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Abbacy of Novacella
Exercises for
Dynamic Equations on Time Scales

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Elvan Akın-Bohner

Martin Bohner

Lynn Erbe

Allan Peterson

Author address:

University of Missouri–Rolla, Rolla, Missouri
E-mail address: akine@umr.edu
URL: http://www.umr.edu/~akine

University of Missouri–Rolla, Rolla, Missouri
E-mail address: bohner@umr.edu
URL: http://www.umr.edu/~bohner

University of Nebraska–Lincoln, Lincoln, Nebraska
E-mail address: lerbe@math.unl.edu
URL: http://www.math.unl.edu/~lerbe

University of Nebraska–Lincoln, Lincoln, Nebraska
E-mail address: apeterso@math.unl.edu
URL: http://www.math.unl.edu/~apeterso

Exercises from Chapter 1 of the First Book

Exercise 1. For each of the following time scales $\mathbb{T}$, find σ, ρ, and μ, and classify each point $t \in \mathbb{T}$ as left-dense, left-scattered, right-dense, or right-scattered:

(i) $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$;
(ii) $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$;
(iii) $\mathbb{T} = \left\{\frac{n}{2} : n \in \mathbb{N}_0\right\}$;
(iv) $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$;
(v) $\mathbb{T} = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$.

Exercise 2. Give examples of time scales $\mathbb{T}$ and points $t \in \mathbb{T}$ such that the following equations are not true. Also determine the conditions on t under which those equations are true:

(i) $\sigma(\rho(t)) = t$;
(ii) $\rho(\sigma(t)) = t$.

Exercise 3. Is $\sigma : \mathbb{T} \to \mathbb{T}$ one-to-one? Is it onto? If it is not onto, determine the range $\sigma(\mathbb{T})$ of σ. How about $\rho : \mathbb{T} \to \mathbb{T}$? This exercise was suggested by Roman Hilscher.

Exercise 4. If $\mathbb{T}$ consists of finitely many points, calculate $\sum_{t\in\mathbb{T}} \mu(t)$.

Exercise 5. Prove that the delta derivative is well defined.

Exercise 6. Sometimes it is convenient to have $f^\Delta(t)$ also defined at a point $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$. Prove that an $f : \mathbb{T} \to \mathbb{R}$ has any $\alpha \in \mathbb{R}$ as its derivative at points $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$.

Exercise 7. (i) Define $f : \mathbb{T} \to \mathbb{R}$ by $f(t) = t^2$ for all $t \in \mathbb{T}$. Find f^Δ.
(ii) Define g by $g(t) = \sqrt{t}$ for all $t \in \mathbb{T}$ with $t > 0$. Find g^Δ.

Exercise 8. Using the definition, show that if $t \in \mathbb{T}^\kappa$ ($t \neq \min \mathbb{T}$) satisfies $\rho(t) = t < \sigma(t)$, then the jump operator σ is not delta differentiable at t.

Exercise 9. Prove the following: Let $t \in \mathbb{T}^\kappa$ be right-dense. If

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number, then f is differentiable at t and

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}.$$

Exercise 10. For each of the following functions $f : \mathbb{T} \to \mathbb{R}$, find f^{Δ}. Write your final answer in terms of $t \in \mathbb{T}$:

(i) $f(t) = \sigma(t)$ for $t \in \mathbb{T} := \{\frac{1}{n} : \ n \in \mathbb{N}\} \cup \{0\}$;
(ii) $f(t) = t^2$ for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{2}} := \{\sqrt{n} : \ n \in \mathbb{N}_0\}$;
(iii) $f(t) = t^2$ for $t \in \mathbb{T} := \{\frac{n}{2} : \ n \in \mathbb{N}_0\}$;
(iv) $f(t) = t^3$ for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{3}} := \{\sqrt[3]{n} : \ n \in \mathbb{N}_0\}$.

Exercise 11. Prove that if x, y, and z are delta differentiable at t, then

$$(xyz)^{\Delta} = x^{\Delta}yz + x^{\sigma}y^{\Delta}z + x^{\sigma}y^{\sigma}z^{\Delta}$$

holds at t. Write down the generalization of this formula for n functions.

Exercise 12. We have

$$(f^2)^{\Delta} = (f \cdot f)^{\Delta} = f^{\Delta}f + f^{\sigma}f^{\Delta} = (f + f^{\sigma})f^{\Delta}. \tag{1}$$

Give the generalization of this formula for the derivative of the $(n+1)$st power of f, $n \in \mathbb{N}$, i.e., for $(f^{n+1})^{\Delta}$.

Exercise 13. Find the derivatives in Exercise 10.

Exercise 14. Find the second derivative of each of the functions given in Exercise 10.

Exercise 15. Find the second derivative of f on an arbitrary time scale:

(i) $f(t) \equiv 1$;
(ii) $f(t) = t$;
(iii) $f(t) = t^2$.

Exercise 16. For an arbitrary time scale, try to find a function f such that $f^{\Delta\Delta} = 1$.

Exercise 17. Find $\Delta^n(fg)$, i.e., $(fg)^{\Delta^n}$ if $\mathbb{T} = \mathbb{Z}$.

Exercise 18. Show that in general, even if both $f^{\Delta^{\sigma}}$ and $f^{\sigma^{\Delta}}$ exist,

$$f^{\Delta^{\sigma}} = f^{\sigma^{\Delta}} \tag{2}$$

does not hold. Is (2) true for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$? Give a sufficient condition that guarantees that (2) holds.

