (C: Littlejohn)	Berg (Chair: Suslov)				
12:30 - 14:00				**Lunch**			
Chairman:		Hamaya	Zhedanov	Krause	Clarkson	Lutz	Geronimo
14:00 - 14:30		Fernandes	Takata	Bau-Sen Du	Atakishiyev	Akin-Bohner	Cachafeiro
14:30 - 15:00		Marini	Rocha	Foley	Alvarez-Nodarse	Kostrov	Cesarano
15:00 - 15:30		Vaz	Pinar	Reinfelds	Lewanowicz	Pituk	Castell
15:30 - 16:00		Cabral	Branquinho	Correira-Ramos	Terwilliger	Hall	Costas-Santos
16:00 - 16:30				**Coffee break**			
Chairman:		Hamaya	Zhedanov	Matsunaga	Clarkson	Lutz	Elaydi
16:30 - 17:00		Gronau	Loureiro	Ivanov	Johnston	Peterson	Akca
17:00 - 17:30		Barrio Blaya	de Vicente	Kulenovic	Takano	Dosla	Sarreira
17:30 - 18:00		Tkachenko	Sadov	Jimenez-Lopez	Chu	Lopez-Fenner	Yen
18:00 - 20:00		**Open Problem Session**					

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Timetable

Tuesday, July 26, 2005

Time	Room 1 (Audimax)	Room 2 (0602)	Room 3 (0606)	Room 4 (0601)	Room 5 (0670)	Room 6 (1601)	Room 7 (2601)
09:00 - 10:00	Sharkovsky (C: Kloeden)						
10:00 - 11:00		Geronimo (C: Kloeden)	Trigiante (C: Marcellan)				
11:00 - 11:30	**Coffee break**						
11.30 - 12.30		Cushing (C: Kloeden)	van Assche (C: Marcellan)				
12:30 - 14:00	**Lunch**						
Chairman:		van Doorn	Everitt	Oliveira	Hietarinta	Dosla	Erbe
14:00 - 14:30		Ohira	Liz	Immink	Kuijlaars	Ruzickova	Nam
14:30 - 15:00		Habibullin	Iwata	Kon	Clarkson	Kent	Oliveira
15:00 - 15:30		Novokshenov	Romanovski	Hamaya	Rodal	Lawrence	Grechko
15:30 - 16:00		Schlichenmaier	Galkin	Balibrea	Ronveaux	Oberste-Vorth	Pinelas
16:00 - 16:30	**Coffee break**						
Chairman:		van Doorn	Everitt	Oliveira	Hietarinta	Dosla	
16:30 - 17:00		Lievens	Januário	Siegmund	Levi	Krause	
17:00 - 17:30		Koelink	Duarte	Stefanidou	Connett	Nesemann	
17:30 -		**ISDE meeting**					

Wednesday, July 27, 2005

Time	Room 1 (Audimax)	Room 2 (0602)	Room 3 (0606)	Room 4 (0601)	Room 5 (0670)	Room 6 (1601)	Room 7 (2601)
09:00 - 10:00	Saff (Chair: Geronimo)						
10:00 - 11:00		Devaney (C: Geronimo)	Stahl (Chair: Noumi)				
11.00 - 12.00		Zhedanov (Geronimo)	van Doorn (C: Noumi)				
	Social Event (for further information see conference homepage)						

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Timetable

Thursday, July 28, 2005

Time	Room 1 (Audimax)	Room 2 (0602)	Room 3 (0606)	Room 4 (0601)	Room 5 (0670)	Room 6 (1601)	Room 7 (2601)
09:00 - 10:00	Joshi (Chair: Nijhoff)						
10:00 - 11:00		Noumi (Chair: Nijhoff)	Suslov (Chair: Chihara)				
11:00 - 11:30	**Coffee break**						
11.30 - 12.30		Sergeyev (C: Nijhoff)	Marcellan (C: Chihara)				
12:30 - 14:00	**Lunch**						
Chairman:		Ismail	Peterson	Hayman	Y. Berezansky	Lopez-Fenner	Elaydi
14:00 - 14:30		Tsujimoto	Pötzsche	Ernst	Abreu	Camouzis	Musa
14:30 - 15:00		Roberts	Schmeidel	Foupouagnigni	Assal	Erbe	Khusainov
15:00 - 15:30		Tongas	Matsunaga	Chaggara	Kamoun	Quinn	Mikolajski
15:30 - 16:00		Papageorgiou	Nowakowska	Lang	Stokman	Papaschinopoulos	Vinagre
16:00 - 16:30	**Coffee break**						
Chairman:		Clarkson	Hamaya	Hayman	Y. Berezansky	Lopez-Fenner	Peterson
16:30 - 17:00		Tsuda	Devault	Coussement	Christiansen	Severino	Hilger
17:00 - 17:30		Masuda	Schultz	Moreno	Pedersen	Schinas	Hilscher
17:30 - 18:00		Ohta	Kalabusic	Minguez-Cheniceros	Otte	Diblik	M. Simon
18:00 - 18:30		Zhu	Thomas	Sanchez-Moreno		Shapira	Ruffing
19:00 -	**Bavarian Evening**						

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Timetable

Friday, July 29, 2005

Time	Room 1 (Audimax)	Room 2 (0602)	Room 3 (0606)	Room 4 (0601)	Room 5 (0670)	Room 6 (1601)	Room 7 (2601)
09:00 - 10:00	Smith (Chair: Ladas)						
10:00 - 11:00		Bohner (Chair: Ladas)	Rains (Chair: Berg)				
11:00 - 11:30				**Coffee break**			
11.30 - 12.30		Valent (Chair: Immink)	Everitt (Chair: Berg)				
12:30 - 14:00				**Lunch**			
14:00 - 15:00		Lutz (Chair: Immink)	Koepf (Chair: Berg)				
Chairman:		Paule	Balibrea	Petkovsek	Ricci	Clarkson	Saff
15:00 - 15:30		de Bruin	Rasmussen	Majidian	Belitskii	Rodkina	Martinis
15:30 - 16:00		Laforgia	Ünal	Minesaki	Heredero	van den Bult	Svrtan
16:00 - 16:30				**Coffee break**			
Chairman:		de Bruin	Balibrea	Gronau	Ricci	Clarkson	Erbe
16:30 - 17:00		Delgado	Manosa	Natalini	Horvath	*Petkovsek*	Khristoforov
17:00 - 17:30		van den Berg	Rehak	Lasecka-Plura	Bouzahir		Tierz
17:30 - 18:00		Ben Cheikh	Györi	Amiraliyev	Coutinho	*Fredet*	L. Berezansky
18:00 - 18:30		Nicolau	Janglajew	Syam	Macedo		Elyseeva
18:30 - 19:30						*Paule*	
19:30 - 20:30						*van Hoeij*	

Saturday, July 30, 2005

Time	Room 1 (Audimax)	Room 2 (0602)	Room 3 (0606)	Room 4 (0601)
09:00 - 10:00	Ismail (Chair: Askey)			
10:00 - 11:00		Hayman (Chair: Askey)	Ricci (Chair: Valent)	
11:00 - 11:30	**Coffee break**			
11.30 - 12.30		Berezansky (C: Joshi)	Kozyakin (C: Kloeden)	
12:30 - 13:00	**Closing**			

name	Contributed Paper
name	Workshop / Computer algebra

Main Talks

The complex moment problem

Yurij M. Berezansky

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We consider the sequence of complex numbers $(s_{m,n})_{m,n=0}^{\infty}$. Such sequence is by definition the complex moment sequence if

$$s_{m,n} = \int\limits_{\mathbb{C}} z^m \bar{z}^n d\rho(z), \qquad m,n = 0, 1 \ldots, \tag{1}$$

where $d\rho(z)$ is a finite Borel measure on the complex plane $\mathbb{C}$.
In the talk the conditions are given which guarantee the representation (1) and also its infinite dimensional analogue.
Explain, that for infinite dimensional situation $s_{m,n}$ are bilinear functionals on nuclear complex space Φ and in (1) $z^m \bar{z}^n$ are replaced by $z^{\otimes m} \widehat{\otimes} \bar{z}^{\otimes n}$ where z belongs to the conjugate space Φ'; $d\rho(z)$ is a measure on Φ'.
The results are obtained by application of generalized eigenvector techniques for an infinite family of commuting normal operators. The connection between moments (1) and the corresponding "Jacobi matrix" is also investigated.

References and Literature for Further Reading

[1] Yu. M. Berenzansky. Some generalizations of the classical moment problem, *Integr. equ. oper. theory*, **44** (2002), 255–289.

[2] Yu. M. Berezansky, M. E. Dudkin. The complex moment problem in the exponential form, *Methods Funct. Anal. Topology*, **10** (2004), No. 4, 1–10.

[3] Yu. M. Berezansky, M. E. Dudkin. On the complex moment problem, *Math. Nachr.* (2005), to appear.

Logarithmic order and type of entire functions associated with indeterminate moment problems

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In this talk I will report on ongoing research with Henrik L. Pedersen about the logarithmic order and type of entire functions in the Nevanlinna matrix for an indeterminate moment problem. In earlier work [1], [2] it was proved that the four entire functions of the Nevanlinna matrix for an indeterminate Hamburger moment problem have the same order, type and Phragmén-Lindelöf indicator. For moment problems connected with q-series the common order is zero.
In this case we study a refined scale which we call logarithmic order and type and prove that these new concepts also agree for the four entire functions. For the q-Meixner indeterminate moment problem the common logarithmic order is 2 and the logarithmic type is $\frac{1}{2\log(1/q)}$.

References and Literature for Further Reading

[1] C. Berg, H. L. Pedersen, On the order and type of the entire functions associated with an indeterminate Hamburger moment problem, *Ark. Mat.* **32** (1994), 1–11.

[2] C. Berg, H. L. Pedersen, Nevanlinna matrices of entire functions, *Math. Nachr.* **171** (1995), 29–52.

Dynamic Equations on Time Scales

MARTIN BOHNER

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Time scales have been introduced in order to unify continuous and discrete analysis and in order to extend those theories to cases "in between". We will offer a brief introduction into the calculus involved, including the so-called delta derivative of a function on a time scale. This delta derivative is equal to the usual derivative if the time scale is the set of all real numbers, and it is equal to the usual forward difference operator if the time scale is the set of all integers. However, in general, a time scale may be any closed subset of the reals. We present some basic facts concerning dynamic equations on time scales (those are differential and difference equations, resp., in the above two mentioned cases) and initial value problems involving them. We introduce the exponential function on a general time scale and use it to solve initial value problems involving first order linear dynamic equation. We also present a unification of the Laplace and Z-transform, which serves to solve any higher order linear dynamic equations with constant coefficients.
Throughout the talk, many examples of time scales will be offered. Among others, we will discuss the following examples:

1. The two standard examples (the reals and the integers).

2. The set of all integer multiples of a positive number (this time scale is interesting for numerical purposes).

3. The set of all integer powers of a number bigger than one (this time scale gives rise to so-called q-difference equations).

4. The union of closed intervals (this time scale is interesting in population dynamics; for example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population).

References and Literature for Further Reading

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001

[2] M. Bohner and A. Peterson (editors), *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003

Periodically Forced Difference Equations and Applications Arising From Population Dynamics

JIM M. CUSHING

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Most models in population dynamics and theoretical ecology are autonomous, even though most biological populations are subject to fluctuating environments (resource availability, vital birth and death rates, etc.). A general mathematical problem is to determine what effects fluctuating model parameters have on the model dynamics. One tenet formulated in the 1970s is that a periodically fluctuating environment is deleterious to a biological population (in the sense that its average population size increases when an oscillating environment is instead held constant at the mean). This talk is motivated by some experimental data that contradicts this tenet and the search for an explanation using non-autonomous, periodically forced difference equation models. Using bifurcation and perturbation methods I will discuss some general existence and stability theorems for periodic solutions of nonlinear, periodically forced difference equations and some results and open problems concerning the average of the periodic solutions.

Sierpinski Curve Julia Sets for Complex Rational Maps

ROBERT L. DEVANEY

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In this lecture we describe the structure of both the Julia sets and the parameter planes for families of rational maps of the form $z^n + C/z^n$ where C is a complex parameter. These sets come in a variety of different shapes, including Sierpinski curves and gaskets, Cantor sets, McMullen rings, and more. On the other hand, unlike the complicated Mandelbrot set, i.e., the parameter plane for the family $z^2 + C$, the parameter plane for these maps has a relatively easy-to-understand structure.

Orthogonal polynomials and birth-death processes with killing

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A birth-death process $\{X(t),\ t \geq 0\}$ is a type of stochastic process on (a subset of) the integers in which transitions take place only to neighbouring states. By a famous result of Karlin and McGregor the transition probabilities $\Pr\{X(t) = j \mid X(0) = i\},\ t \geq 0$, of a birth-death process may, under suitable conditions, be expressed in terms of a sequence of orthogonal polynomials and their orthogonalizing measure. This representation has led to detailed knowledge of many specific birth-death processes and considerable insight into the behaviour of birth-death processes in general.

Evidently, it is of interest to investigate to which extent properties of birth-death processes retain their validity if one allows more general transition structures. Such investigations are usually hampered by the fact that the Karlin-McGregor representation and the analytical tools that go with it are no longer available. The class of processes which is the subject of the lecture – and which comprises an outwardly mild generalization of birth-death processes – does not have this drawback. At the same time the class is interesting because it displays several of the phenomena that occur beyond the setting of the pure birth-death process.

Concretely, I will consider birth-death processes on the set $\{-1, 0, 1, \ldots\}$, with -1 an absorbing bottom state, and the additional feature that absorption in one step (*killing*) may occur from any state rather than just state 0. I will explain why the orthogonal-polynomial representation remains valid in the generalized setting, so that the orthogonal-polynomial toolbox may be used to describe the behaviour of such a process. In particular the existence of *quasi-stationary distributions* (initial distributions with the property that the state distribution of the process, conditional on non-absorption, is constant over t) will be related to the asymptotic behaviour of the orthogonal polynomials involved.

The talk is based on joint work with P. Coolen-Schrijner and A. Zeifman.

References and Literature for Further Reading

[1] P. Coolen-Schrijner and E.A. van Doorn, Birth-death processes with killing: orthogonal polynomials and quasi-stationary distributions (2005), Preprint.

[2] E.A. van Doorn and A.I. Zeifman, Birth-death processes with killing. *Statist. Probab. Lett.* **72** (2005) 33-42.

[3] E.A. van Doorn and A.I. Zeifman, Extinction probability in a birth-death process with killing. *J. Appl. Probab.* **42** (2005) 185-198.

Nonautonomous Difference Equations

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We will present extensions of three fundamental results in autonomous difference equations to periodic nonautonomous difference equations. This task will be accomplished via the techniques of skew-product dynamical systems.

Fourth-order Bessel-type special functions

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The even-order Bessel-type special functions were first defined in the paper [EM] of 1994, by Everitt and Markett. A significant earlier paper [THK], by Koornwinder, on the Jacobi-type and the Laguerre-type orthogonal polynomials was published in 1984.

The Bessel-type special functions have a direct inheritance from the properties of the classical Bessel functions and the Laguerre-type polynomials.

The fourth-order Bessel-type special functions satisfy the fourth-order linear differential equation, see [DEHLM, Section 1],

$$(xy''(x))'' - ((9x^{-1} + 8M^{-1}x)y'(x))' = \lambda^2(\lambda^2 + 8M^{-1})xy(x) \text{ for all } x \in (0, \infty), \quad (1)$$

where $M > 0$ is a positive parameter, and $\lambda \in \mathbb{C}$ is a spectral parameter. This differential equation is Lagrange symmetric (formally self-adjoint) on the open half-line $x \in (0, \infty)$; when considered on the complex plane, with the variable x replaced by $z \in \mathbb{C}$, the equation has a regular singularity at the origin 0 and an irregular singularity at the point at infinity of $\mathbb{C}$. The Frobenius analysis at the origin 0 yields integer indicial roots $-2, 0, 2, 4$.

Two linearly independent solutions of the differential equation (1) are, see [DEHLM, Section 1, (1.4) and (1.6)],

$$J_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]J_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}J_1(\lambda x) \text{ for all } x \in (0, \infty) \quad (2)$$

and

$$Y_\lambda^{0,M}(x) := [1 + M(\lambda/2)^2]Y_0(\lambda x) - 2M(\lambda/2)^2(\lambda x)^{-1}Y_1(\lambda x) \text{ for all } x \in (0, \infty); \quad (3)$$

here J and Y on the right-hand sides of (2) and (3) represent the classical Bessel functions. From a result of Rofe-Beketov, see [DEHLM,Reference [28]], it appears that any additional solutions of (1), which are linearly independent of the solutions (2) and (3), cannot be expressed explicitly in terms of $J_\lambda^{0,M}$ and $Y_\lambda^{0,M}$, say by quadrature, nor in terms of the classical Bessel functions.

We note that $J_\lambda^{0,0}(x) = J_0(\lambda x)$ for all $x \in [0, \infty)$ and all $\lambda \in \mathbb{C}$; similarly for $Y_\lambda^{0,0}$ and Y_0.
The spectral properties of the differential equation (1), including the definition of self-adjoint operators, are studied in the papers [DEHLM] and [EKLM] and considered in the two Hilbert function spaces:

1. The Lebesgue integral weighted space

$$L^2((0,\infty);x) := \left\{ f : (0,\infty) \to \mathbb{C} : \|f\|^2 = \int_0^\infty x\,|f(x)|^2\,dx < +\infty \right\}. \quad (4)$$

2. The Lebesgue-Stieltjes space, here the real parameter $k \in (0, \infty)$,

$$L_k^2(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} : \|f\|_k^2 = k\,|f(0)|^2 + \int_0^\infty x\,|f(x)|^2\,dx < +\infty \right\}. \quad (5)$$

In the space $L_k^2(\mathbb{R})$,when k takes the special value $M/2$, there is a generalised Hankel transform involving the Bessel-type function $J_\lambda^{0,M}$ to compare with the classical Hankel transform for the Bessel function J_0; for the classical case see the text of Titchmarsh [ECT, Chapter VIII, Sections 8.4 and 8.18 both with $\nu = 0$].
For the classical Bessel function J_0 there is a distributional orthogonality result, see [EM,Section 1, (1.7)],

$$\lambda \int_0^\infty J_0(\lambda x) J_0(\mu x) x\ dx = \delta(\lambda - \mu) \text{ for all } \lambda, \mu \in (0, \infty). \quad (6)$$

The corresponding result for the fourth-order Bessel-type function $J_\lambda^{0,M}$, see [EM,Section 4, Corollary 4.3], is

$$\frac{\lambda}{\left[1 + \frac{1}{4} M \lambda^2\right]^2} \left\{ \int_0^\infty J_\lambda^{0,M}(x) J_\mu^{0,M}(x) x\ dx + \tfrac{1}{2} M J_\lambda^{0,M}(0) J_\mu^{0,M}(0) \right\} = \delta(\lambda - \mu), \quad (7)$$

which holds for all $\lambda, \mu \in (0, \infty)$.
The connection between the classical Bessel function J_0 and the Bessel-type function $J_\lambda^{0,M}$ is further extended in noting that the result (7) tends formally to the result (6), as M tends to zero.
The lecture will discuss these results and properties.

Authors

The lecture reports on joint work with Jyoti Das, D.B. Hinton, H. Kalf, L.L. Littlejohn and C. Markett.

References and Literature for Further Reading

[DEHLM] Jyoti Das, W.N. Everitt, D.B. Hinton, L.L. Littlejohn and C. Markett. The fourth-order Bessel-type differential equation. *Applicable Analysis.* **83** (2004), 325-362.

[EKLM] W.N. Everitt, H. Kalf, L.L. Littlejohn and C. Markett. Additional properties of the fourth-order Bessel-type differential equation. (Submitted for publication: November 2004.)

[EM] W.N. Everitt and C. Markett. On a generalization of Bessel functions satisfying higher-order differential equations. *Jour. Computational Appl. Math.* **54** (1994), 325-349.

[THK] T.H. Koornwinder. Orthogonal polynomials with weight function $(1-x)^{\alpha}(1+x)^{\beta} + M\delta(x+1) + N\delta(x-1)$. *Canad. Math. Bull.* **27** (2) (1984), 205-214.

[ECT] E.C. Titchmarsh. *The theory of Fourier integrals.* (Oxford University Press: 1948.)