Exercise 19. Suppose μ is differentiable.

(i) If $f^{\Delta\sigma}$ and $f^{\sigma\Delta}$ both exist, give a formula that actually relates these two functions.
(ii) Give a similar formula that relates the functions (if they exist) $f^{\sigma\sigma\Delta}$, $f^{\sigma\Delta\sigma}$, and $f^{\Delta\sigma\sigma}$.
(iii) Give the corresponding formula that relates the functions (if they exist) $f^{\sigma^n\Delta}$ and $f^{\Delta\sigma^n}$, $n \in \mathbb{N}$.

Exercise 20. Let $q > 1$. For the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, evaluate

(i) σ^Δ;
(ii) μ^Δ.

Exercise 21. Find f^{Δ^3} for the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$. Find f^{Δ^4} and finally find a formula for f^{Δ^n} for any natural number n.

Exercise 22. Show that for any constant c

$$u(t) = ca^t \frac{\Gamma(t-t_1)\Gamma(t-t_2)\cdots\Gamma(t-t_n)}{\Gamma(t-s_1)\Gamma(t-s_2)\cdots\Gamma(t-s_m)}$$

is a solution of the recurrence relation

$$u(t+1) = a\frac{(t-t_1)(t-t_2)\cdots(t-t_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}u(t),$$

where $a, t_1, \ldots, t_n, s_1, \ldots, s_m$ are real constants and $n, m \in \mathbb{N}$. Use this to solve the difference equations

(i) $\Delta u = \frac{4t+6}{t^2+5t+6}u$, where $t \in \mathbb{N}$;
(ii) $\Delta u = \frac{2t+5}{2t+1}u$, where $t \in \mathbb{N}_0$.

Exercise 23. Assume $\alpha, k \in \mathbb{C}$ and $^\Delta$ is differentiation with respect to t on the time scale $\mathbb{T} = \mathbb{Z}$. Show that

(i) $\left[(t+\alpha)^{(k)}\right]^\Delta = k(t+\alpha)^{(k-1)}$;
(ii) $(\alpha^t)^\Delta = (\alpha-1)\alpha^t$;
(iii) $\binom{t}{\alpha}^\Delta = \binom{t}{\alpha-1}$.

Exercise 24. Prove the following well-known formula concerning binomial coefficients:

$$\binom{\alpha}{\beta} + \binom{\alpha}{\beta+1} = \binom{\alpha+1}{\beta+1}.$$

Exercise 25. Introduce some of the above concepts for the time scale $\mathbb{T} = h\mathbb{Z}$ and prove some of the above results for this time scale.

Exercise 26. Are the operators σ, ρ, and μ

(i) continuous;
(ii) rd-continuous;

(iii) regulated?

Exercise 27. For each of the following determine if f is regulated on $\mathbb{T}$, if f is rd-continuous on $\mathbb{T}$, and if f is pre-differentiable. If f is pre-differentiable, find its region of differentiability D.

(i) The function f is defined on a time scale $\mathbb{T}$ and every point $t \in \mathbb{T}$ is isolated.

(ii) Assume $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{1}{t} & \text{if} \quad t \in \mathbb{R} \setminus \{0\}. \end{cases}$$

(iii) Assume $\mathbb{T} = \mathbb{N}_0 \cup \{1 - 1/n : \ n \in \mathbb{N}\}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \mathbb{N} \\ t & \text{otherwise.} \end{cases}$$

(iv) Assume $\mathbb{T} = \mathbb{R}$ and $f(t) = |t|$, $t \in \mathbb{R}$.

(v) Assume $\mathbb{T} = \mathbb{P}_{1,1}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t = 2k+1, \ k \in \mathbb{N}_0 \\ t - 2k & \text{if} \quad t \in [2k, 2k+1), \ k \in \mathbb{N}_0. \end{cases}$$

(vi) Assume $\mathbb{T} = \mathbb{P}_{1,1}$ and

$$f(t) = k, \quad t \in [2k, 2k+1], \quad k \in \mathbb{N}_0.$$

Exercise 28. Show that if $\mathbb{T} = \mathbb{Z}$, $k \neq -1$, and $\alpha \in \mathbb{R}$, then

(i) $\int (t+\alpha)^{(k)} \Delta t = \frac{(t+\alpha)^{(k+1)}}{k+1} + C$;

(ii) $\int \binom{t}{\alpha} \Delta t = \binom{t}{\alpha+1} + C$.

Exercise 29. Let $a \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale and evaluate $\int_a^t 1 \Delta s$. Also evaluate $\int_0^t s \Delta s$ for $t \in \mathbb{T}$, for $\mathbb{T} = \mathbb{R}$, for $\mathbb{T} = \mathbb{Z}$, for $\mathbb{T} = h\mathbb{Z}$, and for $\mathbb{T} = [0,1] \cup [2,3]$.

Exercise 30. Evaluate the integral

$$\int_1^\infty \frac{1}{t^2} \Delta t$$

if $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$.