Two Variable Orthogonal Polynomials and Factorization

Jeffrey S. Geronimo

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Using the lexicographical ordering we will consider two variable polynomials orthogonal on the bicircle with respect to some positive linear functional. Recurrence formulas will be derived and the properties of the recurrence coefficients will be investigated. In particular if the orthogonality measure is the inverse of positive bivariate trignometric polynomial conditions in terms of the recurrence coefficients will be given in order for it to be factorized as the magnitude square of a stable polynmial. This is work done with H. Woerdeman.

On the zeros of a q-Bessel function and some related functions

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This lecture is about a method of getting at zeros via the power series of the functions.

The Ramanujan Continued Fractions

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Ramanujan's note books are full of continued fractions. Many of them are special values of continued J-fractions. We shall discuss some of the orthogonal polynomials that arise from studying continued fractions in the note books. Of special interest are some continued fractionsfrom the lost note book, which do not converge but their convergents of order 3k+j converge to different limits depending on j. This is then generalized from modulo 3 to general moduli.

Analytic results for (ultra-discrete) cellular automata

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Cellular automata (CA) have been widely adopted in the sciences as simple but powerful models of the real world. The complex patterns produced by their long-time behaviours are used to confirm the correctness of scientific hypotheses underlying the model. However, the analysis essential for testing scientific understanding appears to be missing. We show the beginnings of the mathematical analysis needed by reviewing the connection that leads from differential equations to ultra-discrete equations which are extended CA. New results for ultra-discrete equations include a test for integrability and results on solvability through associated linear problems.

Computer Algebra Methods for Orthogonal Polynomials

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In this talk we will show how computer algebra can be used in the study of orthogonal polynomials and special functions. In such computations the following algorithms are most often used: linear algebra techniques, multivariate polynomial factorization and the solution of nonlinear equations, e. g. by Gröbner basis techniques, see e.g. [1].

The classical orthogonal polynomials named after Jacobi, Gegenbauer, Chebyshev, Legendre, Laguerre, Hermite and Bessel can be classified as the polynomial solutions of second order differential equations. Similarly the classical "discrete" orthogonal polynomials named after Hahn, Krawtchouk, Meixner and Charlier are classified as the polynomial solutions of second order difference equations [5].

Using computer algebra one can compute the recurrence equations and hypergeometric representations of these systems, one can convert this process by computing differential and difference equations from the hypergeometric representations automatically, and one can decide whether a recurrence equation has classical orthogonal polynomial solutions ([2], [3], [4]). We will discuss these and related algorithms, and give some on-line demonstrations with *Maple*.

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Asymptotic and convergent representations for solutions of linear and nonlinear difference equations

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For systems of linear difference equations or recurrence relations, asymptotic representations for solutions can be achieved by procedures which are sometimes called "asymptotic diagaonalization". This means that first some explicit transformations are used to bring a given system into the form of a main diagonal term plus a small perturbation and then an implicitly determined transformation completes the diagonalization. Thus the problem of determining asymptotic representations is reduced to one-dimensional equations. In some earlier work with Z. Benzaid we showed the implicitly defined transformations are asymptotic to the identity matrix and now in some recent results with S. Bodine we are able to more accurately estimate the remainders using majorants.

In the second part of the talk, some results obtained jointly with R. Gerard will be discussed, which have to do with the problem of obtaining convergent representations for the error terms for certain classes of difference equations and more general types of operator equations having formal power series solutions.

Hypergeometric tau-functions for the elliptic difference Painlevé equation

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In this talk we will present a survey on analytic properties of a family of polynomials orthogonal with respect to a wide class of Sobolev type inner products in terms of measures with unbounded support on the real line. In particular different kind of asymptotic properties of such polynomials as well as the contracted zero distribution is analyzed. For some basic references see [1] and [2].

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Hypergeometric tau-functions for the elliptic difference Painlevé equation

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There are several ways to formulate the elliptic difference Painlevé equation with affine Weyl group symmetry of type E_8. In this talk I will discuss a system of non-autonomous bilinear equations of Hirota-Miwa type for tau- functions, following the lattice approach of Ohta-Ramani-Grammaticos. In particular I will explain how hypergeometric tau-functions for this system are constructed by means of Casorati determinants from the elliptic hypergeometric integrals in the sense of Spiridonov-van Diejen-Rains. This talk is based on a joint work with K.Kajiwara, Y.Komori, T.Masuda, Y.Ohta, and Y.Yamada.

Multi-variable Gould-Hopper and Laguerre polynomials

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The *monomiality principle* was introduced by G. Dattoli in order to derive properties of special polynomials starting from the corresponding ones of monomials.

The idea of monomiality traces back to the early forties of the last century, when J.F. Steffensen, in a largely unnoticed paper, suggested the concept of *poweroid*.

Y. Ben Cheikh proved that all polynomial families can be viewed as *quasi-monomials* with respect to a suitably defined derivative and multiplication operators. Unfortunately, it turns out that the derivative and multiplication operators for a general set of polynomials are expressed by formal series of the ordinary derivative, so that it is in general impossible to obtain sufficiently simple formulas to handle.

However, for particular polynomials sets, related to suitable classes of generating functions, the above mentioned formal series reduce to finite sums, so that the relevant properties can be easily derived. The leading set in this field is given by the Hermite-Kampé de Fériet or Gould-Hopper polynomials, since many sets of multi-variable or multi-index polynomials have been constructed starting from this important polynomial family.

Multi-index Hermite polynomials were used in order to study the distribution of coherent (or not coherent) radiation fields in quantum optics, the multidimensional coupled systems for electromagnetic radiation problems, and the relevant wave propagation phenomena.

In this article we show a quite general technique for constructing the monomiality operators of many-variables polynomial sets, starting from the corresponding operators of a basic monomial (or quasi-monomial) set of functions. Assuming the basic set of monomials x^n or $x^n/n!$, the many-variables Hermite and Laguerre-type

polynomials are derived and the relevant properties are easily constructed, showing connections with the Bell polynomials.

The Laguerre derivative $D_L := DxD$, and its iterations $D_{nL} := DxDx \cdots DxD$ (containing $(n+1)$ derivatives), are used in order to derive families of higher order multi-variable Laguerre polynomials, which correspond to the multi-variable Gould-Hopper ones.

Finally, an application of the Laguerre derivative in the framework of population dynamics models is shown.

AMS CLASSIFICATION: 44A45, 30D05, 33C45.
KEY WORDS: Exponential operators. Monomiality principle. Gould-Hopper and Laguerre polynomials. Laguerre-type population dynamics.

Elliptic analogues of multivariate beta integrals

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One of the most important "special integrals" is, of course, the beta integral, with its central role in the theory of Gauss' hypergeometric function. I will discuss three particular generalizations of the beta integral, simultaneous generalizations of Selberg-type multivariate beta integrals and Spiridonov's elliptic beta integral. In each case, there is an associated family of difference operators satisfying a sort of adjointness property leading to an easy proof of the integral. I will also discuss the corresponding analogues of the Euler integral, with associated transformation laws.

Discretizing Manifolds via Minimum Energy Points

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For a compact set A in Euclidean space we consider the asymptotic behavior of optimal (and near optimal) N-point configurations that minimize the Riesz s-energy (corresponding to the potential $1/t^s$) over all N-point subsets of A, where $s > 0$. For a large class of manifolds A having finite, positive d-dimensional Hausdorff measure, we show that such minimizing configurations have asymptotic limit distribution (as N tends to infinity with s fixed) equal to d-dimensional Hausdorff measure whenever $s > d$ or $s = d$. In these cases we obtain an expression for the dominant term in the minimum energy. (For $d = 1$ it involves the Riemann zeta function.) Our results are new even for the case of the d-dimensional sphere and are related to best-packing problems. Possible applications to problems in chemistry, physics and biology as well as extensions to weighted Riesz energies will also be discussed.

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Infinity computer and calculus

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All the existing computers are able to execute arithmetical operations only with finite numbers. Operations with infinite and infinitesimal quantities could not be realized. The talk introduces a new positional system with infinite radix allowing us to write down finite, infinite, and infinitesimal numbers as particular cases of a unique framework. This new numeral system gives possibility to introduce a new type of computer able to operate not only with finite numbers but also with infinite and infinitesimal ones. The new approach both gives possibilities to execute calculations of a new type and simplifies fields of mathematics where usage of infinity and/or infinitesimals is necessary (for example, divergent series, limits, derivatives, integrals, measure theory, probability theory, etc.). Particularly, the new approach and the infinity computer are able to do the following:

- to substitute symbols +8 and -8 by spaces of positive and negative infinite numbers, to represent them in the memory of infinity computer and to execute arithmetical operations with them using this computer of a new type as we are used to do with usual finite numbers using traditional computers;
- to substitute qualitative description of the type 'a number tends to zero' by precise infinitesimal numbers, to represent them in the memory of infinity computer and to execute mathematical operations with them using infinity computer as we are used to do with usual finite numbers using traditional computers;
- to calculate limits (including indeterminate forms) as arithmetical expressions using infinity computer;
- to calculate sums of divergent series and improper integrals of various types using infinity computer and to execute operations being indeterminate forms in traditional approaches, e.g. difference and division of divergent series;
- to evaluate functions and their derivatives at infinitesimal, finite, and infinite points (infinite and infinitesimal values of the functions and their derivatives can be also calculated);

- to study divergent processes at different infinite points;
- to extend definition of volume to objects having parts of different dimensions and to calculate these volumes within unique framework using infinite and infinitesimal numbers;
- to introduce notions of numbers of elements in infinite sets compatible with this notion used traditionally for finite sets and to calculate them within unique framework (and not only to distinguish numerable sets from continuum as it happens in traditional approaches);
- to elaborate new mathematical and physical models working simultaneously at micro and macro levels within unique framework.

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Ideal turbulence and problems of its visualization

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Zeros of Orthogonal Polynomials

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I'll discuss recent progress on the fine structure of the zeros of orthogonal polynomials on the real line (OPRL), the unit circle (OPUC) and zeros of paraorthogonal polynomials on the unit circle (POPUC). For OPRL, the focus will be on clock behavior for a large class of examples. For OPUC, we'll discuss clock behavior when there are competing exponentials and also the case of random Verblunsky coefficients. A new variational principle for zeros of OPUC will be discussed and the related issues of approximation of eigenvalues of non-normal matrices. Some of the work described is joint with Yoram Last and Brian Davies and some of it is work by Mihai Stoiciu.

mixed mononotone difference equations

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We consider difference equations in ordered Banach spaces which have increasing and decreasing decompositions.

Orthogonal Polynomials, Padé Approximants and Sets of Minimal Capcity

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The connection between orthogonal polynomials and continued fractions is classical, and at a conference in Munich it may be a good idea to mention the classical book by Oskar Perron on this subject. Nowadays, a large part of the theory of continued fractions is incorporated into the theory of Padé approximation, which is already present in Perron's book, but confined to a small chapter close to the end. The classical results by Markov, Stieltjes, and Hamburger are still given in their original continued fractions formulation, they are concerned with types of functions that correspond in the associated area of orthogonal polynomials with polynomials that are defined on the real axis or on a real interval.

The study of Padé approximation for a wider class of functions leads away from the real axis into the complex plane, and instead of real intervals, sets of minimal capacity come into play. The complements of these sets are called extremal domains, and they are domains in which typically diagonal sequences of Padé approximants converge. In this respect, extremal domains are for Padé approximants what disks are for power series.

In the talk we address results about sets of minimal capacity and their associated extremal domains. Topics of discussion will be their unique existence, the relationship with similar concepts in geometric function theory, and methods for the numerical determination of such sets.

Basic Bernoulli and Euler polynomials and q-zeta function

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We discuss some some properties of q-Bernoulli and q-Euler polynomials and their relations with q-zeta and similar functions.

Pascal matrix, special polynomials and difference equations

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Although Pascal matrix could be considered the oldest matrix in the history of Mathematics, in recent years it has attracted the attention of many specialists in different areas of Mathematics and Applied Mathematics. New and surprising properties have been derived. In this talk, after exploiting some more or less known definitions, its relations to differential and difference equations will be presented along with its use in defining and studying special polynomials such as Bernoulli, Euler, Hermite etc. The matrix can be generalized in many ways, obtaining interesting new matrices, all of them showing peculiar properties. Finally, examples of non trivial partial difference equations which can be treated by using the Pascal matrix will also been discussed.

Heun functions versus elliptic functions

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Heun's differential equation is again under intense study either from the special functions point of view or from the integrability point of view. I will examine some items, missing in the recent book by Ronveaux et al, mainly related to solutions of Heun's equation using the ideas and techniques due to Hermite and Picard for the special functions side. These results will be compared with the so-called finite-gap solutions and Krichever-Hermite solutions intensively studied from the integrability side.

Orthogonal polynomials, recurrence relations and difference equations

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Orthogonal polynomials always satisfy a three term recurrence relation and the spectral theorem for orthogonal polynomials implies that the converse is also true: a sequence of polynomials satisfying a three term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x),$$

with $a_n > 0$ and b_n real, is always a sequence of orthonormal polynomials for some orthogonality measure on the real line. This give rise to two problems: (1) if the orthogonality measure is known, then how can we obtain the recurrence coefficients? (2) If the recurrence coefficients are known, then how can we find the orthogonality measure?
We consider the first problem for exponential weights on the real line. Freud gave a very interesting way to find the recurrence coefficients for weights of the form $w(x) = |x|^\rho \exp(-x^{2m})$. These are symmetric weights so that $b_n = 0$, and Freud found a non-linear recurrence relation for the a_n, which for $m = 2$ corresponds to discrete Painlevé I. Freud was able to obtain the asymptotic behavior of the a_n for $m = 1, 2, 3$ and gave a conjecture for general m, which was later proved by Magnus and Lubinsky, Mhaskar and Saff. In this talk I will explain Freud's technique and some of the later resuls inspired by Freud's conjecture.
A second part of the talk wll be about discrete orthogonal polynomials, in particular Charlier, Meixner, Krawtchouk and Hahn polynomials. These polynomials satisfy a second order difference equation in the variable x, and like all orthogonal polynomials they also satisfy a second order recurrence relation in the variable n. Hence these discrete orthogonal polynomials satisfy a bispectral problem, which in both variables x and n is a discrete problem. Generalized Charlier polynomials are orthogonal on the integers with weights $w_k = a^k/(k!)^m$, where m is an integer and $m = 1$ gives the Charlier polynomials. We show that Freud's idea can be modified so that we get non-linear recurrence relations for the recurrence coefficients b_n and a_n for generalized Charlier polynomials, and for $m = 2$ these can be reduced to discrete Painlevé II.

Explicit Padé interpolation tables and special functions

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We construct numerous explicit examples of the Padé interpolation tables (including schemes with prescribed poles and zeros). These examples give rise to almost all known explicit systems of orthogonal polynomials and biorthogonal rational functions. Some new explicit examples of the biorthogonal rational functions are constructed.

Contributed Talks

Functions q-orthogonal with respect to their own zeros: The analogue of Hardys theorem.

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G. H. Hardy proved that, under certain conditions, the only functions satisfying

$$\int_0^1 f(\lambda_m t)f(\lambda_n t)dt = 0 \tag{1}$$

where the λ_n's are the zeros of f, are the Bessel functions.
Replace the integral in (1) with the q-integral and consider functions such that

$$\int_0^1 f(\lambda_m t)f(\lambda_n t)d_q t = 0 \tag{2}$$

where the λ_n's are the zeros of f. Under the same general assumptions made by Hardy, we prove that, for functions satisfying (2), in one case f must be of the form

$$f(x) \quad = \quad z^{frac12} K J^{(3)}_{\nu-1/2}(Mx;q^2)$$

where $J^{(3)}_\nu$ denotes the third Jackson q-Bessel functions, $M^2 = -aq^{-3}(1-q^2)(1-q^{2\nu+1})$, $a = -2\sum 1/\lambda_n^2$ and K is a real constant. In other case f must satisfy the following q-difference equation:

$$D_q^2 f(z) + \frac{1}{qz}D_q f(z) - \left[\frac{(1-q^\nu)(1-q^{\nu-1})}{(1-q^2)(q^{\nu+1}z^2} - \frac{(1-q^{2\nu+1})(a+1)}{(1+q)q^{\nu+2}z}\right] f(qz) \quad = \quad 0.$$

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A discrete Counterpart of a Continuous-time Additive Hopfield-Type Neural Network with Impulses in an Integral Form

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A discrete analogue of a continuous -time additive Hopfield-type neural network with distributed delay and with impulses in an integral form is formulated and its global stability characteristics are investigated.

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Positive Increasing Solutions of Quasilinear Dynamic Equations

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We consider a quasilinear dynamic equation reducing to a half-linear equation, an Emden-Fowler equation or a Sturm-Liouville equation under some conditions. Any nontrivial solution of the quasilinear dynamic equation is eventually monotone. In other words, it can be either positive decreasing (negative increasing) or positive increasing (negative decreasing). In particular, we investigate the asymptotic behavior of all positive increasing solutions which are classified by certain integral conditions.

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On the Krall-type discrete polynomials

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In this talk we will present a unified theory for studying the so called Krall-type discrete orthogonal polynomials. In particular, the three-term recurrence relation, lowering and raising operators as well as the second order linear difference equation that the sequences of monic orthogonal polynomials satisfy will be established. Some relevant examples of q-Krall polynomials are considered in details as well as the limit relations among them. This is a joint work with J. Petronilho and R. Costas-Santos.

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On the limit behavior of recurrence coefficients for multiple orthogonal polynomials

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Polynomials of multiple orthogonality associated to Angelesco and Nikishin systems have ratio asymptotic under general conditions for the measures. They satisfy a recurrence relation of more than three terms and the ratio asymptotic of the polynomials gives the asymptotic properties of the recurrence coefficients.
In this work we investigate the connection between the properties of the recurrence coefficients and the algebraic equation satisfied by the limit of the ratio for the polynomials of Angelesco and Nikishin systems.

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Traces, Determinants and Dynamical Zeta Functions

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The study of dynamical zeta functions is an important topic in the general theory of Dynamical Systems. The talk concerns to the main role that Linear Algebra plays in that study. Recall that if $f : X \to X$ is a map from an arbitrary set X to itself, one defines the Artin-Mazur zeta function of f as the following power series

$$\zeta_f(t) = exp \sum_{n>0} \sharp\{x \in X : f^n(x) = x\} t^n \diagup n$$

In order to study the Artin-Mazur zeta function of a continuos interval map $f : [0, 1] \to [0, 1]$, Milnor and Thurston introduced the notion of kneading determinant $\Delta_f(t)$. Starting from a main identity, $\zeta_f(t)\Delta_f(t) = 1$, it was possible to show that $\zeta_f(t)$ is meromorphic and as well to compute explicitly the poles of this function. Our goal in this talk is to provide an original proof of this main result different from the standard one of Milnor and Thurston which uses an homotopy argument. We will show that these dynamical zeta functions can be regarded as trace zeta functions of a suitable pair of linear endomorphisms with finite rank. With this we are able to study $\zeta_f(t)$ in a purely Linear Algebra context, simpler than in the previous works.

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Difference Schemes for the Singularly Perturbed Sobolev Equations

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The present study is concerned with the numerical solution, using finite difference method of one-dimensional initial-boundary value problems for linear and quasilinear Sobolev partial differential equations with initial layer. Equations of this type arise in many areas of mechanics and physics. Various approximating aspects of this kind of problems in the regular cases, i.e. when the boundary and initial layers are absent, have been considered by many authors. But the numerical treatment of singular perturbation cases has always been far from standard because of the layer behaviour of the solution. If classical stable numerical methods (finite differences or finite elements) are used on a uniform mesh, it is known that the numerical solution is reliable only if a very large (parameter depending) number of mesh points taken. Thus, it is necessary to use robust numerical methods, i.e. methods that are convergent uniformly in perturbation parameter. In order to obtain an efficient method, to provide good approximations with independence of the perturbation parameter, we have developed a numerical method which combines a finite difference spatial discretization on uniform mesh and the implicit rule on piecewise-uniform and graded meshes for the time variable. The fully discretes scheme are shown to be convergent of order two in space and of order one in time, uniformly in the singular perturbation parameter. Some numerical results confirming the expected behaviour of the method are shown.

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Orthogonal polynomials in two discrete variables solutions of linear partial difference equations of hypergeometric type

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In this talk a systematic study of the orthogonal polynomial solutions of a second order partial difference equation of hypergeometric type of two variables is presented. The Pearson's systems for the orthogonality weight of the solutions and also for the difference derivatives of the solutions are also presented. Some explicit examples related with bivariate Hahn, Kravchuk, Meixner, and Charlier families, and their algebraic and difference properties are given.

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Boundedness of Pseudodifferential Operators on the Laguerre-Sobolev Type-Spaces

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Let $n \in \mathbb{N}$, $\alpha \geq 0$, $\mathbb{K} = [0, \infty[\times \mathbb{R}$ and $\widehat{\mathbb{K}} = \mathbb{R} \times \mathbb{N}$. We denote by

$$\varphi_{(\lambda,m)}(x,t) = e^{i\lambda t}\mathcal{L}_m^\alpha(|\lambda|x^2) \text{ for all } ((x,t),(\lambda,m)) \in \mathbb{K} \times \widehat{\mathbb{K}}$$

where $\mathcal{L}_m^\alpha(x) = e^{-\frac{x}{2}} \frac{L_m^\alpha(x)}{L_m^\alpha(0)}$ and L_m^α being the Laguerre polynomial of order α

We define the Fourier-Laguerre transform $\mathcal{F}_\alpha(f)(\lambda, m)$ of a suitable f at $(\lambda, m) \in \widehat{\mathbb{K}}$ as follows

$$\mathcal{F}_\alpha(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-(\lambda,m),m}(x,t) f(x,t) dm_\alpha(x,t), \tag{3}$$

where $dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}$ is the Haar measure on the Laguerre hypergroup $\mathbb{K}$. And for reasonable function Ψ defined on $\widehat{\mathbb{K}}$, the inverse of the above Fourier Laguerre transform is given by

$$\mathcal{G}_\alpha(\Psi)(x,t) = \int_{\widehat{\mathbb{K}}} \varphi_{(-\lambda,m)}(x,t)\Psi(\lambda,m) d\gamma_\alpha(\lambda,m), \tag{4}$$

where $d\gamma_\alpha(\lambda, m) = L_m^\alpha(0)\delta_m \bigotimes |\lambda|^{\alpha+1} d\lambda$ is the Plancherel on $\widehat{\mathbb{K}}$.
For all $\gamma \in \mathbb{R}$ we define the Sobolev-Laguerre type space $\mathcal{H}_p^\gamma$ as the set of all tempered distributions f such that $\mathcal{F}_\alpha f \in L^p_{loc}(\mathbb{K}, dm_\alpha)$ and $\|f\|_{\mathcal{H}_p^\gamma} < \infty$ where

$$\|f\|_{\mathcal{H}_p^\gamma} = \left(\sum_{m=0}^{+\infty} L_m^\alpha(0) \int_{\mathbb{R}} \Big(1 + |\lambda|\Big(m + \frac{\alpha+1}{2}\Big)\Big)^{p\gamma} \Big|\mathcal{F}_\alpha f(\lambda,m)\Big|^p |\lambda|^{\alpha+1} d\lambda\right)^{1/p}.$$

The aim of this paper is to study the continuity of generalized pseudodifferential operator $B_{\alpha,\sigma}$ and the commutator $[B_{\alpha,\sigma}\ ,\ I_\varphi\ ]$ on the Sobolev-Laguerre type-spaces $\mathcal{H}_p^\gamma$ where σ belongs to a class of generalized symbols defined on $\mathbb{K} \times \widehat{\mathbb{K}}$ and $I_\Phi = \mathcal{G}_\alpha\left(\Phi\mathcal{F}_\alpha\left(\cdot\right)\right)$. Φ is being a suitable function.

The finite Fourier transforms for certain q-polynomial families

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Some q-extensions of Mehta's eigenvectors of the finite Fourier transform are discussed. It is shown that the finite Fourier transform interrelates certain well-known q-polynomial families.

Periodicity of Delayed Difference Equations on Continua with Empty Interior

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We are dealing with delayed difference equations of the form

$$x_{n+k} = f(x_n)$$

with $k \geq 2$ and defined on continua $\mathbb{X}$ (non-empty compact and connected spaces) with empty interior.
There are examples of such type of equations when $\mathbb{X} = \mathbb{I} = [0,1]$ coming from Population Dynamics. For example, there exist species that are *semelparous*, i.e. they reproduce only once and die thereafter but live for several years, it is the case of biennial plants, salmons, cicadas, etc. In special situations, such as when all density dependence is due to competition in the nursery, years classes may develop independently and the population evolution is describe by the former equation.
We are able to find a frame of forcing relations among the periods that a continuous map $f : \mathbb{I} \to \mathbb{I}$ has in the sense of Sharkovskys ordering ([2]) when $\mathbb{X} = \mathbb{I}$ or $\mathbb{S}^1$. It is essentially made in ([1]) and will be reviewed. As an example will see that for $k = 3$ the frame of periods we obtain is

$$\begin{array}{rcl} 3 \cdot 3 & \Rightarrow & (3 \cdot 5 \Leftrightarrow 5) \Rightarrow (3 \cdot 7 \Leftrightarrow 7) \Rightarrow 3 \cdot 9 \Rightarrow (3 \cdot 11 \Leftrightarrow 11) \Rightarrow \dots \\ 3 \cdot 2 \cdot 3 & \Rightarrow & (3 \cdot 2 \cdot 5 \Leftrightarrow 2 \cdot 5) \Rightarrow (3 \cdot 2 \cdot 7 \Leftrightarrow 2 \cdot 7) \Rightarrow 3 \cdot 2 \cdot 9 \Rightarrow \dots \\ & \dots & \end{array}$$

We will deal also with new situations where similar frames can be obtained when the state space is a continuum (*tree, graph, dendrite,* etc) and also explore new line of research in the subject like the consideration of non-autonomous delayed equations, vertical endomorphisms, etc.

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The search for the missing link in dynamics

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In 1986 J. Smital proved that regular or chaotic behavior can be appropriately characterized, in the setting $C(I)$ of continuous maps of a compact interval I into itself, as follows:

Theorem 1: *If $f \in C(I)$ then one and only one of these alternatives must occur:*

- *all points of I are approximable by cycles, that is, for any $x \in I$ and any $\epsilon > 0$ there is a periodic point p such that*

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| < \epsilon$$

  (f is regular);

- *there is a perfect set S which is δ-scrambled for f for some $\delta > 0$. that is,*

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta,$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$$

  for any $x, y \in S$, $x \neq y$ (f is chaotic).

Yet, for an "almost everywhere" point of view (which is the important one for applications) this approach can be misleading, as there are very natural maps (even polynomial ones) which are chaotic in the above sense but for which almost all points (in the sense of Lebesgue measure) are approximable by cycles. The aim of this talk is to show that, under appropriate smoothness conditions for f and an additional technical assumption (non-existence of so-called *strange attractors*) it is possible to propose reasonable definition for "empirical regularity" and "empirical chaos" in the spirit of Smital's theorem. More precisely:

Theorem 2: *Let $f \in C^3(I)$ finite number of points $C(f)$ on which f' vanishes and assume that it has negative Schwarzian derivative*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

in all points $x \notin C(f)$'. Additionally, assume that f has no strange attractors. Then one and only one of these alternatives must occur:

- *almost all point of f are approximable by cycles (f is empirically regular);*
- *there are a perfect set $R \subset I^2$ of bidimensional positive Lebesgue measure and a $\delta > 0$ such that*
$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta,$$
$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$$
for any $(x, y) \in R$ (f is empirically chaotic.

On the other hand, there are no such maps having scrambled sets of positive Lebesgue measure. .

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Multidimensional Difference Equations

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Multidemensional linear difference equations in spaces of smooth functions are considered.Solvability conditions via mutual configuration of shifts are stated.Corollaries and examples are presented.

d-orthogonal polynomial sets of Tchebytchev type

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Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials in $\mathcal{P}$ such that $\deg P_n(x) = n$ for all n and let d be an arbitrary positive integer. The polynomial sequence $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial sequence with respect to a d-dimensional functional $\mathcal{U} = {}^t(u_0, \cdots, u_{d-1})$ if it fulfills:

$$\begin{cases} \langle u_k, P_m P_n\rangle = 0 \ , \quad m > dn + k \ , n \geq 0, \\ \langle u_k, P_n P_{dn+k}\rangle \neq 0 \ , \ n \geq 0, \end{cases}$$

for each integer k belonging to $\{0, 1, \ldots, d-1\}$. For $d = 1$, we recognize the well-known notion of orthogonality.
In this talk, we characterize all d-orthogonal polynomial sets that have generating functions of the type:

$$\frac{A(t)}{1 - xC(t)} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where A and C are two formal power series satisfying: $C(0) = 0$ and $A(0) = C'(0) = 1$. For the obtained polynomials, we discuss their links with known polynomial sets, we give some of their properties, and we express explicitly the corresponding d-dimensional functionals.

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On Oscillation of Some Linear and Nonlinear Difference Equations

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For a scalar delay difference equation

$$x(n+1) - x(n) = \sum_{k=1}^{m} a_k(n)x(h_k(n)), \quad h_k(n) \le n,$$

a connection between the following properties is established: nonoscillation, positiveness of the fundamental function and existence of a nonnegative solution for a some nonlinear difference inequality.

Explicit nonoscillation and oscillation conditions, a comparison theorems for linear and nonlinear equations are presented.

Asymptotics of nonlinear difference equations

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We study equations of type

$$D : Y(X+1) = F(X, Y(X))$$

with $X \longrightarrow \infty$ and F nonlinear and C^1 in Y. We call an *approximate solution* $\widehat{Y}$ a sequence that satisfies the asymptotic functional equation

$$\lim_{X \longrightarrow \infty} \frac{F(X, \widehat{Y}(X)) - \widehat{Y}(X)}{\widehat{Y}(X)\left(\left|F_2'(X, \widehat{Y}(X)\right| - 1\right)} = 0. \tag{5}$$

Note that a fix-point of F satisfies (5), but approximate solutions are good as well in case it is practically impossible to determine such fix-points. It will be argued that equation (5) has a natural geometric interpretation, in fact it expresses that near the approximate solution occurs strong contraction of solutions, attraction if $\left|F_2'(X, \widehat{Y}(X)\right| < 1$ and repulsion if $\left|F_2'(X, \widehat{Y}(X)\right| > 1$.

We present conditions that imply the existence of a true solution $\widetilde{Y}(X)$ of D asymptotic to $\widehat{Y}$. In fact we derive the asymptotic behaviour of all solutions in an asymptotic neighbourhood of $\widetilde{Y}$. The study, which includes joint work with B.L.J. Braaksma (Groningen), is inspired by similar, but not totally analogous, results in differential equations on *rivers* obtained by among others Reeb, F. and M. Diener and the author in the 80's.

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Controllability of Partial Neutral Functional Differential Equations with Infinite Delay

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Our aim is to prove a result about controllability to the following partial neutral functional differential equation with infinite delay

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{D}x_t = A\mathcal{D}x_t + Bu(t) + F(t, x_t),\ t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{6}$$

where the state $x(.)$ takes values in a Banach space $(E, |.|)$, the control $u(.)$ is given in $L^2([0, T], U)$, $T > 0$, the Banach space of admissible control functions with U a Banach space, B is a bounded linear operator from U into E, $A : D(A) \subseteq E \to E$ is a linear operator on E, $\mathcal{B}$ is the phase space of functions mapping $(-\infty, 0]$ into E, which will be specified, $\mathcal{D}$ is a bounded linear operator from $\mathcal{B}$ into E defined, for any $\varphi \in \mathcal{B}$, by $\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi$. $\mathcal{D}_0$ is a bounded linear operator from $\mathcal{B}$ into E and for each $x : (-\infty, T] \to E$, $T > 0$, and $t \in [0, T]$, x_t represents, as usual, the mapping defined from $(-\infty, 0]$ into E, for $\theta \in (-\infty, 0]$, by $x_t(\theta) = x(t + \theta)$. F is an E-valued nonlinear continuous mapping on $\mathbb{R}_+ \times \mathcal{B}$.

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Classical Matrix Multiple Orthogonal Polynomials

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In this talk we present the general theory of multiple orthogonal polynomials. Our departure point is the three term recurrence relation, with matrix coefficients, satisfied by a sequence of vectors polynomials. Characterization of classical multiple orthogonal polynomials are presented. .

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Symbolic dynamics for real rational maps

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We will study the two parameter family of rational maps

$$f_{a,b}(x) = \frac{x^2 + a}{x^2 + b},$$

using symbolic dynamics. Further we define kneading sequences and subshift of finite type (topological Markov chains) with the main goal of studying the dependency of topological entropy of the parameters a and b.

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A system of bi-orthogonal trigonometric polynomials

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In this paper we study a particular type sequence of biorthogonal trigonometric polynomials, in the Szegő's sense. For these type of sequences we study the usual topics in the OP theory. In particular we obtain a new connection with OP on the circle and nice properties like:

1. Short term recurrence relation with a Jacobi type representation.
2. Christoffel-Darboux type formula.
3. Properties about quadrature formulas.
4. Properties about zeros: separation and interlaced.

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Global Behavior of Solutions of the Third Order Rational Difference equation $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}$

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We study the global behavior of solutions of the third order rational difference equations of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}},\ n = 0, 1, ...$$

where the initial conditions x_{-2}, x_{-1}, x_0 and the parameters $\alpha, \beta, \gamma, \delta, A, B, C, D$ are nonnegative real numbers such that the denominator of the equation is always positive.
We study the bounded and unbounded character of all solutions, solutions that converge to a finite limit, periodic solutions and solutions that converge to periodic solutions.

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Basic Fourier Expansions on a q-Linear Grid

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For $0 < q < 1$ define the symmetric q-linear operator acting on a suitable function $f(x)$ by $\delta f(x) = f(q^{1/2}x) - f(q^{-1/2}x)$. The q-linear initial value problem $\frac{\delta f(x)}{\delta x} = \lambda f(x)$, $f(0) = 1$, has two entire functions $C_q(z)$ and $S_q(z)$ as linearly independent solutions, which are orthogonal on a discrete set. Sufficient conditions for pointwise convergence and for uniform convergence of the corresponding Fourier expansion are given. Examples.

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Bessel functions for p-radial functions

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Bessel functions occur in classical analysis in connection with radial symmetry. From the point of view of harmonic analysis they are the zonal spherical functions for the Gelfand pair (M_d, SO_d). As such they characterize the Fourier transform of radial functions on $\mathbb{R}^d$, i.e., the Fourier-Bessel transform.
Considering functions on $\mathbb{R}^d$ which are radial with respect to the ℓ_p-norm in $\mathbb{R}^d$ Richards introduced generalized Bessel functions $J_{p,d}$ characterizing the Fourier transform. Since the ℓ_p-sphere is in general not a group orbit, there is no interpretation of these functions in terms of Gelfand theory. Nevertheless, there are symmetry properties which can be interpreted in terms of group actions.
We will show properties of these functions and give a connection to reflection invariant Bessel functions which result from the action of the symmetry group of the ℓ_p-sphere on $\mathbb{R}^d$.

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New results on Chebyshev polynomials

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The theory and the related formalism of the Monomiality Principle can help us to deduce relevant relations in the field of Chebyshev polynomials. In particular we can state integral represention for first and second kind Chebyshev polynomials by using the properties of the generalized two-variable Hermite polynomials. By mixing the different kinds of the Chebyshev polynomials, we can also investigate new families of polynomials and derive the related properties.

Linearization coefficients for Sheffer Polynomial sets

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The lowering operator σ associated with a polynomial set $\{P_n\}_{n\geq 0}$ is an operator not depending on n and satisfying the relation $\sigma P_n = nP_{n-1}$. In this talk, we express explicitly the linearization coefficients for polynomial sets of Sheffer type using the corresponding lowering operators. We obtain some well known results as particular cases. A Crofton-type formula are derived. As application we give linearization coefficients for Gould-Hopper polynomials as well as reduction formulae for Kampé de Fériet functions .

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Self-adjoint difference operators and indeterminate moment problems

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The polynomials within the q-analogue of the Askey-scheme are eigenfunctions of a second order q-difference operator. When the associated moment problem is indeterminate and several positive measures μ come into play, this operator is only symmetric on $L^2(\mu)$ under certain restrictions on μ.

As an example we consider the Stieltjes–Wigert polynomials. The difference operator in question is essentially self-adjoint on $L^2(\mu)$ when μ is a discrete or absolutely continuous solution to the q-Pearson equation and the corresponding spectral decomposition can be presented very explicitly. In the discrete case the spectral analysis boils down to considering doubly infinite Jacobi operators whereas the spectral analysis for the continuous case involves direct integral techniques.

It is worth mentioning that the analysis for the discrete case also leads to an orthogonal set of q-Bessel functions complementing the Stieltjes–Wigert polynomials to an orthogonal basis for the underlying L^2-space.

The talk is based on joint work with Erik Koelink, TU Delft.

Abel's Lemma on summation by parts and Basic Hypergeometric Series

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Basic hypergeometric series identities are revisited systematically by means of Abel's lemma on summation by parts. Several new formulae and transformations are also established. The author is convinced that Abel's lemma on summation by parts is a natural choice in dealing with basic hypergeometric series.

Special polynomials associated with rational and algebraic solutions of the Painlevé equations

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The Painlevé equations (P_I–P_{VI}) are six nonlinear ordinary differential equations that have been the subject of much interest in the past thirty years, which have arisen in a variety of applications such as random matrices and may be thought of as nonlinear special functions. Rational solutions of the Painlevé equations are expressible in terms of the logarithmic derivative of certain special polynomials. For P_{II} these polynomials were first derived in the 1960's by Yablonskii and Vorob'ev. The locations of the roots of these polynomials have a highly regular triangular structure in the complex plane [1]. In this talk I shall describe the analogous special polynomials associated with rational and algebraic solutions of other Painlevé equations. It is shown that their roots also have a highly regular structure and other properties of these special polynomials will be discussed.

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Structure Theorems for Continuous PolynomialHypergroups in Several Variables

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Let H be a compact set in $\mathbb{R}^d$. Let $(H, *)$ represent a hypergroup with polynomial characters supported on H. A sufficient condition is given for $(H, *)$ to be the tensor product of certain canonical Hermetian and canonical non-Hermetian hypergroups. These conditions involve the location of the hypergroup identity, e, in H, and the relative topology of H near e.

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Factorization of the hypergeometric-type difference equation on non-uniform lattices: dynamical algebra

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We argue that one can factorize the difference equation of hypergeometric type on non-uniform lattices in the general case. It is shown that in the most cases of q-linear spectrum of the eigenvalues, this directly leads to the dynamical symmetry algebra $su_q(1,1)$, whose generators are explicitly constructed in terms of the difference operators, obtained in the process of factorization. Thus all models with the q-linear spectrum (some of them, but not all, previously considered in a number of publications) can be treated in a unified form.

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Fock representations for a quadratic commutation relation

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We characterize the Fock representations of the $*$-algebra generated by the operators X, X^* satisfying the algebraic relation

$$XX^* = f(X^*X)$$

where f is the family of quadratic maps $f(x) = \alpha x^2 + \beta x + \gamma$ with α, β, γ, real numbers. The representations associated to relation (1) are closely connected with the dynamical system $(\mathbb{R}_+, f)$. The Fock representation is a representation on $l^2(\mathbb{N})$ and is directly associated with the solution of the difference equation $x_{n+1} = f(x_n)$ with initial condition $x_0 = 0$. We identify regions in the parameter space for which the Fock representations exist, are bounded/unbounded or finite/infinite.

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A continuum limit of the relativistic Toda lattice: asymptotic theory of discrete Laurent orthogonal polynomials with varying recurrence coefficients

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We consider the continuum limit of the relativistic Toda lattice. In particular, we propose a method in order to 'integrate' this system of nonlinear partial differential equations for some particular initial data and boundary conditions, before possible shocks. First, we recall the relation between the finite relativistic Toda lattice and the theory of discrete Laurent orthogonal polynomials. Our analysis is then based on some results for the asymptotic theory of discrete Laurent orthogonal polynomials with varying recurrence coefficients and the connection with a constrained and weighted extremal problem for logarithmic potentials.

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Planar fronts in bistable coupled map lattices

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Planar fronts in multidimensional coupled map lattices can be studied by reduction to an one-dimensional extended dynamical system that generalises one-dimensional coupled map lattices. This methodology is fully investigated and developed. Continuity of fronts velocity with the coupling strength and with the propagation direction is proven. Examples are provided and illustrated by some numerical pictures.