Exercise 31. Assume $a \in \mathbb{T}$, $a > 0$ and $\sup \mathbb{T} = \infty$. Evaluate

$$\int_a^\infty \frac{1}{t\sigma(t)} \Delta t.$$

Exercise 32. Assume that $\mathbb{T} = \mathbb{Z}$ and $f(t) = g(t) = t^2$. Show that for all $t \neq 0$,

$$(f \circ g)^{\Delta}(t) \neq f^{\Delta}(g(t))\, g^{\Delta}(t).$$

Exercise 33. Assume that $\mathbb{T} = \mathbb{Z}$ and $f(t) = g(t) = t^2$. Find directly the c so that $(f \circ g)^{\Delta}(2) = f'(g(c))\, g^{\Delta}(2)$ and be sure to note that c is in the interval guaranteed.

Exercise 34. Let $\mathbb{T} = \mathbb{N}_0$, $\nu(t) = t^2$, $\tilde{\mathbb{T}} = \nu(\mathbb{T})$, and $w(t) = 2t^2 + 3$. Show directly that

$$(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

Exercise 35. Find a time scale $\mathbb{T}$ and a strictly increasing function $\nu : \mathbb{T} \to \mathbb{R}$ such that $\nu(\mathbb{T})$ is not a time scale.

Exercise 36. Evaluate the integral

$$\int_0^t 2\tau(2\tau - 1)\Delta\tau$$

for $t \in \mathbb{T} := \{\frac{n}{2} : \ n \in \mathbb{N}_0\}$.

Exercise 37. Evaluate the integral

$$\int_0^t \left[(\tau^3 + 1)^{\frac{2}{3}} + \tau(\tau^3 + 1)^{\frac{1}{3}} + \tau^2\right] 2^{\tau^3} \Delta\tau$$

for $t \in \mathbb{T} := \{\sqrt[3]{n} : \ n \in \mathbb{N}_0\}$.

Exercise 38. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for some $q > 1$. For $n \in \mathbb{N}$, evaluate

$$\int_0^t s^n \Delta s.$$

Exercise 39. Find the functions $h_k(\cdot, 0)$, for $k = 0, 1, 2, 3$ if $\mathbb{T} := [0, 1] \cup [3, 4]$.

Exercises from Chapter 2 of the First Book

Exercise 40. Prove that

$$\lim_{h\to 0}[\operatorname{Re}_h z + \mathring{\imath}\operatorname{Im}_h z] = \operatorname{Re}(z) + i\operatorname{Im}(z).$$

Also try to visualize this by looking at what happens to the Hilger complex plane as $h \to 0$.

Exercise 41. Show that the addition $\oplus$ on $\mathbb{C}_h$ satisfies the associative law.

Exercise 42. Assuming that $z \in \mathbb{C}_h$ and $w \in \mathbb{C}$, simplify the expression $z \oplus \frac{w}{1+hz}$.

Exercise 43. If $n \in \mathbb{N}$ and $z \in \mathbb{C}_h$, then we define the "circle dot" multiplication $\odot$ by

$$n \odot z := z \oplus z \oplus z \oplus \cdots \oplus z,$$

where we have n terms on the right-hand side of this last equation. Show that

$$n \odot z = \frac{(zh+1)^n - 1}{h}.$$

Exercise 44. Show directly that if $z \in \mathbb{C}_h$, then $\ominus(\ominus z) = z$.

Exercise 45. If $z, w \in \mathbb{C}_h$ with $h \geq 0$, show directly that

(i) $z \ominus z = 0$;
(ii) $z \ominus w = \frac{z-w}{1+wh}$;
(iii) $z \ominus w = z - w$ if $h = 0$.

Exercise 46. Show that if $z \in \mathbb{C}_h$, then $\overline{z} = \ominus z$ iff $z \in \mathbb{I}_h$.

Exercise 47. Show that $\ominus(\mathring{\imath}\omega) = \overline{\mathring{\imath}\omega}$.

Exercise 48. Show that the cylinder transformation ξ_h when $h > 0$ maps open rays emanating from the point $-\frac{1}{h}$ in $\mathbb{C}$ onto horizontal lines on the cylinder $\mathbb{Z}_h$. Also show that circles with center at $-\frac{1}{h}$ are mapped onto vertical lines on the strip $\mathbb{Z}_h$ (circles on the cylinder $\mathbb{Z}_h$).

Exercise 49. Show that the inverse transformation of the cylinder transformation ξ_h when $h > 0$ is given by

$$\xi_h^{-1}(z) = \frac{1}{h}\left(e^{zh} - 1\right)$$

for $z \in \mathbb{Z}_h$. Also find $\xi_0^{-1}(z)$.

Exercise 50. Show that $\mathcal{R}$ is an Abelian group under the "circle plus" addition $\oplus$ defined by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa,$$

$p, q \in \mathcal{R}$. This group is called the *regressive group*.

Exercise 51. Show that if $p, q \in \mathcal{R}$, then the function $p \oplus q$ and the function $\ominus p$ defined by

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

are also elements of $\mathcal{R}$.

Exercise 52. We define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa. \tag{3}$$

Suppose $p, q \in \mathcal{R}$. Show directly from the definition that

(i) $p \ominus p = 0$;
(ii) $\ominus(\ominus p) = p$;
(iii) $p \ominus q \in \mathcal{R}$;
(iv) $p \ominus q = \frac{p-q}{1+\mu q}$;
(v) $\ominus(p \ominus q) = q \ominus p$;
(vi) $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Exercise 53. If $p \in \mathcal{R}$ and $q : \mathbb{T}^\kappa \to \mathbb{R}$, simplify $p \oplus \frac{q}{1+\mu p}$.