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Some aspects of $(0, 1, 2 \ldots, r, r + m)$ and $(0, m, m + 1, m + 2, \ldots, m + r)$ interpolation

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Let $r \geq 0$ and $m \geq 2$ be integers and consider:

A. $(0, 1, 2 \ldots, r, r + m)$ **interpolation**

- given n pairwise distinct complex numbers $z_1, \ldots, z_n$ (nodes),
- given $n(r + 2)$ complex numbers c_j^i $(1 \leq j \leq n,\ i \in \{0, 1, 2, \ldots, r, r + m\})$ (data),

find a polynomial $P_N(z)$ of degree at most $N = n(r + 2) - 1$ satisfying

$$P_N^{(i)}(z_j) = c_j^i \ (1 \leq j \leq n,\ i \in \{0, 1, 2, \ldots, r, r + m\}).$$

B. $(0, m, m + 1, m + 2, \ldots, m + r)$ **interpolation**

- given n pairwise distinct complex numbers $z_1, \ldots, z_n$ (nodes),
- given $n(r+2)$ complex numbers d_j^i $(1 \leq j \leq n,\ i \in \{0, m, m+1, m+2, \ldots, m+r\})$ (data),

find a polynomial $P_N(z)$ of degree at most $N = n(r + 2) - 1$ satisfying

$$P_N^{(i)}(z_j) = d_j^i \ (1 \leq j \leq n,\ i \in \{0, m, m + 1, m + 2, \ldots, m + r\}).$$

Some aspects of the connection between the length m of the gap and regularity (the polynomial P_N is unique for all sets of data) will be studied.

Information Entropy of Gegenbauer Polynomials

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Let $\{p_n(x)\}$ denote a sequence of polynomials orthogonal on Δ with respect to the weight function $\omega(x)$. The information entropy E_n of the polynomials $p_n(x)$ is defined as

$$E_n = -\int_{\Delta} p_n^2(x)\omega(x)\log p_n^2(x)dx. \tag{1}$$

In the last ten years much attention has been paid to the evaluation of these integrals for several families of orthogonal polynomials, because of their interest in the context of quantum theory. However, in most cases only asymptotic expansions and/or algorithms for numerical evaluation are known. Closed analytical formulas have only been obtained for Gegenbauer polynomials of parameter 0 (Chebyshev polynomials of the first kind), 1 (Chebyshev polynomials of the second kind) and 2. This last result was achieved using an algorithmic formula for the entropy of Gegenbauer polynomials of integer parameter. In this work we present a new approach to the evaluation of the information entropy of Gegenbauer polynomials of parameter $\lambda \in \mathbb{N}$ in terms of finite sums. We derive closed formulas for $\lambda = 2$, reproducing the result known before, and for $\lambda = 3$. We also discuss the application of this method to other values of the parameter.

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Symmetric semiclassical linear functionals of class 1 and generalized coherent pairs

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Let u and v be two symmetric quasi–definite linear functionals and denote by $\{P_n\}$ and $\{R_n\}$ their corresponding systems of monic orthogonal polynomials. In the theory of Sobolev orthogonal polynomials, the concept of coherent pairs has played an important role. As a generalization of this concept, we assume these two sequences are related by an algebraic-differential equation in the form

$$(n+1)P_n + a_n P_{n-2} = R'_{n+1} + b_n R'_{n-1},\ n \geq 2,$$

with $b_n \neq 0$ for every $n \geq 2$. If $a_n = 0$ for all $n \geq 2$, then (u, v) is said to be a symmetric coherent pair. In our contribution we present a study of the case when the linear functional v is symmetric semiclassical of class 1 and we describe the possible companion linear functionals. We also conjecture that in this situation, one of the functionals, either u or v, must be a semiclassical linear functional of class at most 2.

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Global Behavior of Solutions of the Nonlinear Difference Equation $x_{n+1} = p_n + x_{n-1}/x_n$

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We study the global asymptotic behavior of solutions of the nonautonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n},\ n = 0, 1, ...,$$

where $\{p_n\}_{n=0}^{\infty}$ is a positive bounded sequence and the initial conditions x_{-1} and x_0 are positive. We obtain sufficient conditions for the boundedness and persistence of solutions and for the existence of unbounded solutions. In addition we obtain global attractivity results. The results are applied to the case when $\{p_n\}_{n=0}^{\infty}$ is periodic with prime period k. We also state some open questions related to the equation.

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Asymptotic Behavior of Solutions of Discrete Equations

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A survey of author's latest results achieved (together with his collaborators) in asymptotic theory of discrete equations is presented. Consider a system of nonlinear ordinary difference equations

$$u(k+1) = F(k, u(k)), \tag{2}$$

where $u = (u_1, \dots, u_n)$, $F : N(a) \times \mathbb{R}^n \to \mathbb{R}^n$ is a one valued mapping, $N(a) = \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $\mathbb{N} = \{0, 1, \dots\}$ and k is independent variable assuming values $k = a, a+1, a+2, \dots$. Define, for every $k \in N(a)$, a set $\Omega(k)$ as *n-dimensional*, *open*, *bounded* and *simply connected subset* of the set $S(k) := \{(k, u) : u \in \mathbb{R}^n\}$. Let a point $M = (k, u^0) \in S(k)$ with $k \in N(a)$ be given. We say that a point $M_1^c = (k+1, u^1)$ with $u^1 = F(k, u^0)$ is the *first consequent point* to the point M and we write $M_1^c = \mathcal{C}[M]$. Let a set $\mathcal{S} \subset S(k)$ with $k \in N(a)$ be given. We say that a set $\mathcal{S}_1^c$ is the *first consequent set* to the set $\mathcal{S}$ if $\mathcal{S}_1^c := \{M_1^c, M \in \mathcal{S}\}$ and we write $\mathcal{S}_1^c = \mathcal{C}[\mathcal{S}]$. Define a mapping $\mathcal{F} : N(a) \times \mathbb{R}^n \to N(a) \times \mathbb{R}^n$ by the formula $\mathcal{F}(k, u) = (k+1, F(k, u))$, $k \in N(a)$, $u \in \mathbb{R}^n$.

Theorem: *Let mapping $F : N(a) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous with respect to its second coordinate and the mapping $\mathcal{F} : \partial\Omega(s) \to \mathcal{C}[\partial\Omega(s)]$ is bijective for every fixed $s \in N(a)$. Suppose that for every fixed $s \in N(a)$ the set $\mathcal{C}[\partial\Omega(s)]$ is a boundary of n-dimensional closed domain $\mathcal{D}(s+1)$, homeomorphic with n-dimensional closed ball $\mathcal{B}(s+1)$, and $\overline{\Omega}(s+1) \subset \mathcal{D}(s+1)$. Then there exists at least one initial problem $u_*(a) = u_*^a$ with $(a, u_*^a) \in \Omega(a)$ such that corresponding solution $u = u_*(k)$, $k \in N(a)$ of (2) satisfies $(k, u_*(k)) \in \Omega(k)$ for every $k \in N(a)$.*

Example: Consider a nonlinear system

$$\begin{aligned} u_1(k+1) &= k^2 u_2(k), \\ u_2(k+1) &= -k^2 u_1(k) + \frac{1 + \operatorname{arctg} u_2(k)}{k} \end{aligned} \tag{3}$$

for $k \in N(2)$. With the aid of above theorem one can prove that the system (3) has a solution $u_*(k) = (u_{*1}(k), u_{*2}(k))$, satisfying $u_{*1}^2(k)+u_{*2}^2(k) < 1$ for every $k \in N(2)$.

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New summation inequalities and their applications to difference equations

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We present new summation inequalities with applications to the half-linear difference equation

$$\Delta\big(a_n|\Delta x_n|^\alpha \operatorname{sgn} \Delta x_n\big) = b_n|x_{n+1}|^\alpha \operatorname{sgn} x_{n+1}.$$

We show that the half-linear equation exhibits not only similarities but also some discrepancies with the linear case, especially as regards the possible coexistence of the so-called extremal solutions. Some of our results are new also for the linear difference equation. (Joint work with M. Cecchi and M. Marini, University of Florence, and I. Vrkoč, Czech Academy of Sciences)

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A Simple Non-directed Graph Proof of Sharkovsky's Theorem

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A self-contained non-directed graph proof of the well-known result of Sharkovsky on the periods of coexistent periodic points of continuous interval maps using the following lemma is presented.

Lemma. *Let P be a period-m orbit of f with $m \geq 3$. Then there exist a fixed point z of f, a period-2 point y of f, and a point v such that $f(v) \in P$ and* $\max\{f^2(v), y\} < v < z < \min\{f(y), f(v)\}$*. Furthermore, $f(x) > z$ and $f^2(x) < x$ for all $y < x < v$.*

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The effect of coupling on identical bursting cells

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Bursting activity is an interesting feature of the temporal organization in many cell firing patterns. This complex behavior is characterized by clusters of spikes (action potentials) interspersed with phases of quiescence. The interpretation of experimental results, particularly the study of the influence of coupling on chaotic bursting oscillations, is of great interest from a physiological and mathematical perspective. In this paper, we apply techniques of symbolic dynamics to study the topological entropy of a map used to examine the role of coupling on identical bursting cells. Finally, the strength of coupling leads to the introduction of a second topological invariant that allows us to distinguish isentropic dynamics. We exhibit numerical results about the effect of coupling strength on the variation of the topological invariants.

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Symmetric Operators for conjoined bases of Symplectic difference systems

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We study the symmetric operators $\Lambda[W]$ associated with conjoined bases of symplectic difference systems $Y_{i+1} = W_i Y_i$, $W^T J W = J$. Main property of $\Lambda[W]$ defined for any symplectic $W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is that $\Lambda[WV] = V^T(\Lambda[W] - \Lambda[V^{-1}])V$ for any symplectic W, V is congruent to the variation $\Lambda[W] - \Lambda[V^{-1}]$. In this work we consider the separation results connected with the variation of a conjoined basis Y. Thus, by analogy with the definition of the multiplicities of focal points introduced by W. Kratz, we consider so called comparative index $\mu(i)$ defined for two conjoined bases Y, $\hat{Y}$. We show that $\mu(i)$ describes the index (the number of the negative eigenvalues) of the generalized variation of the factors $L\mathfrak{N}$ in the symplectic factorization $Y = L\mathfrak{N}[M^T\ 0]^T$ which holds for any conjoined basis of the symplectic system (see[2]). We derive (under some assumptions) the formula $\Delta\mu(i) = m(i) - \hat{m}(i)$ which connects the numbers of focal points $m(i)$, $\hat{m}(i)$ of $Y, \hat{Y}$ with $\mu(i+1) - \mu(i)$ in $(i, i+1]$. We also state other index results which generalize well-know separation results in Sturmian theory.

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Results on Oscillation for Dynamic Equations

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We consider the oscillation properties of linear and nonlinear dynamic equations on time scales. In addition to a discussion of various Hille-Wintner comparison criteria, we present some explicit oscillation results.

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A renaissance for a q-umbral calculus

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Let the q-shifted factorial be defined by $\langle a;q\rangle_n = \prod_{m=0}^{n-1}(1-q^{a+m})$,

and let the tilde operator be defined by $\langle \widetilde{a};q\rangle_n = \prod_{m=0}^{n-1}(1+q^{a+m})$. We present a q-umbral calculus in the spirit of Rota and M.Ward with the following ingredients:

1. The Nalli-Ward-Alsalam q-addition.
2. The Jackson-Hahn-Cigler q-addition.
3. A close connection to the Jackson q-gamma function and the Heine Ω function.

In this way we can find q-analogues of the results of Nörlund for generalized Bernoulli- and Euler polynomials, which are more general than the recent article by M. Schlosser.

This kind of q-calculus is a mix of the work of among others Heine and his pupil Thomae (who reintroduced the q-integral), P. Appel 1879, Ashton & Lindemann 1909, Smith & Pringsheim 1911, Daum 1941, Gasper & Rahman.

The Jacobi-Neville elliptic functions can easily be expressed in terms of the Heine Ω function and its tilde version.

I hope that this talk will also bring attention to some interesting results in q-calculus previously published in different languages.

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Laplacians and the Cheeger constants for discrete dynamical systems

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We consider discrete laplacians for iterated maps on the interval and examine their eigenvalues. We have introduced a notion of conductance (Cheeger constant) for a discrete dynamical system, now we study their relations with the spectrum. We compute the systoles and the first eigenvalue of some families of discrete dynamical systems.

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Linear Difference Equations and Associated Polynomials

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This talk will outline some simple combinatorial techniques used recently to obtain sharp bounds for linear difference equations with coefficients in $[-A, 0]$ for various orders. Generating functions associated with the discrete bounding process are seen to be generalizations of Fibonacci and Pell polynomials. The talk concludes with a discussion of the properties of these polynomials including some new results.

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On difference equations for associated classical orthogonal polynomials on quadratic and q-quadratic lattices

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Let (P_n) be a familly of classical orthogonal polynomial on the quadratic or q quadratic lattice $x(s)$. As orthogonal polynomial sequence, it satisfies a three-term recurrence relation given in the monic form by

$$P_{n+1}(x(s)) = (x(s) - \beta_n)\, P_n(x(s) - \gamma_n\, P_{n-1}(x(s)), \quad n \geq 1,$$
$$P_0(x(s)) = 1, \; P_1(x(s)) = x(s) - \beta_0, \tag{4}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0$.
(P_n) also satisfies a second-order divided-difference equation

$$\sigma(x(s))\, \mathbb{F}_{x(s)}(Y(x(s)) + \tau(x(s))\, \mathbb{M}_{x(s)}(Y(x(s)) + \lambda_n\, Y(x(s)) = 0,$$

where λ_n is a constant, σ is a polynomial of degree at most 2 and τ is a first degree polynomial. $\mathbb{F}_{x(s)}$ and $\mathbb{M}_{x(s)}$ are the divided difference operators defined by

$$\mathbb{F}_{x(s)}(f(s)) = \frac{\Delta}{\Delta x(s-\frac{1}{2})} \frac{\nabla f(s)}{\nabla x(s)}, \quad \mathbb{M}_{x(s)}(f(s)) = \frac{1}{2}\left(\frac{\Delta f(s)}{\Delta x(s)} + \frac{\nabla f(s)}{\nabla x(s)}\right),$$

where Δ and ∇ are the forward and the backward operators given by

$$\Delta(f(s)) = f(s+1) - f(s), \;\; \nabla(f(s)) = f(s) = f(s-1).$$

Given a positive integer r, the rth associated of the sequence (P_n) is the sequence denoted $(P_n^{(r)})$ and given by the three-term recurrence relation obtained from (4) by replacing n by $n+r$ in the coefficients β_n and γ_n

$$P_{n+1}^{(r)}(x(s)) = (x(s) - \beta_{n+r})\, P_n^{(r)}(x(s) - \gamma_{n+r}\, P_{n-1}^{(r)}(x(s)), \quad n \geq 1,$$
$$P_0^{(r)}(x(s)) = 1, \; P_1^{(r)}(x(s)) = x(s) - \beta_r. \tag{5}$$

We prove that the $(P_n^{(r)})$ satisfy a fourth-order difference equation of the form

$$I_4(x,n)\,\mathbb{F}^2_{x(s)}\,(y(s)) + I_3(x,n)\,\mathbb{F}_{x(s)}\,\mathbb{M}_{x(s)}(y(s)) + I_2(x,n)\,\mathbb{F}_{x(s)}(y(s)) \\ +I_1(x,n)\,\mathbb{M}_{x(s)}(y(s)) + I_0(x,n)\,y(s) = 0, \qquad (6)$$

where $I_j(x,n)$ are polynomials in the variable $x(s)$.
Next factorize and find a basis of four independent solutions of these fourth-order difference equations and give explicitely the coefficients $I_j(x,n)$ for r associated Askey-Wilson and Raccah polynomials.

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Geometric Properties of Special Functions Determined by Polynomial Differential Equations

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Polynomial differential equations are considered. The main problem of qualitative theory of such equations is Hilbert's Sixteenth Problem on the maximum number and relative position of limit cycles, special functions determined by these equations. There are three local bifurcations of limit cycles: 1) Andronov – Hopf bifurcation; 2) separatrix cycle bifurcation; 3) multiple limit cycle bifurcation. We connect all these local bifurcations by means of the Wintner – Perko termination principle, construct canonical systems with field-rotation parameters which correspond to our equations and, using geometric properties of these parameters, develop a new global approach to the solution of the Problem in the quadratic case of polynomial equations [1].

By means of Erugin's two-isocline method, we construct also a canonical Kukles-type cubic system with field-rotation parameters and apply it for studying limit cycle bifurcations. In particular, we consider a special case of the Kukles system which corresponds to a generalized Liénard equation and is very important for applications, classify separatrix cycles and study global bifurcations of limit cycles [2].

Finally, we carry out the global qualitative analysis of a cubic centrally symmetric equation which can be used as a learning model of planar neural networks [3] and study a general Liénard equation of $2k + 1$ degree proving that it can have at most k limit cycles.

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Difference equations and Monte Carlo method for the Smoluchowski process of the coagulation kinetical theory [1]

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We consider the physical system consisting of N particles possessing non-negative masses in the volume $V(N)$. The coagulation process is supposed during collision of couples of particles [1-3]. The main goal of the paper is the mathematical simulation of coagulation process based on difference equations. The background of the Monte Carlo process which leads to to the solution of above difference equations was obtained. We describe the behavior of the solutions in the Grad-Boltzmann-Smoluchowski limit when $N \longrightarrow \infty$, $V(N) \longrightarrow \infty$. The convergence of approximations to the solution of the Smoluchowski kinetical equation was established.

The special case of Boltzmann gas was considered for the orthogonal collisions and respective difference equations were described. The Monte Carlo approximate solution for above system was compared with exact solution and background of Smoluchowskii approach for difference kinetics was obtained.

Consider a system of N particles of nonnegative mass in a volume $V(N)$. The particles are assumed to move chaotically in $V(N)$ and experience pairwise collisions, in which they can merge to form particles of aggregate mass (coagulation). An accepted mathematical model of the coagulation process as N and $V(N)$ is the Smoluchowski kinetic equation [1, 2]. If we assume that the initial masses of the particles in the system are $m_k = km$, where $k \in \mathbb{Z}^+$ and $m_1 > 0$, then the mass of the particle formed in the coagulation of a pair of particles remain the same. Without loss of generality, we assume that $m_1 = 1$. Let $u_k(t)$ denote the concentration of particles of mass $k \in \mathbb{N}$ at time $t \geq 0$ and $\Phi_{k,l}$ be the coagulation intensity for particles of masses k and l, with $\Phi_{k,l} = \Phi_{l,k} \geq 0$. The values of the Smoluchowski collision operator S [1, 2] specify the rate of change in the concentrations u_k and

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are defined by the relation

$$S_{\Phi}^{(k)}(u) \overset{def}{=} \sum_{l+r=k} \Phi_{l,r} u_l u_r - 2u_k \sum_{l=1}^{\infty} \Phi_{k,l} u_l, \quad k \in \mathbb{N}.$$

The Smoluchowski kinetical equation

$$\frac{du_k}{dt} = S_{\Phi}^{(k)}(u), \quad t > 0,\ k \in \mathbb{N}, \tag{1}$$

is used in numerous applications for description of coagulation process, but there exists gap between equation (1) and direct simulation of the above process, based on difference equations.

This study analyzes the relationship between solutions of the difference equations and the asymptotic behavior (as $N \longrightarrow \infty$, $V(N) \longrightarrow \infty$) of the results obtained by Monte Carlo direct simulation of coagulation [3] based on random draws of coagulation acts for individual particles. Following this approach, the Smoluchowski equation is rigorously derived for a wide class of coagulation intensities Φ and initial data.

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Dynamical Properties of the Symmetrical Images Generated by Discrete Chaotic Dynamics Difference Equations

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Chemical reactions discrete chaotic dynamics (DCD) is the theory describing multi-component systems evolution in discrete time and space. Interactions between the systems constituents (agents), in addition to the physicochemical transformations, include into consideration *information exchange*. DCD's basic equations which are derived from the first physicochemical principles have a form of the system of non-linear difference equations [1].

Discrete space distributed patterns are presented on the squared lattices where each cell, at each discrete moment of time, is filled with corresponded to the calculated by DCD difference equations solutions colors. Therefore we are receiving discrete time evolution of the images (colored cells within the lattice).

In this work we will present and analyze evolution of the sequences of the 2-D symmetrical images. It will be shown that the symmetrical images evolution presents three different types of dynamical behavior: convergence to the one pattern (steady state), convergence to the periodically changing patterns (period two, three, etc.) and convergence to the asymmetrical disordered patterns.

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Difference equations, Euler's summation formula and Hyers-Ulam stability

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In connection with the difference equation of the square root spiral (spiral of Theodorus) we consider an approach of Herbert Kociemba who gave an approximation of this spiral using Euler's summation formula. Aspects of stability in the sense of Hyers-Ulam are indicated.

These observations can also be applied to more general linear difference equations of first order.

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Dynamical Properties for Economic Models Described by Non-linear Difference Equations Having Two Decision Parameters

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The studies related to onset of chaos in one-dimensional discrete systems modelled by non-linear maps have been quite intense and exhaustive during the last two decades. This paper proposes an analysis of creation mechanism of bubbling sequences and bistability regions in bifurcation scenario of a special class of one-dimensional, two-parameter map. The simplest cases where those are present are maps with at least two control parameters, one that controls the non-linearity and the other that is a constant additive one, i.e. maps of the type:

$$X_{n+1} = f(X_n, a, b) = f_1(X_n, a) + b$$

Analyses of character of observed bifurcations in dynamic systems are more frequent in the literature now. The bubbling scenario is seen in the bifurcation diagrams of many non-linear systems like coupled driven oscillators, oscillatory chemical reactions, diode circuits, lasers, insect populations, traffic flow systems, etc. Bistability is an equally interesting and common feature associated with many non-linear systems like a ring laser and a variety of electronic circuits.

There is also proposed an answer is also proposed to the question: which one of those behaviors is more typical in economic systems. In the paper there is the analytical and graphical analysis of the dynamic properties for economic models (demographic models, production and cost models, growth models) described by non-linear difference equations having two decision parameters.

Subexponential Decay in Linear Volterra Difference Equations

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In this lecture, which is a joint work with J. A. D. Appleby and D. Reynolds, we study general linear Volterra convolution difference equations on finite dimensional spaces, and give results which ensure that their solutions are bounded, asymptotically stable. These results are then used to obtain the exact rate of decay of solutions, by considering suitable weighted solutions: indeed exact asymptotic representations for the solutions are obtained. It is also shown that the hypotheses of our results are used as weights. Using subexponential functions, for such equations we can obtain the exact rate of convergence of solutions in cases of the solution not converging exponentially, and of their being no characteristic roots.

Characteristic Lie algebras of the discrete hyperbolic type equations

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It is well known that the characteristic Lie algebras, introduced by A.B.Shabat in 1980, play the crucial role in studying the hyperbolic type partial differential equations. For example, if the characteristic algebra of the equation is of finite dimension, then the equation is solved in quadratures, if the algebra is of finite growth then the equation is integrated by the inverse scattering method. Recently, it has been observed by A.V.Zhiber that the characteristic algebra provides an effective tool to classify the nonlinear hyperbolic equations. However, the characteristic algebras has not yet been used to study the discrete equations. In the talk the notion of the characteristic Lie algebra of the discrete hyperbolic type equation will be introduced. An effective algorithm to compute the algebra for the equation given is suggested. Examples and further applications will be discussed.

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Change in Time Scales: The Effects of Parameterization on Dynamics

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In this paper, we introduce the idea of using time scales as a parameter to understand the changes in dynamics between difference and differential equations and we investigate the changes that occur for particular sequences of time scales. More specifically, we use times scales converging to $\mathbb{R}_+$ to examine how altering the time scale sequence changes the dynamics as we vary μ from constant to nonconstant values.

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On the asymptotic behaviour of solutions of neuronic difference equations

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We consider the global attractivity of the nonlinear delay difference equations which appear as the neural networks model.

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Higher order Veselov discretization of field theories

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We discuss multisyplectic discretisation of Lagrangian systems of high order. It consists in discretising the Lagrangian of a Lagrangian system before deriving the Euler-Lagrange equations for the stationary solutions. The numerical schemes generated posess geometric properties similar to the continuous system, i.e. they constitute a (multi) symplectic system. On integrable systems this could mean the preservation of features such as symmetries and conservation laws. We give the example of a higher order system that contains the integrable nonlinear Schrödinger equation as a reduction.

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Ladder formalisms for the discrete heat and harmonic oscillator equation

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Discrete versions of the heat and harmonic oscillator equation will be studied within a general concept of ladder formalisms. The combination of the underlying ladders will allow to introduce Kravchuk polynomials as operators acting on the ladders. It will turn out that the whole theory is the natural generalization to the discrete case of the algebraic background of those basic equations of mathematical physics.

Time scales symplectic systems without normality

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We present a theory of the definiteness (nonnegativity and positivity) of quadratic functionals $\mathcal{F}$ related to *time scales symplectic systems* (S). These are time scales linear systems which generalize and unify the linear Hamiltonian differential systems and discrete symplectic systems. Under a *time scale* $\mathbb{T}$ we mean any nonempty closed subset of $\mathbb{R}$ and, in particular for this talk, we consider a bounded time scale denoted by $[a, b]$. The novel approach is in removing the assumption of *normality* in characterizations of the nonnegativity and positivity of $\mathcal{F}$. This requires a new notion of generalized focal points for conjoined bases (X, U) of (S), results on piecewise constant kernel of $X(t)$, and Picone-type results under such piecewise constant kernel condition. These results generalize many recent ones, for example in the references below.

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Order preservation of semiflows and their discretizations'

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In this talk we consider order preserving semiflows generated by differential equations in Banach spaces and we investigate whether their discretizations with arbitrary Runge-Kutta methods are alike order preserving. To this aim, first we derive sufficient and necessary conditions that guarantee the continuous semiflow be order preserving with respect to an arbitrary order cone. Then we construct explicit formulas for the discretization parameter, the step size of the Runge-Kutta method that imply the discrete semiflow be order preserving. We remark that in many situations of practical importance this construction provides us with the largest step size of this property. From these results we see that the order preservation of the discrete semiflow is possible only for few methods and even for them under severe time step restrictions. To avoid these to some extent we construct a subcone of the original order cone such that the order preservation with respect to this new cone requires much weaker conditions on the method and its step size.

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Gevrey type solutions of analytic difference equations

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We discuss the existence of Gevrey-type solutions in different (unbounded) domains of the complex plane: domains containing a part of the real axis and domains contained in an upper or lower half plane. The results are of importance in the study of (accelero-)summability of formal solutions.

Global Attractivity in Difference Equations and Dynamics of Certain Functional Differential Equations

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We relate several classes of functional differential equations with specially constructed difference equations. The latter is defined by explicit or implicit multi-dimensional maps. Some of the basic properties of the maps, such as global attractivity, are then translated back to the functional differential equations. We show in particular that the global attractivity in difference equations implies the global asymptotic stability in the corresponding functional differential equations. The instability implies the existence of periodic solutions, under some additional natural assumptions. The general approach applies in particular to the well-known delay differential equation

$$\mu\dot{x}(t) = f(x(t-\tau)) - g(x(t))$$

which serves a mathematical model of a number of real life phenomena. Some of the results on the latter equation were recently obtained in a joint work with E. Liz and S. Trofimchuk [1].

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The analysis of multiple species coexistence with a lottery model

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To clarify the mechanism promoting biodiversity is very important problem. Here we pay attention to the coexistence of plant species. It is well known that $n+1$ species cannot survive under n resources theoretically[1]. There is dilemma between nature and theory, and we consider it by using the lottery model[2]. Chesson and Warner (1981) showed that on the lottery model two species can coexist under the fluctuating environment. However, in tropical rain forest, environment for plant species seems to be almost constant. In this talk, we introduce a new lottery model and consider the species coexistence under constant environment.

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Construction of an integral manifold for delay differential-difference equations

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A linear system of differential-difference equations with delays and containing a small parameter is studied. It is assumed that this system satisfies the dichotomy conditions when the small parameter vanishes.

The sufficient conditions for the existence of an asymptotic integral manifold described by a system of differential equations without delays are obtained. For the coefficients of this system the expansions are given in powers of parameter provided that parameter is small enough. An estimate for this parameter is obtained.

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Chaotic behavior in a two-dimensional business cycle model

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We consider a discrete-time economic model which is a particular case of the Kaldor-type business cycle model and it is described by a two-dimensional dynamical system. Under certain conditions the map can be reduced to a skew map whose components, the base and the fiber map, both have entropy. Our proposal is to study and measure the complexity of the system using symbolic dynamics techniques and the topological entropy.

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Positive Lyapunov exponents or positive metric entropy imply sensitivity

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Let $f : I = [0,1] \to I$ be a map and let μ be a probability measure on the Borel subsets of I absolutely continuous with respect to the Lebesgue measure and/or invariant for f. We say that $x \in I$ is *sensitive to initial conditions* if there exists a number $\delta > 0$ with the property that for any neighbourhood U of x there is some $n = n(x, U)$ such that $\mathrm{diam} f^n U > \delta$.

Under rather mild assumptions on f and μ we show that if the set of points with positive Lyapunov exponent has positive μ-measure or the metric entropy of f is positive then the set of points having sensitivity to initial conditions has positive μ-measure. In particular we improve previous results from [1].

This is an account of a joint work with Alejo Barrio Blaya, Universidad de Murcia, Spain [2].

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Asymptotic properties of some $_3F_2$ hypergeometric polynomials

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The Euler integral representation of the $_2F_1$ Gauss hypergeometric function is well known and plays a prominent role in the derivation of transformation identities and in the evaluation of $_2F_1(a, b; c; 1)$, among other applications. The general $_{p+k}F_{q+k}$ hypergeometric function has an integral representation where the integrand involves $_pF_q$. We give a simple and direct proof of an Euler integral representation for a special class of $_{q+1}F_q$ functions for $q \geq 2$. We then obtain an asymptotic expansion for the Euler integral representation of some $_3F_2$ hypergeometric polynomials which leads us to our main result that, asymptotically, the zeros of these $_3F_2$ hypergeometric polynomials cluster on the loops of specified lemniscates.

On the Trichotomy Behavior of Equations

$$x_{n+1} = \frac{q_n x_n + r_n x_{n-1}}{1 + x_n}$$

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Abstract

We extend the known results of the non-autonomous solutions of equation in the title to the situation where any of parameters are period two sequence with non-negative values and the initial conditions are positive. In particular we find regions of local stability and regions where unbounded solution exists

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Calderón's Reproducing Formula Associated With Partial Differential Operators On Half-plane

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For $\alpha > 0$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$, we consider the functions

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\text{ch}y)^{\lambda}\varphi_{\mu}^{(\alpha,\lambda)}(y)\,, \quad (y,\theta) \in]0,+\infty[\times\mathbb{R}\,,$$

where $\varphi_{\mu}^{(\alpha,\lambda)}$ is the Jacobi function defined by

$$\varphi_{\mu}^{(\alpha,\lambda)}(y) \;=\; {}_2F_1\left(\frac{\alpha+\lambda+1+i\mu}{2}, \frac{\alpha+\lambda+1-i\mu}{2}; \alpha+1; -\text{sh}^2 y\right)\,,$$

${}_2F_1$ being the Gaussian hypergeometric function. $\varphi_{\lambda,\mu}$ are eigenfunctions of the following partial differential operators

$$\begin{cases} D = \dfrac{\partial}{\partial\theta} \\[2ex] D_\alpha = \dfrac{\partial^2}{\partial y^2} + [(2\alpha+1)\text{coth}y + \text{th}y]\dfrac{\partial}{\partial y} - \dfrac{1}{\text{ch}^2 y}\dfrac{\partial^2}{\partial\theta^2} + (\alpha+1)^2 \end{cases}$$

A generalized convolution # on the half-plane is associated with the operators D and D_α. A Calderón's reproducing formula associated with # and which involves finite Borel measures is studied.

On $x_{n+1} = \max\left\{\dfrac{A_n}{x_n}, \dfrac{B_n}{x_{n-1}}\right\}$ With Period-Four Parameters[2]

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We investigate the periodic character of positive solutions of the difference equation $x_{n+1} = \max\left\{\dfrac{A_n}{x_n}, \dfrac{B_n}{x_{n-1}}\right\}$, where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are both periodic sequences of positive numbers with (not necessarily prime) period four. We show that every positive solution is eventually periodic with (not necessarily prime) period eight. This is work-in-progress, and utimately our objective is to extend the proof of this investigation (and others) and generalize our results to the case when $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive numbers with period $p \in \{1, 2, \ldots\}$, $\{B_n\}_{n=0}^{\infty}$ is a periodic sequence of positive numbers with period $q \in \{1, 2, \ldots\}$, and neither p nor q is a multiple of three, in which case we intend to show that every positive solution is eventually periodic with period $2 \cdot \mathrm{lcm}(p, q)$.

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Analysis methods of multiple eigenvalues of spheroidal wave functions

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Spheroidal wave functions are eigenfunctions of special differential operator of second order. They play an important role in many applied problems of diffraction, astrophysics, etc.
When the spheroid slenderness parameter c is real or pure imaginary, the numerable set of different real eigenvalues $\lambda_n(c)$ corresponds to different spheroidal functions. Otherwise, when the differential operator is not self-adjoint, the values of $\lambda_n(c)$ can coincide. This case originates in the complex plane c the singular points c_b — square-root branch points of eigenvalues.
Several modern methods are proposed for the structure of these singularities analysis. By means of effective symbolic evaluations the explicit power series expansions $\lambda_n = \lambda_n(c)$ were obtained and the character of singularities on the boundary of convergence disk was studied. The strong cancellation of digits appearing during c_b evalution was overcome. Application of Padè approximants and its generalization — Hermite-Pade approximants – allowed to compute branch points c_b and eigenvalues $\lambda_n(c_b)$ with high precision and computation speed.

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Relative Controllability of Linear Discrete Systems with Constant Coefficients and Pure Delay

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We deal with linear discrete systems with delay

$$\Delta x(k) \quad = \quad Bx(k-m) + bu(k), \quad k \in \mathbb{Z}_0^\infty := \{0, 1, \ldots\}, \tag{1}$$

where $\Delta x(k) = x(k+1) - x(k)$, $x : \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$, B is a square $n \times n$ constant matrix, $m \geq 1$ is an integer, b is a constant nonzero vector and $u : \mathbb{Z}_0^\infty \to \mathbb{R}$ is a control function. The system (1) is called to relatively controlled, if for arbitrary initial conditions $x(k) = \varphi(k)$, $k \in \mathbb{Z}_{-m}^0$, arbitrarily finite $k_1 \geq k^*$, where $k^* \in \mathbb{Z}_1^{k_1-1}$ is a fixed number, and arbitrary finite state $x^* \in \mathbb{R}^n$ there exists function $u^* : \mathbb{Z}_0^{k_1-1} \to \mathbb{R}$, such that system (1) with $u = u^*$ has solution $x : \mathbb{Z}_{-m}^{k_1} \to \mathbb{R}^n$, satisfying initial conditions and the terminal condition $x(k_1) = x^*$. We solve this problem with the aid of discrete matrix function e_m^{Bk} defined as

$$e_m^{Bk} \quad = \quad \begin{cases} \Theta \text{ if } k < -m \\ I + \sum_j = 1^l \dfrac{B^j}{j!} \; \dfrac{[k-(j-1)m]!}{[k-(j-1)m-j]!} \\ \text{ if } k = (l-1)(m+1)+1, \ldots, l(m+1),\ l = 0, 1, \ldots \end{cases}$$

We prove that system (1) is relatively controllable if and only if $k^* = 1+(n-1)(m+1)$ and $\operatorname{rand}(b, Bb, B^2b, \ldots, B^{n-1}b) = n$. Corresponding control function has e.g. the form

$$
\begin{aligned}
u^*(k) = \; &= b^T(e_m^{B(k_1-m-k-1)})^T \left[\sum_{j=1}^{k_1} e_m^{B(k_1-m-j)} bb^T (e_m^{B(k_1-m-j)})^T\right]^{-1} \\
&\times \left[x^* - e_m^{Bk_1}\varphi(-m) - \sum_{j=-m+1}^{0} e_m^{B(k_1-m-j)} \Delta\varphi(j-1)\right], \quad k = 0, 1, \ldots, k_1 - 1.
\end{aligned}
$$

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Duality for dynamical quantum groups and elliptic hypergeometric series

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Dynamical quantum groups are analogues of Hopf algebras, where the base field is extended from $\mathbb{C}$ to meromorphic functions on a complex vector space in a non-trivial way. Such dynamical quantum groups can be constructed from solutions of the quantum dynamical Yang-Baxter equation, and they come naturally equipped with a kind of self-duality, see [3]. This duality leads to natural actions of a dynamical quantum group on itself. In case of the elliptic $U(2)$ quantum group, corresponding to an elliptic solution of the Yang-Baxter equation, we can establish the duality between matrix elements of corepresentations in terms of elliptic hypergeometric series. This can be used for a dynamical quantum group theoretic proof of the bi-orthogonality relations, Bailey's transform and the hexagon relation for very-well-poised elliptic hypergeometric series extending the approach of [2]. This work [1] is joint with Yvette van Norden (TU Delft).

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Class coexistence and synchronous phenomena in imprimitive matrix population models

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The Leslie matrix model for a semelparous population often exhibits a synchronous phenomenon in which all but one year-class is missing [1,2]. In this talk, we investigate the condition leading to a synchronous phenomenon in the Leslie matrix model. The synchronous phenomenon in matrix population models is characterized by an orbit lying on the boundary of the non-negative cone $\mathbb{R}^n_+$, which is called a synchronous orbit. It is known that a synchronous orbit does not exist in matrix population models with primitive life cycle graph [3]. Therefore, we concentrate on matrix population models with imprimitive life cycle graph. In particular, we provide a condition for attractivity of (periodic or aperiodic) synchronous orbits in the Leslie matrix model for a semelparous biennial population.

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On the periodic character of solutions of the third order rational difference equation

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We investigate the periodic character of solutions of the third order rational difference equation

$$x_{n+1} \quad = \quad \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}} \quad n = 0, 1, \ldots$$

In particular, we determine all special cases which posses an *essentially unique* period-two solution.

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Convergence of the Agreement Algorithm in Higher Dimensions

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The agreement algorithm studied by Tsitsiklis et al. ([1, 4]) is extended to higher dimensions by the nonautonomous dynamical system

$$x^j(t+1) \;=\; \sum_{j=1}^{n} a_{ij}(t)x^j(t), \quad 1 \leq i \leq n,\; t \in \mathbb{N}$$

where $x^j(\cdot) \in D \subset \mathbb{R}^d$ and the matrix $A(t)$ of coecients of interaction $a_{ij}(t)$ is row-stochastic. Conditions on $t \to A(t)$ are specied which guarantee that to every $x(0) \in D$ there exists some $c \in D$ such that $\lim_{t\to\infty} x^j(t) \to c$ for all $1 \leq i \leq n$. This result extends a convergence result proved in [1] for the scalar case and constant communication strength.

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On a solution of the Painlevé II equation with no real poles

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This talk is based on joint work with Tom Claeys and Maarten Vanlessen.

The Painlevé II equation $q'' = sq + 2q^3 - \alpha$ with $\alpha \neq 0$ has a unique solution $q = q_\alpha$ with the asymptotic behavior $q_\alpha(s) \sim \alpha/s$ as $s \to +\infty$ and $q_\alpha(s) \sim \sqrt{-s/2}$ as $s \to -\infty$. We show that for $\alpha > -1/2$ this solution has no poles on the real line. This result is used in the study of random matrix ensembles

$$\frac{1}{Z_{n,N}} \left|\det M\right|^{2\alpha} e^{-N\mathrm{Tr}V(M)} dM$$

defined on $n \times n$ Hermitian matrices M, when the limiting mean eigenvalue density associated with V vanishes quadratically at the origin. Then as $n, N \to \infty$ so that $n^{2/3}(\frac{n}{N} - 1)$ remains constant, the eigenvalue correlation kernel has a double scaling limit which is expressed in terms of ψ-functions associated with q_α. These ψ-functions exist exactly because the Painlevé transcendent has no real poles. The q_α function itself describes the double scaling limit of the recurrence coefficients of the orthogonal polynomials associated with the weight $|x|^{2\alpha} e^{-NV(x)}$ on the real line.