Exercise 54. Assume $\frac{2}{t}$, $\frac{5}{t}$ are regressive on $\mathbb{T} \cap (0, \infty)$ and let $t_0 \in \mathbb{T} \cap (0, \infty)$. Evaluate the integral

$$\int_{t_0}^{t} \frac{e_{\frac{5}{s}}(s, t_0)}{s e^{\sigma}_{\frac{2}{s}}(s, t_0)} \Delta s.$$

Exercise 55. Show that if $\alpha \in \mathbb{R}$ is constant and the exponentials below exist, then

$$\frac{e_{\frac{\alpha^2}{t} - \frac{(\alpha-1)^2}{\sigma(t)}}(t, t_0)}{e^2_{\frac{\alpha}{t}}(t, t_0)} = \frac{t_0}{t}$$

by first showing that

$$\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)} = e_{\frac{\alpha-1}{\sigma(t)}}(t,t_0)$$

and then showing that

$$\frac{e_{\frac{\alpha-1}{\sigma(t)}}(t,t_0)}{e_{\frac{\alpha}{t}}(t,t_0)} = e_{-\frac{1}{\sigma(t)}}(t,t_0) = \frac{t_0}{t}$$

for $t, t_0 \in \mathbb{T} \cap (0, \infty)$.

Exercise 56. Let $s, t \in \mathbb{T}$. Assuming we can interchange the order of differentiation and integration, show that

$$\frac{d}{dz}\left[e_z(t,s)\right] = \left(\int_s^t \frac{1}{1+\mu(\tau)z}\Delta\tau\right) e_z(t,s)$$

for those $z \in \mathbb{C}$ satisfying $1 + \mu(\tau)z \neq 0$ for τ between s and t.

Exercise 57. Show that if the constant $a < -1$ and $\mathbb{T} = \mathbb{Z}$, then the exponential function $e_a(\cdot, 0)$ changes sign at every point in $\mathbb{Z}$. In this case we say the exponential function $e_a(\cdot, 0)$ is *strongly oscillatory* on $\mathbb{Z}$.

Exercise 58. Show that if $\mathbb{T}$ has constant graininess $h \geq 0$ and if α is constant with $1 + \alpha h \neq 0$, then $e_\alpha(t+s, 0) = e_\alpha(t, 0)e_\alpha(s, 0)$ for all $s, t \in \mathbb{T}$.

Exercise 59. Find $e_\alpha(t, t_0)$ for $t, t_0 \in q^{\mathbb{N}_0}$ with constant $\alpha \in \mathcal{R}$.

Exercise 60. For $\alpha \neq -1$, find $e_\alpha(\cdot, 0)$, where $\mathbb{T} = \mathbb{P}_{1,1}$.

Exercise 61. Find the exponential function $e_{\frac{\lambda}{t}}(\cdot, 1)$, where λ is a real constant for $\mathbb{T} = [1, \infty)$ and $\mathbb{T} = \mathbb{N}$. If $\mathbb{T} = \mathbb{N}$ and $\lambda \in \mathbb{N}$, simplify your answer.

Exercise 62. Use the variation of constants formula to solve the following initial value problems on the indicated time scales:

(i) $y^\Delta = 2y + t$, $y(0) = 0$, where $\mathbb{T} = \mathbb{R}$;
(ii) $y^\Delta = 2y + 3^t$, $y(0) = 0$, where $\mathbb{T} = \mathbb{Z}$;
(iii) $y^\Delta = p(t)y + e_p(t, t_0)$, $y(t_0) = 0$, where $\mathbb{T}$ is an arbitrary time scale and $p \in \mathcal{R}$.

Exercises from Chapter 3 of the First Book

Exercise 63. If $p \in \mathrm{C}_{\mathrm{rd}}$ and $-\mu p^2 \in \mathcal{R}$, simplify

(i) $\cosh_p(t,s) + \sinh_p(t,s)$;
(ii) $\cosh_p(t,s) - \sinh_p(t,s)$;
(iii) $\frac{\cosh_p(s,t_0)\cosh_p(t,t_0) - \sinh_p(s,t_0)\sinh_p(t,t_0)}{\cosh_p^2(s,t_0) - \sinh_p^2(s,t_0)}$;
(iv) $\frac{\cosh_p(s,t_0)\sinh_p(t,t_0) - \sinh_p(s,t_0)\cosh_p(t,t_0)}{\cosh_p^2(s,t_0) - \sinh_p^2(s,t_0)}$.

Exercise 64. Show that if $\gamma > 0$ with $-\gamma^2\mu \in \mathcal{R}$, then $\cosh_\gamma(\cdot, t_0)$ and $\sinh_\gamma(\cdot, t_0)$ are solutions of

$$y^{\Delta\Delta} - \gamma^2 y = 0. \tag{4}$$

Exercise 65. Show that if $\gamma > 0$ and $-\gamma^2\mu \in \mathcal{R}$, then the solution of the IVP

$$y^{\Delta\Delta} - \gamma^2 y = 0, \quad y(t_0) = y_0, \quad y^\Delta(t_0) = y_0^\Delta$$

is given by

$$y(t) = y_0 \cosh_\gamma(t,t_0) + \frac{y_0^\Delta}{\gamma}\sinh_\gamma(t,t_0).$$

Exercise 66. Let α be constant such that $|\alpha h| \neq 1$. For $\mathbb{T} = h\mathbb{Z}$ find $\cosh_\alpha(\cdot, 0)$ and $\sinh_\alpha(\cdot, 0)$. Is it true that

$$[\cosh_\alpha(t,0)]^2 - [\sinh_\alpha(t,0)]^2 = 1$$

for $t \in h\mathbb{Z}$?