Competitive-Exclusion versus Competitive-Coexistence for Systems of Difference Equations in the Plane

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We investigate the attractivity of an equilibrium of a system

$$\begin{cases} x_n &= x_n f(x_n, y_n) \\ y_n &= y_n g(x_n, y_n) \end{cases}$$

where f, g are continuous functions and $f(x,y), g(x,y)$ are non-increasing in y and x respectively in some domain A. We will assume the existence of unique equilibrium $\overline{e}$ in the interior of A. We will consider two cases depending of local asymptotic stability of the equilibrium $\overline{e}$. In case where $\overline{e}$ is locally asymptotically stable, we will prove that it is also globally asymptotically stable (competitive coexistence). In case where $\overline{e}$ is locally saddle point, we will prove that the equilibrium points on the boundary are attractors of all points which are above or below the global stable manifold $W^s(\overline{e})$ (competitive exclusion).

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Some problems on Gamma and Polygamma functions

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We consider some problems on gamma and polygamma functions. These are related to inequalities, functional properties, monotonicity and convexity (concavity) properties, Turan-type inequalities.

Two Normal Ordering Problems and certain Sheffer polynomials

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The first normal ordering problem involves bosonic harmonic oscillator creation and annihilation operators (*Heisenberg* algebra). It is related to the problem of finding the finite transformation generated by $L_{k-1} := -z^k\,\partial_z, k \in \mathbf{Z}$, $z \in \mathbf{C}$ (conformal algebra generators). It can be formulated in terms of a subclass of *Sheffer*-polynomials called *Jabotinsky*-polynomials. The coefficients of these polynomials furnish generalized *Stirling*-number triangles of the second kind, called $S2(k; n, m)$ for $k \in \mathbf{Z}$. Generalized *Stirling*-numbers of the first kind, $S1(k; n, m)$ are also defined.

The second normal ordering problem appears in thermo-field dynamics for the harmonic Bose oscillator. Again *Sheffer*-polynomials appear. They relate to *Euler* numbers and iterated sums of squares. In a different approach to this problem one solves the differential-difference equation

$$f_{n+1} = f'_n + n^2\, f_{n-1}\,, n \geq 1, \text{ with certain inputs } f_0 \text{ and } f_1 = f'_0.$$

In this case the integer coefficients of the special *Sheffer*-polynomials which emerge have an interpretation as sum over multinomials for some subset of partitions.

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Generalization of static problem for beams formulated by finite element method using difference equations

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In the paper the bending problem of one-dimensional systems loaded by an arbitrary set of forces is solved. Equilibrium equations are formulated by finite element method. If a beam is divided into a regular mesh of identical finite elements, the set of algebraical equations can be replaced by the set of two difference equations:

$$A_1(E^{-1} - E)\phi_r + A_2\Delta^2 w_r = A_P P_r$$

$$(A_3\Delta^2 + A_4)\phi_r - A_1(E^{-1} - E)w_r = A_m m_r$$

where ϕ_r is a discrete function of nodal rotations, w_r is a discrete function of nodal displacements, E^n is the shifting operator, P_r and m_r are the nodal forces and moments, $\Delta^2 = (E + E^{-1} - 2)$ is the second-order difference operator, A_i are the constants depending on the type of used finite element.

A solution of the difference equations is determined in the analytical form by using the eigenfunction method. This method requires the determination of a set of eigenvalues and eigenfunctions for each type of boundary conditions. This advantage is eliminated by introductions of elastic supports at the beam ends. Reactions in these supports are linearly depending on displacements. An advantage of the obtained solution for non-homogeneous boundary conditions is that using the eigenvalues for clamped-clamped beam it is possible to build the analytical function of generalized displacements for any combination of boundary conditions.

The presented method is a generalization of the static problem solution for one-dimensional discrete systems, which is obtained in the analitical form using the difference equations.

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Solutions of Dynamic Equations with Varying Time Scales

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Consider the dynamic equations $x^{\Delta} = 4x(3/4 - x)$ with the initial condition $x(0) = x_0$ on different time scales. The differential equation on the time scale $\mathbb{R}_+$ yields smooth, continuous solutions, whereas solutions of the difference equation, for the time scale $\mathbb{Z}_+$, are generically chaotic.

Our goal is to understand the behavior of such solutions of dynamic equations of the same function over different time scales as bifurcations and limits over their underlying domains–the time scales R_+ and $\mathbb{Z}_+$Z, respectively. For this purpose we consider the hyperplane $Cl(\mathbb{R}_+)$–the set of time scales in $\mathbb{R}_+$–as a parameter space for a given equation.

We will discuss some results in this direction.

Multiscale Analysis of Discrete Nonlinear Equations

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In this presentation we consider multiple lattices and functions defined on them. We introduce slow varying conditions and define a multiscale analysis on the lattice, i.e. a way to express the variation of a function in one lattice in terms of an asymptotic expansion with respect to the others. We apply these results to the case of the multiscale expansion of the potential lattice Korteweg–de Vries equation.

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An extension of the q-Bernstein polynomials

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The generalized Bernstein basis in the space of polynomials of degree at most n, being an extension of the q-Bernstein basis introduced by Phillips [2], is given by the formula [1]

$$B_i^n(x;\omega|q) := \frac{1}{(\omega;\, q)_n} \begin{bmatrix} n \\ i \end{bmatrix}_q x^i\, (\omega x^{-1};\, q)_i\, (x;\, q)_{n-i} \qquad (0 \le i \le n),$$

where ω is a parameter (Phillips' case corresponds to $\omega = 0$). Notice that also the classical Bernstein, and discrete Bernstein basis polynomials [3] are limiting forms of the polynomials $B_i^n(\cdot\,;\omega|q)$. Formulas relating $B_i^n(\cdot\,;\omega|q)$, big q–Jacobi and q–Hahn (or dual q-Hahn) polynomials are presented. The dual basis functions $D_k^n(\cdot\,;a,b,\omega|q)$ for the polynomials $B_i^n(\cdot;\omega|q)$ are given explicitly in terms of big q-Jacobi polynomials $P_k(\cdot\,;a,b,\omega/q;q)$, a and b being parameters; the connection coefficients are evaluations of the q-Hahn polynomials. Further, an alternative formula is given, representing the polynomial $D_j^n(\cdot\,;a,b,\omega|q)$ $(0 \le j \le n)$ as a "short" linear combination of $\min(j, n-j) + 1$ big q-Jacobi polynomials with shifted parameters.

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The Underlying Group Structure of some Three Term Transformations for Basic Hypergeometric Series

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The study of invariance groups associated with two term transformations between (basic) hypergeometric series has received its fair share of attention, and indeed, for most two term transformations between (basic) hypergeometric series the invariance group is explicitly known (see [2] and references therein). In this talk, we will present the group structure underlying some three term transformation formulae, which can e.g. be found in (the appendix of) [4] or in the list of identities accompanying [3]. The main result is that the group structure underlying a three term identity comprises a "local" invariance group H, of which the action leaves each of the terms invariant, and a "invariance group" $G \supset H$. Each term in the three term identity is characterized by a coset of H in G. We show that conversely for each triple of cosets there exists a three term identity.
The group structure is determined by giving explicit and simple realizations that prove very helpful in determining whether two of these transformation formulae are equivalent are not. Using the group theoretical context allows us to summarize, for each of the cases studied, the various three term formulae that appear in the literature in one simple statement. The presented results are available in [1].

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Some recent global stability results for higher order difference equations

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We give a brief overview of some recent results for the global asymptotic stability of a difference equation

$$x_{n+1} = f(n, x_n, \dots, x_{n-k}),\ n \geq 0,$$

assuming that the zero solution is the unique equilibrium. Different techniques are used, involving the use of new discrete inequalities, monotonicity arguments, and delay perturbation methods. For the particular case of linear autonomous difference equations, we present new explicit sufficient conditions for the exponential stability. Some related conjectures and open problems are discussed.

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On (h, k) hyperbolic induced splittings for a class of noninvertible difference equations

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We consider a class of linear difference equations for which non invertibility plays a central rôle. We present general assumptions under which forward splittings are considered as generalized (h, k) hyperbolicity, thus naturally extending the theory developed for exponential dichotomies.
This work base on previous results obtained by Aulbach et al in the field of non-invertible difference equations, but it also comes from the theory of generalized difference and differential equations (via the so called differential equations with impulse effect), following the spirit of extensions of the notion of dichotomies, like (μ_1, μ_2) dichotomies of Muldowney.

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The generalized Bochner condition about classical orthogonal polynomials revisited

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We bring a new proof for showing that an orthogonal polynomial sequence is classical if and only if any of its polynomial fulfils a certain differential equation of order $2k$, for some $k > 1$. So, we build those differential equations explicitly. If $k = 1$, we get the Bochner's characterization of classical polynomials. With help of the formal computations made in Mathematica, we explicitly give those differential equations for $k = 1$, 2 and 3 for each family of the classical polynomials. Higher order differential equations can be obtained similarly.

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General quadratic decomposition

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The main goal of this work is to generalize the quadratic decomposition presented in [1].
If we have a monic polynomial sequence $\{W_n\}_{n\geq 0}$, it is always possible to associate with it two monic sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ and two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ such that:

$$W_{2n}(x) = P_n(x^2) + xa_{n-1}(x^2)$$

$$, \ n \geq 0 \,,$$

$$W_{2n+1}(x) = b_n(x^2) + xR_n(x^2)$$

where $\deg a_n \leq n$, $\deg b_n \leq n$, $a_{-1}(x) = 0$.
More generally, given $\varpi(x) = x^2 + px + q$, $p, q \in \mathbb{C}$ and $a \in \mathbb{C}$, it is possible to obtain the following unique decomposition:

$$W_{2n}(x) = P_n(\varpi(x)) + (x-a)a_{n-1}(\varpi(x))$$

$$, \ n \geq 0 \,.$$

$$W_{2n+1}(x) = b_n(\varpi(x)) + (x-a)R_n(\varpi(x))$$

We give the general quadratic decomposition of a MPS and we use it to the special case of the sequence $\{x^n\}_{n\geq 0}$.
We study some particular cases where the sequence $\{W_n\}_{n\geq 0}$ is orthogonal and finally we fully characterize the sequence $\{W_n\}_{n\geq 0}$ whose coefficients $\{\beta_n\}_{n\geq 0}$ satisfy

$$\beta_{2n+1} = \beta_1 \quad \text{and} \quad \beta_{2n+2} = -(p+\beta_1) \,, \ n \geq 0 \,,$$

showing that, if $\{W_n\}_{n\geq 0}$ is a MOPS, then the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are also MOPS, and conversely.

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On the Asymptotic Behavior of a System of Difference Equations

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Recently, M. F. Kondratieva and S. Sadov have introduced an accurate and robust numerical algorithm for diffraction problems in mid-high frequency range. In this note, a sketch of proof of an asymptotic formula for the ratio of two neighboring Hankelos function as the index goes to infinity has given. This proof is based on a difference equation with variable coefficients. Study of the asymptotic behavior of this equation, is a very difficult problem because the coefficients varies as a noncontinuous function of the index. We treat this difficulty by introducing two system of difference equations with constant coefficients, which sandwich the main difference equation for all indexes. Then, we proof that the only equilibrium point of either of these systems of equations is globally asymptotical stable. The proof is applied only for upper bound system; since there is only a slight difference in the formulation of these two systems, the proof can be used for the lower bound system too. We show this in a simple corollary.

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Periodic and dense solutions of the third order Lyness equation via invariants and Möbius transformations.

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We present some new results concerning the dynamics of the real third order lyness equation $x_{n+3} = (a + x_{n+2} + x_{n+1})/x_n$, where a is a real parameter and the initial conditions are in $\mathbb{R}^3$. Among others, we prove the existence of continua of initial conditions giving rise to periodic orbits of period 2, 4, 5, 6, 7 and $4p$ for $p > 2$, as well as continua of initial conditions giving rise to dense orbits in $\mathbb{R}$. The main results are obtained by using the well known first integral (invariant) of the discrete dynamical system associated with the difference equation: $V(x, y, z) = (x + 1)(y + 1)(z + 1)(a + x + y + z)/(xyz)$. The dynamics at one of the level sets of V can be studied using a new system for which it can be found a new first integral. The key point is that on each leaf of the foliation induced by this new first integral the dynamics is given by a real Möbius transformation.

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Asymptotic properties of recessive solutions for half-linear difference equations

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Recessive and dominant solutions of the half-linear difference equation are investigated. It is shown that the property of the recessive solutions to be smallest solutions in a neighbourhood of infinity continue to holds also in the half-linear case. In addition it is proved that recessive solutions can be fully characterized by means of two different summation criteria, which reduce to that one well-known in the linear case. Using some summation inequalities stated in [3], some applications are offered and new results on the asymptotic behavior of these solutions are presented.
The presented results have been achieved in a joint research with Mariella Cecchi (University of Florence) and Zuzana Dosla (University of Brno).

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Spectral Analysis of Polygonal Billiards

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Classical dynamics of polygonal billiards is neither integrable nor chaotic, it is pseudointegrable [1], which implies that in the quantum case the Bohr-Sommerfeld's method of quantization cannot be used to find the energy levels of the system. Polygonal billiards have, except the Hamilton's function H, also a second constant of motion K [2] if all angels α between sides are rational, i.e. $\alpha = m\pi/n$. In this case the motion is restricted to two-dimensional invariant surfaces. We study the energy spectra (energy levels) of cyclic polygonal billiards in terms of the side lengths. There are attempts to express the area of a polygon in terms of its side lengths [3]. This possibility enables us to find a close connection between the properties of the energy spectra and the polygon area. Cyclic polygons can also be considered as a polygonalization of a circle billiard [4] which is integrable.

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q-difference Painlevé VI equation arising from q-UC hierarchy

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We study the q-difference analogue of the sixth Painlevé equation (q-P_{VI}) by means of tau functions associated with affine Weyl group of type D_5. We prove that a solution of q-P_{VI} coincides with a self-similar solution of the q-UC hierarchy. As a consequence, we obtain in particular algebraic solutions of q-P_{VI} in terms of the universal character which is a generalization of Schur polynomial attached to a pair of partitions. This is a joint work with Dr. Teruhisa Tsuda.

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Asymptotic Behavior of Solutions of a Linear Delay Difference System

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Recently, several authors have investigated the asymptotic behavior of solutions of linear difference systems under the assumption that the associated characteristic equations have a dominant real root. In this talk we will classify the limits of solutions of a linear delay difference system completely. In paticular, if the solution tends to an equiliblium point or a periodic orbit, we will give the explicit expressions in terms of the initial conditions and delay parameters.

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Properties of solutions of the third and fourth order linear difference equations

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Some third and fourth order linear difference equations are considered. Asymptotic properties of their solutions, particularly oscillation and boundedness, are described. In the case of the constant coefficients we compare our solutions with solutions of suitable differential equations. These equations with the same roots of their characteristic equations can have solutions with different asymptotic properties.

Accurate Numerical Integration Method for the Kepler Motion

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The two and three-dimensional discrete Kepler motions were presented in [2, 3] which conserve all of the constants of motion such as the Hamiltonians, the angular momenta and the Runge-Lenz vectors. Sequences of points described by the discrete Kepler motions exactly lie in the continuous orbits of the Kepler motions. However, the time evolution of the discrete Kepler motions is different from that of the continuous Kepler motions. In this talk formulas are found which describe the time when the two and three-dimensional continuous Kepler motions arrive at the points located by the discrete Kepler motions. As an application time adjustments of the discrete Kepler motions are performed. Consequently an accurate numerical integration method for the Kepler motions is designed.

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Fourier-Padé Approximants for Angelesco Systems

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In this paper we study Fourier-Padé approximation for Angelesco systems of functions. This construction is similar to that of Hermite-Padé approximation. Instead of considering power series expansions of the functions in the system, we take their expansion in a series of orthogonal polynomials. The main results of the present paper concern the diagonal Hermite-Padé approximants of orthogonal expansions. We give the rate of convergence of these approximants for a system of Markov funtions whose supports do no intersect, that is, Angelesco systems. The answer is given in terms of the solution of the extremal solutions of certain vector valued equilibrium problems for the logarithmic potential.

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Riemann-Hilbert Problem for a Generalized Nikishin System

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Recently it has been shown that the multiple orthogonal polynomials, related to the Hermite-Padé approximation of type I or type II, to a system of Markov functions are solutions of a Riemann-Hilbert problem. It is also be given the normalization of this Riemann-Hilbert problem for two particular well known systems of Markov functions, the Angelesco and the Nikishin systems. In this paper we extend this result to the class of Generalized Nikishin systems and we also give the normalization for this Riemann-Hilbert problem using a solution of an equilibrium problem.

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A Delayed-Feedback Queueing System with Splitting

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We consider the multi-processor model illustrated in Figure 1. The system consists of a service and a delay station set in tandem. The service station consists of a buffer, server unit and a splitting unit. The server unit consists of c individual servers, identical and set in parallel. The buffer capacity is $M - c \geq 0$ and is set before the server unit. Next to each server, there is an exit way from the system, a runway to a splitter that is set in tandem with the server, and a feedback way to the delay station. Next to each splitter there is also an exit way from the system, and a feedback way to the delay station. We will solve the steady-state system of differential-difference equations analytically. To check the validity of the result, not only we compare the special cases with those that exist in the literature, but we will also simulate the system and compare the analytical and simulated results.

Finite-difference methods for solution of non-local boundary value problems

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Research of finite-difference analogues for non-local problems have been conducted by various scientists (see e.g. [1]-[6]). In these works the authors basically studied convergence issues of finite-difference schemes on classes of smooth solutions for either second order ordinary differential equations or the simplest partial differential equations, particularly heat conductivity and Poisson's equations in quadrangle areas.

This talk is devoted to the construction and investigation of various types of difference schemes relevant to non-local boundary value problems stated for multidimensional elliptic equations. Suitable tools for performing our investigation are developed, including difference analogue of Green's formula and inequalities corresponding to non-local boundary conditions.

More specifically, analogically to the methods of lines, we construct difference schemes on irregular grids for non-local problems for elliptic equations. The convergence of the discretized solution to the solution of the initial problem is proved. The speed of convergence is estimated. It is worth to note that the irregularity of the grids allows considering non-local conditions without approximation. The same topics are covered in the case of a regular grid. But in the latter case it becomes necessary to approximate non-local conditions on the grid.

It is also covered all the above considered issues in the case of full discretization. We investigate and estimate the speed of convergence in this case as well.

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A Generalization of Wiener's Lemma and its Application to Volterra Difference Equations on a Banach Space

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Some stability properties for the zero solution of linear Volterra difference equations on a Banach space are studied in connection with the summability of the fundamental solution and the invertibility of the characteristic operator. A key is to extend Wiener's lemma for absolutely convergent scalar sequences to summable sequences of bounded linear operators on a Banach space by applying certain result in commutative Banach algebras.

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An Approach for Global Asymptotic Stability for some Classes of Positive Rational Difference Equations

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This paper reports on results about the global asymptotic stability of the equilibrium $x^* > 0$ on some classes of positive rational difference equations, e.g. the Putnam difference equation

$$x_{n+1} = \frac{x_n + x_{n-1} + x_{n-2}x_{n-3}}{x_n x_{n-1} + x_{n-2} + x_{n-3}},\ n = 0, 1, 2, \ldots.$$

In the last years many authors proved global asymptotic stability for various families of positive rational difference equations by different approaches. Some classes of these equations have in common that they are contractive with respect to the part-metric.

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On the inverse problem of the product of a form by a monomial: the case $n = 4$.

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A form (linear functional) u is called regular if there exists a unique sequence of monic polynomials $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, which is orthogonal with respect to u. On certain regularity conditions, the product of a regular form by a polynomial is still a regular form. In this work, we consider the particular inverse problem: given a regular form v, find all the regular forms u which satisfy the equation $x^4 u = -\lambda v$, $\lambda \in \mathbb{C} \setminus \{0\}$. We give the second order recurrence relation of the orthogonal polynomial sequence with respect to u. Some examples are studied.

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Asymptotics in the complex plane of the third Painlevé transcendent

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An uniform asymptotics in the complex plane for the third Painlevé transcendent is constructed and proved. The leading term of asymptotics as $|z| \to \infty$ is given by the *Boutroux ansatz*, i.e. by an elliptic function with its modulus depending on $\arg z$. A functional equation for the modulus is universal for PIII equation and does not depend on initial conditions. It can be solved as an Abel problem of inversion of elliptic integrals.

Another component of the Boutroux ansatz is the phase shift in the elliptic function. It depends on initial data, and we calculate it with the help of Isomonodromic Deformation Method (IDM) [3]. By solving a direct monodromy problem for a relevant Lax pair operators, we fit given monodromy data with their approximations, coming from the leading term of asymptotics. This leads to explicit formulas both for the modulus and the phase shift. Since a monodromy data for PIII transcendent can be expressed explicitly through the initial conditions at $z = 0$ (see [2]), we come to a *connection formulas* linking the PIII transcendent asymptotics at infinity and at the origin.