Exercise 67. State and prove the four formulas for the trigonometric functions analogous to the formulas in Exercise 63 for the hyperbolic functions. In particular, also show *Euler's formula*

$$e_{ip}(t,t_0) = \cos_p(t,t_0) + i\sin_p(t,t_0). \tag{5}$$

Exercise 68. Find $\sin_\alpha(\cdot, 0)$ and $\cos_\alpha(\cdot, 0)$ for constant $\alpha \in \mathcal{R}$ for the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively.

Exercise 69. Find $e_1(t,0)$, $\sin_1(t,0)$, and $\cos_1(t,0)$ if $\mathbb{T} = \mathbb{Z}^2 = \{k^2 : k \in \mathbb{Z}\}$.

Exercise 70. Show that the identity

$$[\sin_p(t,t_0)]^2 + [\cos_p(t,t_0)]^2 = 1$$

need not hold.

Exercise 71. Factor the following dynamic equations and then use the method of factoring to solve the equation:

(i) $y^{\Delta\Delta} - (5+t)y^{\Delta} + 5ty = 0$;
(ii) $y^{\Delta\Delta} - (6+t)y^{\Delta} + (5+5t)y = 0$;
(iii) $y^{\Delta\Delta} - (t+\sigma(t))y^{\Delta} + (t^2-1)y = 0$;
(iv) $y^{\Delta\Delta} - 3(t+1)y^{\Delta} + (9t-3)y = 0$ on $\mathbb{T} = \mathbb{R}$;
(v) $y^{\Delta\Delta} - (t+6)y^{\Delta} + (5t-1)y = 0$ on $\mathbb{T} = \mathbb{N}_0$.

Exercise 72. Prove that the solution of the initial value problem

$$x^{\Delta\Delta} - q^{②}(t)x^{\sigma} = 0, \quad x(t_0) = x_0, \quad x^{\Delta}(t_0) = x_0^{\Delta}$$

is given by

$$x(t) = e_q(t,t_0)\left[x_0 + (x_0^{\Delta} - qx_0)\int_{t_0}^{t} \frac{e_q^2(t_0,\tau)}{1+q\mu(\tau)}\Delta\tau\right].$$

Exercise 73. Find a general solution of

$$x^{\Delta\Delta} - 4^{②}(t)x^{\sigma} = 0$$

and simplify your answer when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Exercise 74. In this exercise we present several formulas that are satisfied by the functions ch and sh. The reader is asked to prove these results:

(i) $e_{p+q} = \operatorname{ch}_{pq} + \operatorname{sh}_{pq}$;
(ii) $e_{p-q} = \operatorname{ch}_{pq} - \operatorname{sh}_{pq}$;
(iii) $|\operatorname{ch}_{pq}| \geq 1$;
(iv) parity condition for sh: $\operatorname{sh}_{pq}(t,s) = -\operatorname{sh}_{pq}(s,t)$;
(v) parity condition for ch: $\operatorname{ch}_{pq}(t,s) = \operatorname{ch}_{pq}(s,t)$;
(vi) difference formula for sh:

$$\operatorname{sh}_{pq}(t,s) = \operatorname{sh}_{pq}(t,r)\operatorname{ch}_{pq}(r,s) - \operatorname{ch}_{pq}(t,r)\operatorname{sh}_{pq}(s,r);$$

(vii) difference formula for ch:

$$\operatorname{ch}_{pq}(t,s) = \operatorname{ch}_{pq}(t,r)\operatorname{ch}_{pq}(s,r) - \operatorname{sh}_{pq}(t,r)\operatorname{sh}_{pq}(s,r);$$

(viii) sum formula for sh:

$$\operatorname{sh}_{pq}(t,r) = \operatorname{sh}_{pq}(t,s)\operatorname{ch}_{pq}(s,r) + \operatorname{ch}_{pq}(t,s)\operatorname{sh}_{pq}(s,r);$$

(ix) sum formula for ch:

$$\operatorname{ch}_{pq}(t,r) = \operatorname{ch}_{pq}(t,s)\operatorname{ch}_{pq}(s,r) + \operatorname{sh}_{pq}(t,s)\operatorname{sh}_{pq}(s,r).$$

Exercise 75. Simplify the expression

$$e_{\frac{\alpha}{t}}(t,t_0)\int_{t_0}^{t}\frac{1}{\tau+\alpha\mu(\tau)}\Delta\tau$$

for $t\in\mathbb{T}$ and $t_0=1$ when $\mathbb{T}=[1,\infty)$ and $\mathbb{T}=\mathbb{N}$, respectively.

Exercise 76. Solve the following Euler–Cauchy dynamic equations on a general time scale $\mathbb{T}\subset(0,\infty)$. Simplify your answers when $\mathbb{T}=[1,\infty)$ and when $\mathbb{T}=\mathbb{N}$.