Finally, the IDM technique provides the proof of the above constructions, giving an analog of the Bolibrukh-Its-Kapaev theorem proved earlier for the similar asymptotic description of PII transcendent in the complex plane [4].

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Oscillation Theorems for Difference Functional Equations of Higher Order

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We give some oscillation theorems for difference functional equation of the form

$$\Delta_g x(t) = \sum_{i=1}^{m+1} a_i(t)x(g^i(t)) + \sum_{j=1}^{l} b_j(t)x(g^{-j}(t)),$$

where $t \in \Im \subset \Re_+$, $m, l \in \aleph$. The difference operator Δ_g is defined by $\Delta_g x(t) = x(g(t)) - x(t)$. The functions $a_i, b_j : \Im \to \Re_+$, $(i = 1, 2, ..., m+1,\ j = 1, 2, ..., l)$ and $g : \Im \to \Im$ are given and x is an unknown real-valued function. By g^m we mean the m-th iterate of the function g, i.e.

$$g^0(t) = t, \qquad g^{m+1}(t) = g(g^m(t)), \qquad t \in \Im, \qquad m = 0, 1, ...$$

By g^{-1} we mean the inverse function to g and $g^{-m-1}(t) = g^{-1}(g^{-m}(t))$.
We also show an application of our results to difference equations.

Solutions of Dynamic Equations with Varying Time Scales

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Consider the dynamic equations $x^{\Delta} = 4x(3/4 - x)$ with the initial condition $x(0) = x_0$ on different time scales. The differential equation on the time scale $\mathbb{R}_+$ yields smooth, continuous solutions, whereas solutions of the difference equation, for the time scale $\mathbb{Z}_+$, are generically chaotic.
Our goal is to understand the behavior of such solutions of dynamic equations of the same function over different time scales as bifurcations and limits over their underlying domains–the time scales R_+ and $\mathbb{Z}_+$Z, respectively. For this purpose we consider the hyperplane $Cl(\mathbb{R}_+)$–the set of time scales in $\mathbb{R}_+$–as a parameter space for a given equation.
We will discuss some results in this direction.

Hankel Determinants of Generalized Catalan Numbers, Toda Equation and Orthogonal Polynomials

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Hankel determinants of certain combinatorial numbers are calculated explicitly in terms of the Toda equation and orthogonal polynomials. These combinatorial numbers contain the so-called generalized Catalan numbers. It is shown that the determinant form of the τ-function of the Toda equation is composed of such kind of combinatorial numbers. Hankel determinants of the combinatorial numbers with shifted index are then presented. Next some orthogonal polynomials are considered whose moments are represented by combinatorial numbers. Hankel determinants of sum of adjacent combinatorial numbers are derived through the Christoffel transform of orthogonal polynomials.

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A Generalization of the Discrete UC Hierarchy and Its Reductions

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We discuss about a generalization of the UC hierarchy which was recently introduced by Tsuda as an extension of the KP hierarchy. By generalizing the determinant expression of universal characters with the independent variable transformation given by Tsuda, the bilinear equations for the multicomponent determinants are constructed through the Laplace expansion technique. It is also shown that the periodic reductions derive discrete integrable 2+1 dimensional equations.

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Bifurcations of non continuous discrete dynamical systems on the real line

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In this paper we study the iteration of symmetric non continuous one real parameter family of mappings on the real into itself

$$g(x) = -\beta \tan(\beta \tan x).$$

The classic bifurcations obtained when we vary the real parameter β are: saddle node, period doubling and pitchfork.

We obtain bifurcations associated to the non continuous nature of the family, non classic period doubling (fusion of orbits) and non classic period halving (splitting of orbits).

The global structure of the bifurcation scheme is studied and a cascade of bifurcations observed.
When the parameter is increased the complexity increases and consequently the topological entropy.

Diagonalization of the Hilbert matrix

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It is shown that the infinite Hilbert matrix H commutes with a special self-adjoint Jacobi matrix, which turns out to be an elementary function of H. The corresponding orthogonal polynomials are the continuous dual Hahn polynomials. Thus, one obtains explicitely a complete set of formal eigenvectors of H by dint of those polynomials or, equivalently, by the associated difference equation. This yields formulae that are considerably simpler than those of Rosenblum [2], who wrote H as an integral operator and computed formal eigenvectors in terms of Laguerre coefficients of Whittaker functions. Finally, an application to Hilbert's inequality is discussed.

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Compatible Discrete Integrable Partial Difference Equations on multidimensional lattices

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We review the structure of integrable partial difference equations using their multidimensional compatibility. In particular we present the effect of slicing the multidimensional orthogonal grid in order to obtain planar quadrilateral graphs and show the relation of some of them with non quadrilateral ones. The presentation of some of the integrable partial difference equations as coupled maps is also discussed.

On a $(k+1)$-th order difference equation with a coefficient of period $k+1$

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We study the boundedness, the periodicity and the attractivity of the positive solutions of the non-autonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, ...$$

where k is an odd number, p_n, $n = 0, 1, ...$ is a positive sequence of period $k+1$. Moreover we investigate the global asymptotical stability of the above equation for $k = 3$.

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The Chen-Rubin conjecture in a continuous setting

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The median $m(x)$ of the Γ-distribution with parameter $x > 0$ is defined implicitly as

$$\int_0^{m(x)} e^{-t}t^{x-1}\,dt = \frac{1}{2}\int_0^{\infty} e^{-t}t^{x-1}\,dt.$$

We shall show that

$$0 < m'(x) < 1$$

for all $x > 0$. This implies a conjecture of Chen and Rubin from 1986: the sequence $m(n) - n$ decreases for $n \geq 1$. The proof is based on a mixture of real and complex methods and the theory of Pick functions plays an essential role.

We shall relate our results to work of Szegö, Watson and others concerning Ramanujan's error function. We shall also describe the behaviour of m at zero and at infinity.

This is joint work with Christian Berg at the University of Copenhagen.

The Henstock-Kurzweil Delta and Nabla Integrals

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We will define the Henstock-Kurzweil delta and nabla integrals and give several of their properties. This enables one to study more general dynamic equations on time scales.

A matrix Rodrigues formula for classical orthogonal polynomials in two variables

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Classical orthogonal polynomials in one variable can be characterized as the only orthogonal polynomials satisfying a Rodrigues formula. In this paper, using the symmetrized Kronecker power of a matrix, a Rodrigues formula is introduced for classical orthogonal polynomials in two variables.

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Nonoscillations in odd order difference systems of mixed type

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The aim of this note is to discuss the existence of nonoscillations for the difference system of mixed type

$$\Delta x(n) \;=\; \sum_{i=1}^{l} P_i x(n-i) + \sum_{j=1}^{m} Q_j x(n+j), \qquad n = 0, 1, 2, \ldots$$

in terms of some matrix measures envolving the matrices P_i and Q_j ($i = 1, \ldots, l$, $j = 1, \ldots, m$). Those measures are formulated in basis upon arbitrary logarithmic norms of matrices.

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Extensions of Perron's Theorems for Differential and Difference Equations

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Recent extensions of Perron's theorems about the strict Lyapunov exponents of the solutions of differential and difference equations will be discussed.

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Discrete Inertial Manifolds

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Various evolutionary processes in physics and other sciences can be described using nonlinear dissipative partial differential equations generating infinite dimensional dynamical systems. The reduction of such a system in an appropriate infinite dimensional state space to a finite one preserving its long-time behavior, is a relevant and interesting problem in both pure and applied mathematics.

The description of the long-term behavior of a dynamical system ultimately means to determine its attractor. It has been found out that in many cases the global attractor can be embedded into exponentially attractive finite dimensional manifolds (cf., e.g. [2]). Consequently, it turned out that so-called *inertial manifolds* are often an appropriate tool for the studies related to the long-term behavior of evolutionary equations, which allow for the reduction of the dynamics to a finite dimensional ordinary differential equation.

Motivated by an analytical discretization theory for evolutionary equations, this talk deals with such questions of existence and exponential attraction to invariant manifolds for nonautonomous (ordinary) difference equations, instead of evolutionary differential equations. We discuss their essential properties, like smoothness, the existence of an asymptotic phase and normal hyperbolicity in a nonautonomous framework of pullback attractors (cf., e.g. [1]).

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Exact and quasi-exact solvability of two-dimensional superintegrable quantum systems

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In this note we show that the separation of variables for second-order superintegrable systems in two-dimensional sphere and two and three dimensional Euclidean spaces generates both exactly solvable (ES) and simple quasi-exactly solvable (QES)problems in quantum mechanics.

On the ergodic and spectral properties of generalized Boole transformations. Part 1 Ergodic and spectral properties

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The invariant ergodic measures for generalized Boole type discrete dynamical systems are studied making use of the invariant quasi-measure approach, based on some special solutions to the Frobenius-Perron operator. Some generalizations are suggested for the transformatuions of general form.

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Period-Two Convergence in a Third Order Rational Difference Equation

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We investigate the periodic character of solutions of the third-order difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n}, \quad n = 0, 1, \ldots .$$

In particular, we discuss the convergence to period-two solutions.

Morse Decompositions of Nonautonomous Dynamical Systems

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The global asymptotic behavior of dynamical systems on compact metric spaces can be described via Morse decompositions. Their components, the so-called Morse sets, are obtained as intersections of attractors and repellers. In this talk, we introduce special notions of attractor and repeller for nonautonomous dynamical systems which are designed to establish nonautonomous generalizations of the Morse decomposition. We discuss the dynamical properties of these Morse decompositions and consider Morse decompositions of one-dimensional and linear systems.

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Discrete Functional Boundary Value Problems on Infinite Intervals

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A method for solving discrete functional boundary value problems on infinite intervals will be presented and applied to the investigation of the existence of positive unbounded solutions of coupled nonlinear difference systems.

Dynamical equivalence of nonautonomous difference equations

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With the use of Green's type map, sufficient conditions under which the the nonautonomous semilinear difference equation in the Banach space $\mathbf{X} \times \mathbf{Y}$ is simpler than the given one in terms of decoupling and linearization are obtained. The second system splits into two parts. The first of them does not contain the variable $y \in \mathbf{Y}$, while the second one does not contain the variable $x \in \mathbf{X}$ and is linear. This result allows one to replace the given system by a much simpler one. Relevant results concerning partial decoupling and simplifying of the noninvertible nonautonomous difference equations are given also.

An algebraic geometric approach to integrable birational maps and difference equations

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We consider birational maps of the plane, autonomous birational second order difference equations being a special case. Moreover, we assume there is a rational integral (invariant) of the motion so that the dynamics occurs on the integral's level curves. Famous examples include the McMillan map or the Lyness second order difference equation (well known, respectively, in the integrable dynamics community and the difference equation community).

We take an algebraic geometric approach which elucidates the dynamics of such systems. We show that the dynamics of a birational map on an elliptic curve over a field is, typically, conjugate to addition by a point (under the associated group law). When the field is taken to be the function field of rational complex functions of one variable, this amounts to an algebraic geometric version of the Arnol'd-Liouville integrability theorem for integrable planar maps or second order difference equations. When the result is applied to finite fields, it helps underpin the theoretical basis for a quick and very accurate test of whether a rational map or difference equation has an integral in the first place.

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On Asymptotic Behavior of Solutions to Stochastic Difference Equation of the First Order

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We consider stochastic nonlinear difference equation

$$x_{n+1} = x_n \Big(1 + a_n f(x_n) + g(x_n)\xi_{n+1} \Big) + S_n, \quad n = 0, 1, \dots, \tag{1}$$

with nonrandom initial value $x_0 > 0$, independent random noises ξ_n and bounded nonlinear functions f and g. Applying the martingale inequality approach we obtain sufficient criteria for a.s. asymptotic stability of the solutions of (1). We study two cases: when the drift part, $a_n f(x_n)$, is non-positive and ensures stability, and when the diffusion part, $g(x_n)\xi_{n+1}$, is big enough to stabilize the equation (in the second case the drift part $a_n f(x_n)$ can even be positive). We investigate the balance between summability of the free coefficient S_n and the strength of the noises ξ_n needed to guarantee asymptotic stability of the solution (cf. [1]–[3]). We also obtain results on the rate decay of the solutions and asymptotic behavior of moments $\mathbf{E}x_n$.

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Local Bifurcations of Periodic Points of Algebraic Maps

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Consider the equation

$$w + z + \sum_{i+j=2}^{n} a_{ij} z^i w^j = 0. \tag{1}$$

In a neighborhood of the origin it implicitly defines an analytic map $z \to w$ of the form

$$w = f(z) = -z - \sum_{n=1}^{\infty} b_n(a_{ij}) z^{n+1}, \quad z \in \mathbf{R}. \tag{2}$$

One possible way to investigate the behavior of trajectories of the map (1) near the origin is a transformation to the normal form

$$z \mapsto -z(1 + d_1 z^2 + d_2 z^4 + \dots).$$

If the first coefficient, which differs from zero, is d_k and if $d_k > 0$ then

$$f^2(z) = z + 2d_k z^{2k+1} + o(z^{2k+1}),$$

which implies the instability of $z = 0$, otherwise if $d_k < 0$ then the singular point $z = 0$ is asymptotically stable.

Another possible way is based on making use of Lyapunov functions, that is the functions of the form

$$\Phi(z) = z^2(1 + \sum_{k=1}^{\infty} b_k z^k) \tag{3}$$

with the property

$$\Phi(f(z)) - \Phi(z) = g_2 z^4 + g_4 z^6 + \dots + g_{2m} z^{2m+2} + \dots. \tag{4}$$

The coefficients g_{2k} are called *focus quantities.*

Variations of the coefficients a_{ij} of the equation (1) change stability of the equilibrium point $z = 0$ of the map (2) yielding bifurcations of periodic points in a neighborhood of the origin. Therefore there arises the problem to estimate the maximum number of periodic points of map (2) bifurcating from $z = 0$ with variations of coefficients of equation (1). This is a discrete analog of the so-called local 16th Hilbert problem.

We present a general method to investigate the problem and apply it to the case of equation (1) with $n = 3$, namely, to the map defined by the equation

$$\Psi(z, w) = z + w + Az^2 + Bzw + Cw^2 + Dz^3 + Ez^2w + Fzw^2 + Gw^3 = 0. \qquad (5)$$

We show that it is sufficient to compute at most three first focus quantities in order to resolve the problem of stability for the map defined by (5), namely, if $g_2(A^*, \dots, G^*) = g_4(A^*, \dots, G^*) = g_6(A^*, \dots, G^*) = 0$ then the $z = 0$ is stable, but not asymptotically, that is, $f^2(z) \equiv z$, otherwise, $z = 0$ is asymptotically stable if the first different from zero number among $g_2(A^*, \dots, G^*), g_4(A^*, \dots, G^*)$, $g_6(A^*, \dots, G^*)$ is negative, and unstable if the first different from zero term in the sequence $g_2(A^*, \dots, G^*)$, $g_4(A^*, \dots, G^*)$, $g_6(A^*, \dots, G^*)$ is positive.

We also prove that if at least one of the numbers $g_2(A^*, \dots, G^*), g_4(A^*, \dots, G^*)$, $g_6(A^*, \dots, G^*)$ is different from zero, then all maps from a neighborhood of the point $(A^*, \dots, G^*)$ have at most two periodic points in a small neighborhood of $z = 0$.

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Orthogonal Polynomials and the Bezout Identity

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We present in this talk a new way to compute the BEZOUT polynomials $A(x), B(x)$ solving the Bezout's problem $A(x)P_n(x) + B(x)Q_m(x) = 1$, when $P_n(x)$ and $Q_m(x)$ relatively prime, belong to two Orthogonal Polynomials families. We extend first results given by P. Humbert for $Q_m = D\{P_n(x)\}$ ($P_n(x)$ classical) with $D = d/dx$, and generalize when D is replaced by respectively, the difference or the q-difference operator. All these cases generate new families of Orthogonal polynomials linked to the first associated of $P_n(x)$. We examine also some cases where $A(x)$ and $B(x)$ satisfy also a three term recurrence relation like $P_n(x)$ and $Q_m(x)$, and situations allowing to build a differential equation for $A(x)$ and $B(x)$ when both $P_n(x)$ and $Q_m(x)$ are only solution of differential equations.

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Difference Schrödinger Operators for Harmonic Oscillators on a Unitary Lattice

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The formalism of raising and lowering operators is developed for the difference operator analogue of a quantum harmonic oscillator which acts on functions on a discrete support. The grid under consideration is a mixed version of an equidistant lattice and a basic linear grid. Several properties of the grid are described. The grids under consideration are referred to by the name unitary linear lattices. The ladder difference operators are derived and compared with the continuum situation. The arising spectral problems for these operators are dealt by using the theory of bilateral Jacobi operators in weighted $l^2(\mathbb{Z})$ spaces. Eventual applications to mathematical physics and numerical Schrödinger theory are briefly discussed.

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Riccati inequality and other results for discrete symplectic systems

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We present a characterization of the positivity of a quadratic functional for discrete symplectic systems with separable endpoint constraints in terms of the existence of a solution of the Riccati inequality, and a characterization of the nonnegativity of the functional with general endpoint constraints in terms of conjoined bases of the discrete symplectic system and solutions of an implicit Riccati equation.

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Fisher information of hyperspherical harmonics

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The hyperspherical harmonics have been shown to determine the angular part of the wavefunctions which describe the quantum-mechanical states of the physical systems with a central potential in a D-dimensional space. The spatial spreading of these systems can be measured by means of the tralationally-invariant Fisher information, which is an information-theoretic quantity of local character. Here we prove that this quantity is controlled by means of the heretoforth called Fisher information of the involved hyperspherical harmonics. Moreover, we explicitly calculated this mathematical quantity in a compact and closed form.

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Chaotic discrete learning systems

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We apply concepts and tools from nonlinear dynamics and chaos to the modeling and study of higher brain functions.
Better understanding of brain functions can have implications in the study of the dynamics of the learning.
We model coupled neurons in basic functional units by parametrically coupled chaotic maps, governed by difference equations. Our results show that these systems can exhibit self-organization and learning.

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On a difference equation with 3-periodic coefficient

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We study the periodicity, the boundedness and the asymptotic behavior of the positive solutions of the non-autonomous difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, ...$$

where p_n, $n = 0, 1, ...$ is a positive sequence of period 3.

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Higher genus affine Lie algebras of Krichever-Novikov type

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Classical affine Lie algebras appear e.g. as symmetries of infinite dimensional integrable systems and are related to certain differential equations. They are central extensions of current algebras associated to finite-dimensional Lie algebras $\mathfrak{g}$. In geometric terms these current algebras might be described as Lie algebra valued meromorphic functions of the Riemann sphere with two possible poles. They carry a natural grading. In this talk the generalization to higher genus Riemann surfaces and more poles is reviewed. In case that the Lie algebra $\mathfrak{g}$ is reductive (e.g. $\mathfrak{g}$ is simple, semi-simple, abelian, ...) a complete classification of (almost-) graded central extensions is given. In particular, for $\mathfrak{g}$ simple there exists a unique (almost-)graded extension class. The considered algebras play a role in Conformal Field theory.

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Asymptotic Behavior of Solutions of a Class of Even Order Nonlinear Neutral Difference Equations with Quasidifferences

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We consider a class of higher order nonlinear neutral difference equations with quasidifferences

$$\Delta(a_n^{(k-1)}\Delta(a_n^{(k-2)}\Delta(...a_n^{(1)}\Delta(y_n + p_n y_{n-\tau})))) + f(n, y_{n-\sigma}) = 0,$$

where sequence (p_n) is any real sequence. The classification of nonoscillatory solutions of the above equation is obtained. For a class of even order nonlinear neutral difference equations, we establish conditions under which the eventually positive solutions of this equation can be classified into three nonempty distinct categories. Sufficient conditions under which considered equation has solution which converges to zero, which tends to nonzero constants and which diverges to infinity are given.

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On the Dynamics of $x_{n+1} = \dfrac{\beta x_n + \gamma x_{n-1}}{Bx_n + Dx_{n-1}}$

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We investigate the difference equation in the title with positive parameters and positive initial conditions. We obtain sufficient conditions for the equation to have unbounded solutions and sufficient conditions for the existence of invariant and attracting intervals. We also present some conjectures concerning the global character of the solution.