(i) $t\sigma(t)y^{\Delta\Delta}-5ty^{\Delta}+8y=0$;
(ii) $t\sigma(t)y^{\Delta\Delta}+ty^{\Delta}-4y=0$;
(iii) $2t\sigma(t)y^{\Delta\Delta}+7ty^{\Delta}-5y=0$;
(iv) $t\sigma(t)y^{\Delta\Delta}-3ty^{\Delta}+4y=0$;
(v) $t\sigma(t)y^{\Delta\Delta}-5ty^{\Delta}+9y=0$.

Exercise 77. Use the method of variation of parameters to solve each of the following dynamic equations:

(i) $y^{\Delta\Delta}-6y^{\Delta}+8y=e_3(t,t_0)$, where $t,t_0\in\mathbb{T}$;
(ii) $y^{\Delta\Delta}-3y^{\Delta}+2y=e_2(t,t_0)$, where $t,t_0\in\mathbb{T}$;
(iii) $t\sigma(t)y^{\Delta\Delta}-4ty^{\Delta}+4y=e_{\frac{6}{t}}(t,t_0)$, where $t,t_0\in\mathbb{T}\cap(0,\infty)$;
(iv) $t\sigma(t)y^{\Delta\Delta}+ty^{\Delta}-y=e_{\frac{4}{t}}(t,t_0)$, where $t,t_0\in\mathbb{T}\cap(0,\infty)$.

Exercise 78. Show that if α and β are constants, then

$$(D-\alpha I)(D-\beta I)=(D-\beta I)(D-\alpha I).$$

Exercise 79. Use the method of annihilators to solve each of the following dynamic equations:

(i) $y^{\Delta\Delta}-6y^{\Delta}+8y=e_3(t,t_0)$, where $t,t_0\in\mathbb{T}$;
(ii) $y^{\Delta\Delta}-3y^{\Delta}+2y=e_2(t,t_0)$, where $t,t_0\in\mathbb{T}$;
(iii) $y^{\Delta\Delta}+y^{\Delta}-2y=2+t$, where $t,t_0\in\mathbb{T}\cap(0,\infty)$;
(iv) $y^{\Delta\Delta}+y=e_3(t,t_0)$, where $t,t_0\in\mathbb{T}$.

Exercise 80. Show that if $\mathbb{T}_0$ is the real interval $[0,\infty)$, then the Laplace transform is the familiar formula for the continuous case. Show that if $\mathbb{T}_0=\mathbb{N}_0$, then

$$(z+1)\mathcal{L}\{x\}(z)=\mathcal{Z}\{x\}(z+1), \tag{6}$$

where $\mathcal{Z}\{x\}$ is the Z-transform of x, which is defined by

$$\mathcal{Z}\{x\}(z)=\sum_{t=0}^{\infty}\frac{x(t)}{z^t}$$

for those complex values of z for which this infinite sum converges.

Exercise 81. Prove the following: Assume f and g are regulated on $\mathbb{T}_0$ and α and β are constants. Then

$$\mathcal{L}\{\alpha x + \beta y\}(z) = \alpha\mathcal{L}\{x\}(z) + \beta\mathcal{L}\{y\}(z)$$

for $z \in \mathcal{D}\{x\} \cap \mathcal{D}\{y\}$.

Exercise 82. Show that under suitable assumptions

$$\mathcal{L}\{x^{\Delta\Delta}\}(z) = z^2\mathcal{L}\{x\}(z) - zx(0) - x^{\Delta}(0). \tag{7}$$

Write down the formula for $\mathcal{L}\{x^{\Delta^n}\}$.

Exercise 83. Show that

$$\mathcal{L}\{\sinh_\alpha(\cdot, 0)\}(z) = \frac{\alpha}{z^2 - \alpha^2}.$$

Exercise 84. Show that

$$\mathcal{L}\{\sin_\alpha(\cdot, 0)\}(z) = \frac{\alpha}{z^2 + \alpha^2}.$$

Exercise 85. Assume that $\mathbb{T}_0$ has constant graininess $\mu(t) \equiv h \geq 0$. Show that under suitable assuptions

$$\mathcal{L}\{x^\sigma\}(z) = (1 + hz)\mathcal{L}\{x\}(z) - h(1 + hz)x(0)$$

holds.

Exercise 86. Use Laplace transforms to solve the following initial value problems on $\mathbb{T}_0$:

(i) $x^{\Delta\Delta} + x^{\Delta} - 2x = 0, \quad x(0) = 1, \quad x^{\Delta}(0) = 1;$
(ii) $x^{\Delta\Delta} - 9x = 0, \quad x(0) = 0, \quad x^{\Delta}(0) = 1;$
(iii) $x^{\Delta\Delta} + 2x^{\Delta} - 3x = 0, \quad x(0) = 5, \quad x^{\Delta}(0) = 1;$
(iv) $x^{\Delta\Delta} + 16x = 0, \quad x(0) = 0, \quad x^{\Delta}(0) = 3;$
(v) $x^{\Delta\Delta} - 6x^{\Delta} + 25x = 0, \quad x(0) = 1, \quad x^{\Delta}(0) = 2;$
(vi) $x^{\Delta\Delta} + 4x^{\Delta} + 13x = 0, \quad x(0) = -1, \quad x^{\Delta}(0) = 1;$
(vii) $x^{\Delta\Delta\Delta} + x^{\Delta} = e_1(t, 0), \quad x(0) = x^{\Delta}(0) = x^{\Delta\Delta}(0) = 0;$
(viii) $x^{\Delta\Delta} + 6x^{\Delta} + 25x = 0, \quad x(0) = 2, \quad x^{\Delta}(0) = 1.$