Critical groups for iterated maps

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The critical group G of a connected graph is a finite abelian group, which is closely related to the discrete laplacian. The order of G is the number of spanning trees in the graph.
Using the Markov partition associated to the itinerary of the critical points of one-dimensional maps we define and study critical groups for m-modal discrete dynamical systems.

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Computational Problems of the Inverse Task of Discrete Chaotic Dynamics

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Inverse task of the discrete chaotic dynamics (DCD) have been formulated as a search for the parameters sets of DCD difference equations which will result to the desired images and oscillations. The solution of this task is extremely time consuming due to the necessity to solve numerically the system of non-linear algebraic equations on each step of the searching procedure as well as to the general problem of finding the regions of high dimensional parameter space corresponded to the desired solutions.

In this work we justify the use of parallel programming for solution of the DCD inverse task. Since this task allows decomposition of the entire parameter space into the smaller regions, the principles of parallel programming are applied. For that purposes we used the Linux cluster of 16 interconnected nodes. It shown that the total computational time can be reduced as much as the number of slave processing units.

It's known that the systems of difference equations usually contain numerous of chaotic solutions which makes results of the computations to be strongly dependent on computer accuracy, types of data representation, program language and operational system. The impact of these characteristics on DCD inverse task is analyzed and the results will be presented.

Vortex Merger — Analysis of Transient Behaviour of Discrete Data

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Nowadays huge amounts of observational and numerical data are available to reconstruct and predict the behaviour of an underlying dynamical system, e.g. velocity fields in oceanography. However, available data is discrete in time and space and available only over a finite time interval. As a consequence the classical asymptotic methods of the theory of dynamical systems do not apply.

We study such an example namely the merging of two vortices. The vortices merge in time and the data describing this process is given only numerically as the solution of a PDE, discrete in time and space. The solution of the PDE gives rise to a time-dependent (nonautonomous) difference equation which is known only on a finite-time interval. This nonautonomous difference equation undergoes a bifurcation as two vortices come close together and merge to one big vortex. However, classical methods do not apply, since the bifurcation does not depend on a parameter but on time. The description of this merging process as a bifurcation in time leads to a new aspect of nonautonomous bifurcation theory, an actual topic of research.

Sufficient and Necessary Conditions for the Square Integrability of Generalized $\mathbb{T}$-Gaussian Bells

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We shall consider a certain generalization $\mathbf{E}(x)$ of the Gaussian bell $e^{-x^2/2}$ on symmetric time scales $\mathbb{T}$, as has been treated among other topics in the monograph [2]. In particular, we are interested in the following question: What conditions must the time scale $\mathbb{T}$ satisfy in order to guarantee $\mathbf{E}(x)$ is square integrable? We will provide sufficient and necessary conditions on the growth of the time scale's graininess. Indeed not every symmetric time scale possesses a square integrable Gaussian bell. However, the counterexamples have a huge growth of graininess — meaning that at least for applications we seem to be on the right side. The Gaussian bells topic has been a joint work with L. Erbe and A. Peterson, see [1].

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Periodicity of a Fuzzy Max-Difference Equation

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In this paper we study the periodic nature of the solutions of the following fuzzy max-difference equations:

$$x_{n+1} = \max\left\{\frac{A}{x_{n-k}}, \frac{B}{x_{n-m}}\right\},$$

where k, m are positive integers, A, B are positive real numbers and the initial values x_i, $i = -\pi, -\pi+1, ..., 0$, where $\pi = \max\{k, m\}$, are positive fuzzy numbers.

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On the spectral problem for Askey-Wilson second order difference operators

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The Askey-Wilson second order q-difference operator is an important quantum analogue of the Gauss' hypergeometric differential operator. Its polynomial eigenfunctions are the celebrated Askey-Wilson polynomials.
In the first part of the talk I discuss the (non-polynomial) spectral problem for the Askey-Wilson second order q-difference operator when the base $q = \exp(2\pi i\tau)$ has modulus less than one. I will focus on the constructions of eigenfunctions as Barnes or Euler type integrals.
In the second part of the talk I discuss the common spectral problem for two commuting Askey-Wilson second order difference operators. One of the operators is defined with respect to the base $q = \exp(2\pi i\tau)$, the other operator is defined with respect to the modular inverted base $\tilde{q} = \exp(-2\pi i/\tau)$. Quite remarkably, common eigenfunctions exist which admit analytic continuation to $\tau \in \mathbb{R}_{>0}$, in which case $|q| = |\tilde{q}| = 1$. Two such eigenfunctions will be considered, one due to Ruijsenaars [2] defined as a Barnes type integral, the other defined as an Euler type integral. The interrelations between different eigenfunctions will be discussed.
The talk is based on joint work with Fokko van de Bult.

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Generalized Chebishev multivariable polynomials

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Numerical Solution for Solving Two Point Boundary Value Problems

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Some of the most common problems in applied sciences and engineering are usually formulated as two point boundary value problems. A well known fact is that exact solutions in closed form of such problems do not exist. This fact makes numerical solutions of special interest. More than that and unlike initial value problems which are normally uniquely solvable, boundary value problems can have no solution or several solutions. In fact under very restrictive conditions, one can show the unique solvability of certain boundary value problems.
We will consider a nonlinear two point boundary value problem of the form

$$[p(t)u'(t)]' + q(t)u(t) = g(t); \quad a \leq t \leq b.$$

with he functions $p(t)$, $p'(t)$, $q(t)$ and $g(t)$ are continuous. This can be written in the form

$$p(t)u''(t) + p'(t)u'(t) = h(t, u, u')$$

subject to

$$u(a) = \alpha, \quad u(b)) = \beta.$$

We will present an efficient shooting method for solving two point boundary value problems. The Adomian decomposition method will be utilized to obtain a series solution of the initial value problems involved. Numerical examples and comparison of the work of others will also be done.

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The Hypergeometric Series and Trigonometric Sums

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Let us consider the following Gauss hypergeometric series,

$$F(-n, 2m; 2m+n+1; z) = \Sigma_{k=0}^{n} \frac{(2m)_k(-n)_k z^k}{(2m+n+1)_k k!}$$

where m is a positive constant and n is a positive integer. Let

$$R(-n, 2m; 2m+n+1; t) = \Re\big\{F(-n, 2m; 2m+n+1; e^{it})\big\}$$

and

$$I(-n, 2m; 2m+n+1; t) = \Im\big\{F(-n, 2m; 2m+n+1; e^{it})\big\}.$$

On the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; e^{it})$ we can derive some formulas of trigonometric sums.

Theorem 1: *It holds that*

$$\begin{aligned} & |F(-n, 2m; 2m+n+1; e^{it})|^2 \\ = \; & \frac{(n+1)_n}{(2m+n+1)_n} \Sigma_{s=0}^{n} \frac{(-n)_s(n+1-s)_{n-s}(2m)_s y^s}{(-2n)_s(2m+n+1)_{n-s} s!} \\ = \; & \Sigma_{s=0}^{n} \frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}} \binom{n}{s} \binom{2n-s}{n} (2(n-s))! y^s, \qquad (6) \end{aligned}$$

where $y = 2(1-\cos t)$.

Theorem 2: *It holds that*

$$
\begin{aligned}
& n\Big\{I(-n+1,2m;2m+n+1;t)R(-n,2m;2m+n+1;t) \\
& -R(-n+1,2m;2m+n+1;t)I(-n,2m;2m+n+1;t)\Big\} \\
= \; & \frac{(n+1)_n}{(2m+n+1)_n}\Sigma_{s=1}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s s y^{s-1}}{(-2n)_s(2m+n+1)_{n-s}s!}\sin t \\
= \; & \Sigma_{s=1}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!sy^{s-1} \\
& \cdot \sin t, \qquad (7)
\end{aligned}
$$

where $y = 2(1-\cos t)$.

The 4-terms Recurrence Relation of Certain Multiple Orthogonal Polynomials and a Discretization of the Bessel Equation

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The Mehler-Heine type formula tells a scaling asymptotic property of orthogonal polynomials near the endpoints of the interval of orthogonality. Jacobi-Jacobi type multiple orthogonal(polyorthogonal) polynomials have the same property, which can be proved by a direct calculation[6]. We note that they satisfy a 4-terms recurrence relation. We consider a certain 4-terms recurrence formula as a discretization of the Bessel equation, and prove the Mehler-Heine type formula for the broader class of multiple orthogonal polynomials.

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The compact representation of a Leonard pair

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Let $\mathbb{K}$ denote a field, and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both conditions below.

1. There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
2. There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. There is a natural correspondence between the Leonard pairs and a class of orthogonal polynomials consisting of the q-Racah and related polynomials in the Askey Scheme. Let A, A^* denote a Leonard pair on V. Associated with this pair is a certain parameter q that is used to describe the eigenvalues. For the case $q \neq 1$, $q \neq -1$, we display a basis for V with respect to which the matrix representing $AA^* - qA^*A$ is upper triangular and the matrix representing $A^*A - qAA^*$ is lower triangular. With respect to this basis the matrices representing A, A^* are tridiagonal, with entries of a very simple and attractive form. We call this basis the *compact basis*. This is joint work with Hjalmar Rosengren.

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Continuous Invariants
For A Class Of Difference Equations

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In this talk we investigate the behavior of the second order nonlinear difference equation $x_{n+1} = 1 + \frac{x_{n-1}}{x_n}$. This equation has been studied in the context of periodicity by Grove and Ladas, as well as others. We re-examine the dynamics of the system using geometric techniques applied to the associated planar diffeomorphism. We also answer a question of Kulinovic by showing that the equation possesses a continuous invariant which very effectively distinguishes between different solutions. We show, however, that it possesses no nontrivial invariant that is a rational function of the variables. The results are extended to the system $x_{n+1} = c + \frac{cx_{n-1}}{x_n}$, $c > 0$, and some unanswered questions are discussed.

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The Stieltjes-Wigert orthogonal polynomials in Chern-Simons theory

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We show how the Stieltjes-Wigert orthogonal polynomials appear in Chern-Simons theory. More precisely, we show their usefulness as a tool for nonperturbative computations of the quantum topological invariants associated to Chern-Simons gauge theory. Explicit examples are worked out and we also discuss the connection with random matrix theory and the moment problem.

On global attractivity of scalar difference equations with delay

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We consider the nonlinear difference equation with delay which arises in many contexts in mathematical biology

$$x_{n+1} = qx_n + f_n(x_n, ..., x_{n-k}), \quad n \in \mathbb{Z}, \tag{1}$$

where $q \in (0, 1)$ and $f_n : \mathbb{R}^{k+1} \to \mathbb{R}$ satisfy the following assumptions **(H)**:

(H1) There exists $\vartheta : \mathbb{R} \to \mathbb{R}$ such that $f_n(\phi) \leq \vartheta(z)$ for every $\phi \in \mathbb{R}^{k+1}$ with $\min \phi_i \geq z$.
(H2) There are $b \geq 0,\ a < 0$ such that

$$\frac{a\mathcal{M}(\phi)}{1 + b\mathcal{M}(\phi)} \leq f_n(\phi) \leq \frac{-a\mathcal{M}(-\phi)}{1 - b\mathcal{M}(-\phi)}$$

for all $\phi \in \mathbb{R}^{k+1}$ such that $\min_i \phi_i > -b^{-1} \in [-\infty, 0)$. The monotone continuous functional $\mathcal{M} : \mathbb{R}^{k+1} \to \mathbb{R}_+$ is defined as $\mathcal{M}(\phi) = \max_i\{0, \phi_i\}$.

Eq. (1) has the unique steady state solution $x = 0$. Hypothesis **(H2)** can be easily verify if f_n has negative Schwarzian. If **(H2)** holds with $b = 0$, then **(H1)** is satisfied automatically with $\vartheta(z) = -aM(-z)$.

Theorem 1. Suppose that $q \in (0,1)$ and f_n satisfy **(H)**. If $b \neq 0$ and the condition

$$q^{k+1} > -\frac{a}{1-q} \ln \frac{a^2 - a(1-q)}{a^2 + (1-q)^2},$$

holds, then $\lim_{n\to\infty} x_n = 0$ for every solution $\{x_n\}$ of Eq. (1).

Theorem 2. Assume that $b \neq 0$ and f_n satisfy the hypotheses **(H)**. Then for every positive integer k there exists $q_k \in (0,1)$ such that for $q \in (0, q_k]$ the inequality

$$\frac{a}{1-q} \geq -\frac{1+q^{k+1}}{1-q^{k+1}} \tag{2}$$

assures the global attractivity of eq. (1). Condition (2) is sharp within the class of eqs. (1) determined by **(H)** and the assumptions $b \neq 0$ and $q \in (0, q_k]$.

We propose formulae from which q_k can be found explicitly, let us list some values of q_k: $q_1 = 0.887; q_2 = 0.796; q_3 = 0.788; q_4 = 0.795; q_5 = 0.805; q_6 = 0.815; q_7 = 0.825; q_8 = 0.834; q_9 = 0.842; q_{10} = 0.849; q_{100} = 0.965; q_{1000} = 0.994$. Upper restrictions on q are intrinsic to (1) and cannot be omitted for $k > 1$ since (2) do not guarantee even the local stability for q close to 1. Notice that the above mentioned values of q_k are not optimal (exact). If $b = 0$, then a result similar to Theorem 2 holds with q_k^* being the positive root of $q^{k+1}(q + ... + q^k) = 1$. In this sublinear case the values of q_k^* proved to be the best possible and we conjecture that Theorem 2 also holds with $q_k = q_k^*$.

We apply our results to discrete versions of Nicholsons blowflies equation, the Mackey-Glass equations, and the Wazewska and Lasota equation.

We give conditions for global attractivity of eq. (1) in the case $q = 1$ and show that these conditions are sharp in the class of all non-autonomous equations. As application, we consider Ricker's equation with delayed-density dependence and generalization of the Pielou equation.

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On the symmetries of integrable partial difference equations

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We investigate symmetries and invariant solutions of partial difference equations on elementary quadrilaterals, possessing parametres assigned to the lattice directions. For the relevant constructions we exploit the link between partial difference equations and compatible partial differential equations which is provided once the continuous lattice parametres are treated as independent variables. We use several examples to demonstrate the symmetry based techniques for certain linear and nonlinear integrable partial difference equations.

Universal Characters and an Extension of the KP Hierarchy

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The universal character is a polynomial attached to a pair of partitions and is a generalization of the Schur polynomial. In this talk, we introduce an infinite-dimensional integrable system characterizing the universal characters as its homogeneous polynomial solutions, called the *UC hierarchy*; we regard it as an extension of the KP hierarchy. We study also discrete analogues of the UC hierarchy and relationships to the Painlevé equations through similarity reductions.

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Determinant structure of the discrete integrable systems associated with biorthogonal rational functions

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The discrete integrable systems associated with the biorthogonal rational functions is studied using Hirota's bilinear method. The bilinear method is one of the most effective method to clarify algebraic structures of integrable systems. By the method, integrable systems are transformed to bilinear equations of τ functions. The τ functions reveal the underlying algebraic structures of integrable systems and let us know relations with other integrable systems.

The Bilinear equations of the discrete integrable system which is derived by Spiri - donov-Zhedanov, the R_{II} chain, is discussed and a particular solution for the R_{II} chain on a semi-infinite lattice are constructed in terms of Casorati-type determinants. It is also discussed how the system relates to other Toda systems including the discrete relativistic Toda equation.

This is Joint work with Atsushi Mukaihira.

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Kneser-Type Oscillation Criteria for Second-Order q-difference Equations

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Comparison type oscillation and nonoscillation criteria, that is, necessary and/or sufficient conditions for real-valued solutions of the given differential/difference equation to have (or not to have) an infinite number of zeros, have long been studied in both the continuous and discrete settings. In this study, after making a review of q-difference equations, we obtain certain oscillation criteria for second-order q-difference equations, among them, a q-difference version of the famous Kneser theorem. Other Kneser-type oscillation criteria are also given.

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The modular double of $U_q(\mathfrak{sl}_2)$ and second order Askey-Wilson difference equations

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The Askey-Wilson polynomials [1] and the trigonometric Askey-Wilson functions can be constructed as matrix coefficients of representations of $\mathcal{U}_q = U_q(\mathfrak{sl}_2(\mathbb{C}))$. We will discuss how this construction can be extended to a hyperbolic analog of the Askey-Wilson function by using the modular double [3] of $\mathcal{U}_q$. We obtain a simultaneous solution to two Askey-Wilson second order difference equations in different step sizes w_1 and w_2 with $w_1/w_2 \in \mathbb{R}$, equal to Ruijsenaars [5] solution to this set of equations.
The results presented in this talk are based on the paper [2].

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Pisot and Salem growth numbers in n-order difference equations

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To each β-shift we associate a n-order difference equation in $\mathbb{N}$. Then we characterize the difference equations when β is a Pisot or Salem number. Finally we study the dependency of the growth number properties of these recurrences with the type of β.

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Computing topological invariants in boundary value problems reducible to difference equations

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Among boundary values problems (BVP) for partial differential equations there are certain classes of problems reducible to difference equations. Effective study of such problems has became possible in the last 20-30 years ([1-5]) owing to appreciable advances done also in the theory of difference equations with discrete time, specifically given by one-dimensional maps. Here we apply how this reduction method may be used in simple nonlinear BVP, determined by a bimodal map. We consider two-dimensional linear hyperbolic system with constant coefficients, with nonlinear boundary conditions and usual initial conditions. The objective is to characterize the dependence of the motions of the vortice solutions with the topological invariants of the bimodal map.

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Floquet Theory for Linear Implicit Difference Systems

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The aim of this report is to develop the Floquet theory for linear implicit nonautonomous difference systems. It is proved that index-1 linear implicit difference systems (LIDS) can be transformed into their Kronecker normal forms. Then the Floquet theorem on the representation of fundamental matrices for index-1 LIDS has been established. As an immediate consequence, it is shown that any index-1 periodic LIDS can be transformed into a LIDS in Kronecker normal form with constant coefficients.

A 2+1 non-isospectral integrable lattice hierarchy related to a generalized discrete second Painlevé hierarchy

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In this article, by considering a 2+1 dimensional discrete non-isospectral linear problem, a new 2+1 dimensional integrable lattice hierarchy is constructed. It is shown that a generalization of the discrete second Painlevé hierarchy can be obtained as a reduction. Other reductions include new 1+1 dimensional integrable lattice hierarchies.

Computer Algebra Session

Solving linear ODE's in terms of solutions of linear ODE's of lower order

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Let L be a linear differential operator, corresponding to a linear ordinary differential equation L(y) = 0. If L can be factored then solving the corresponding linear ODE can be reduced to solving linear ODE's of lower order. We will discuss algorithms for factoring differential operators L with rational function coefficients.
If L can not be factored, and if the order of L is 3, then in some cases the solutions of L(y) = 0 can still be expressed in terms of solutions of second order linear ODE's. In such cases, Singer showed that solving L(y) = 0 reduces to finding a point on a conic with rational function coefficients. An algorithm for this step was recently implemented. In the talk we will show how to use this algorithm to perform the reduction from order 3 to order 2 explicitly.

Contiguous Relations and Creative Telescoping

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Contiguous relations are a fundamental concept within the theory of hypergeometric series and orthogonal polynomials. Their study goes back to Gauss who gave a list of 15 'fundamental' relations for the 2F1 case. Applications range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series. Creative telescoping is the underlying principle of Zeilberger's extension of Gosper's algorithm. The resulting algorithm for definite summation of terminating hypergeometric series constitutes a major break-through in symbolic summation. Besides surveying these concepts, the main theme of the talk is to establish a new connection between them. Namely, all classical contiguous relations between terminating or non-terminating hypergeometric series can be computed by creative telescoping. This, for instance, allows computer proofs of summation formulas involving also non-terminating hypergeometric sums, or the automatic discovery of new recurrences for given combinatorial sums. Various illustrative examples derived by corresponding computer algebra programs are given.

Solution Spaces of Hypergeometric Systems and the Structure of Hypergeometric Terms

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A *discrete hypergeometric system* is a system of first-order linear homogeneous partial difference equations with polynomial coefficients, containing a single unknown multivariate function. Algebraically, the solution space of a consistent hypergeometric system has dimension one. Here we consider the unknowns as discrete functions which are defined either everywhere on $\mathbb{Z}^d$, or at the nonsingular points of the system. Contrary to the algebraic framework, in our setting the dimension of the corresponding solution spaces can be, under some conditions, anything between 1 and ∞. This has some interesting consequences for structure theorems, as well as for summation algorithms that deal with hypergeometric terms.

This is joint work with S. A. Abramov.

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Index