Exercise 87. Assume f is a regulated function and

$$g(t) := \int_0^{\sigma(t)} \frac{1}{1 + \mu(\tau)z}\Delta\tau.$$

Show that if we can change the order of differentiation and integration for an appropriate integral, then

$$\mathcal{L}\{gf\}(z) = -\frac{d}{dz}\mathcal{L}\{f\}(z) \quad \text{for} \quad z \in \mathcal{D}(f).$$

Show that this gives a well-known formula for Laplace transforms in the case if $\mathbb{T}_0$ is the real interval $[0, \infty)$ and a well-known formula for Z-transforms by taking $\mathbb{T}_0 = \mathbb{N}_0$.

Exercise 88. It can be shown that if $f(t) = t$ for $t \in \mathbb{T}_0$, then $1 * f = f * 1$. Prove this for the special case $\mathbb{T}_0 := \overline{q^{\mathbb{Z}}}$ by evaluating the integrals you obtain for this special case.

Exercise 89. Prove the convolution theorem when f is any of the functions $\sinh_\alpha(\cdot, 0)$, $\cos_\alpha(\cdot, 0)$, $\sin_\alpha(\cdot, 0)$ and g is a regulated function. What assumptions on α are you making in each case?

Exercise 90. Solve each of the following integral equations:

(i) $x(t) = 2e_3(t, 0) - 5\int_0^t e_4(t, \sigma(s))x(s)\Delta s$;
(ii) $x(t) = \cos_2(t, 0) + 3\int_0^t \sin_3(t, \sigma(s))x(s)\Delta s$;
(iii) $x(t) = e_2(t, 0) + 4\int_0^t x(s)\Delta s$;
(iv) $x(t) = t + 4\int_0^t (t - \sigma(s))x(s)\Delta s$;
(v) $x(t) = h_3(t, 0) + 9\int_0^t h_2(t, \sigma(s))x(s)\Delta s$.

Exercise 91. Prove that

$$e_\alpha(\cdot, 0) * \sinh_\beta(\cdot, 0) = \sinh_\beta(\cdot, 0) * e_\alpha(\cdot, 0).$$

Exercise 92. Use Laplace transforms and the convolution theorem to solve the following initial value problems:

(i) $x^{\Delta\Delta} + 9x = \sin_2(t, 0)$, $x(0) = x^\Delta(0) = 0$;
(ii) $x^{\Delta\Delta} + 16x = \sin_4(t, 0)$, $x(0) = x^\Delta(0) = 0$;
(iii) $x^{\Delta\Delta} - 2x^\Delta + x = 9e_4(t, 0)$, $x(0) = 0$, $x^\Delta(0) = 3$;
(iv) $x^{\Delta\Delta} - x^\Delta - 2x = 3e_2(t, 0)$, $x(0) = 0$, $x^\Delta(0) = 1$;
(v) $x^{\Delta\Delta} - 3x^\Delta = 3e_3(t, 0)$, $x(0) = 0$, $x^\Delta(0) = 1$;
(vi) $x^{\Delta\Delta} + 25x = h_2(t, 0)$, $x(0) = x^\Delta(0) = 0$;
(vii) $x^{\Delta\Delta} + x = h_3(t, 0)$, $x(0) = x^\Delta(0) = 0$.

Exercises from Chapter 1 of the Second Book

Exercise 93. Show that if $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, $q > 1$, then

$$(\log t)^\Delta = \frac{\log q}{q-1} \cdot \frac{1}{t}.$$

Exercise 94. Let $p \in \mathcal{R}$. Show that $p \oplus (\ominus p) \equiv 0$ and that $\ominus p$ is the only solution of the equation $p \oplus x \equiv 0$.

Exercise 95. Define the set of *positively regressive* functions $\mathcal{R}^+$ as the set consisting of those $p \in \mathcal{R}$ satisfying

$$1 + \mu(t)p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Show that $(\mathcal{R}^+, \oplus)$ is a subgroup of the regressive group.

Exercise 96. Prove the following: If $p, q \in \mathcal{R}$, then

(i) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;
(viii) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$;
(ix) $[e_p(c,\cdot)]^\Delta = -p[e_p(c,\cdot)]^\sigma$, where $c \in \mathbb{T}$.

Exercise 97. Let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. Let $\alpha \in \mathcal{R}$ be constant. Verify

$$e_\alpha(t,0) = (1 + \alpha h)^{\frac{t}{h}} \quad \text{for all} \quad t \in \mathbb{T}.$$

Exercise 98. Consider the time scale

$$\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : \ n \in \mathbb{N}_0\}$$

and verify

$$e_1(t,0) = 2^{\sqrt{t}}(\sqrt{t})! \quad \text{for} \quad t \in \mathbb{T}.$$

Exercise 99. Define

$$\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}, \quad \text{where} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Let $\alpha \geq 0$ be constant. Verify

$$e_\alpha(H_n, 0) = \binom{n+\alpha}{n}.$$

Exercise 100. We consider the time scale $\mathbb{T} = q^{\mathbb{N}_0}$. Let $p \in \mathcal{R}$. Verify

$$e_p(t, 1) = \prod_{s \in \mathbb{T} \cap (0,t)} (1 + (q-1)sp(s)).$$

Exercise 101. For the time scale

$$\mathbb{T} = \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1]$$

and $\alpha \neq 0$, find the exponential function $e_\alpha(t, 0)$.

Exercise 102. Let $p > 0$ and consider the time scale

$$\mathbb{T} = \mathbb{N}_0^p := \{0, 1, 2^p, 3^p, 4^p, \ldots\}.$$

Find the forward jump operator and the graininess for $\mathbb{T}$. If $p \in \mathbb{N} \setminus \{1\}$, give a formula for e_λ (for λ constant) in terms of products and quotients of Gamma functions. Use Stirling's formula to give asymptotic representations of that exponential function. Try to obtain such asymptotic representations also for the case that $p \in (0, 1/2)$ and also for the case that $p \in (1/2, 2/3)$.

Exercise 103. Verify in detail that $(\mathcal{R}_\alpha, \oplus)$ is an Abelian group.

Exercise 104. Verify in detail that $\mathcal{R}_\alpha^+$ is a subgroup of $(\mathcal{R}_\alpha, \oplus)$, where

$$\mathcal{R}_\alpha^+ = \{p \in \mathcal{R}_\alpha : 1 + p(t)\mu_\alpha(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Exercise 105. Verify that under appropriate existence and uniqueness assumptions

$$\left(\frac{1}{e_p(\cdot, s)}\right)^\Delta = \frac{-p}{e_p^\alpha(\cdot, s)}$$

holds.

Exercises from Chapter 2 of the Second Book

Use the variation of constants formula to solve the following IVPs on the indicated time scales:

(i) $y^\Delta = 2y + t$, $y(0) = 0$, where $\mathbb{T} = \mathbb{R}$;
(ii) $y^\Delta = 2y + 3^t$, $y(0) = 0$, where $\mathbb{T} = \mathbb{Z}$;
(iii) $y^\Delta = p(t)y + e_p(t, t_0)$, $y(t_0) = 0$, where $\mathbb{T}$ is an arbitrary time scale and $p \in \mathcal{R}$.

Exercise 106. Calculate $2 \odot p$ for $p \in \mathcal{R}$. Show also that $2 \odot p = p \oplus p$.

Exercise 107. Let $n \in \mathbb{N}$ and $p \in \mathcal{R}$. Find a formula for $n \odot p$.

Exercise 108. Let $p \in \mathcal{R}^+$. Derive

$$\frac{1}{2} \odot p = \frac{p}{1 + \sqrt{1 + \mu p}}. \tag{8}$$

Show directly

$$2 \odot \left(\frac{1}{2} \odot p \right) = p.$$

Exercise 109. Let $p \in \mathcal{R}^+$. Find formulas for $\frac{1}{4} \odot p$ and $\frac{3}{4} \odot p$. Then use these two results to show the formulas

$$\left(\frac{3}{4} \odot p \right) \oplus \left(\frac{1}{4} \odot p \right) = p \quad \text{and} \quad \left(\frac{3}{4} \odot p \right) \ominus \left(\frac{1}{4} \odot p \right) = \frac{1}{2} \odot p.$$

Exercise 110. The three formulas

$$\frac{(xy)^\Delta}{xy} = \frac{x^\Delta}{x} \oplus \frac{y^\Delta}{y},$$
$$\frac{(x/y)^\Delta}{x/y} = \frac{x^\Delta}{x} \ominus \frac{y^\Delta}{y},$$

and

$$\frac{(x^\alpha)^\Delta}{x^\alpha} = \alpha \odot \frac{x^\Delta}{x}$$

could be used to define a *logarithm* as

$$\log_f(t,t_0)=\int_{t_0}^t \frac{f^\Delta(\tau)}{f(\tau)}\Delta\tau,$$

but then the usual logarithm rules would not hold as we had formulas involving an additional integral, e.g.,

$$\log_{xy}(t,t_0)=\log_x(t,t_0)+\log_y(t,t_0)+\int_{t_0}^t \frac{\mu(\tau)x^\Delta(\tau)y^\Delta(\tau)}{x(\tau)y(\tau)}\Delta\tau.$$

Try to define a logarithm that satisfies somehow "smoother" logarithm rules.

Exercise 111. Solve the initial value problem

$$x'=-x+tx^4,\quad x(0)=1. \tag{9}$$

Exercise 112. Solve the initial value problem

$$x'=-2x+tx^5,\quad x(0)=1.$$

Exercise 113. Solve the initial value problem

$$\Delta x=\frac{5x-x^3+5x\sqrt{1+x^2}}{1+x^2+\sqrt{1+x^2}},\quad x(0)=1. \tag{10}$$

Exercise 114. Let $p\in\mathcal{R}$. Show that

(i) $p^{②}=\frac{p^2}{1+\mu p}$;
(ii) $(\ominus p)^{②}=p^{②}$;
(iii) $1+\mu p=\frac{p^2}{p^{②}}$;
(iv) $p+(\ominus p)=\mu p^{②}$;
(v) $p\oplus p^{②}=p+p^2$.

Exercise 115. Find solutions of the Clairaut dynamic equations

(i) $y=ty^\Delta+\frac{1}{y^\Delta}$;
(ii) $y=ty^\Delta+e^{y^\Delta}$.

Bibliography

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