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# Open Problems and conjectures in discrete dynamical systems

## PART I

## 0.1 Sharkovsky's Theorem

One of the fascinating theorems about one dimensional maps is Sharkovsky's Theorem [**?**]. To state this theorem, we need Sharkovsky's ordering of the positive integers, it is given by

$$\begin{aligned} &3 \prec 5 \prec 7 \prec ... \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec ... \\ &\prec 2^n \cdot 3 \prec 2^n \cdot 5 \prec 2^n \cdot 7 \prec ... \prec 2^n \prec ... \prec 2^2 \prec 2 \prec 1. \end{aligned} \tag{1}$$

Observe that all positive integers are in this ordering, and for two positive integers $r$ and $k$,, $r$ precedes $k$is denoted by $r \prec k$. We state Sharkovsky's Theorem

**Theorem 0.1.1 (Sharkovsky's Theorem).** *Let $f : I \to I$ be a continuous map on a closed interval $I$. If $f$ has a periodic point of period $k$, then it has a periodic point of period $r$ for all $r$ with $k \prec r$.*

**Problem 1** Let $f : I \to I$ be a continuous map on a closed interval $I$.. Suppose that every point in $I$ is of period $r$, i.e. $f^r(x) = x$, and assume that $r$ is minimal. Prove or disprove the following conjecture: $r = 1$ or $r = 2$.

**Open Problem 2** Extend Sharkovsky's theorem to the delay difference equations
(a)

$$x_{n+1} = f(x_{n-1})$$

(b)

$$x_{n+1} = f(x_{n-k})$$

**Problem 3** Define $\mathcal{A}_q = \{x : \ lcm(x, p) = pq\}$. □

Now to each positive integer $p \geq 1$, we associate the following ordering, which we call the *p-Sharkovsky's ordering.*

$$
\begin{aligned}
&\mathcal{A}_3 \rhd \mathcal{A}_5 \rhd \mathcal{A}_7 \rhd ...\\
&\mathcal{A}_{2\cdot 3} \rhd \mathcal{A}_{2\cdot 5} \rhd \mathcal{A}_{2\cdot 7} \rhd ...\\
&\vdots\\
&\mathcal{A}_{2^n\cdot 3} \rhd \mathcal{A}_{2^n\cdot 5} \rhd \mathcal{A}_{2^n\cdot 7} \rhd ...\\
&\vdots\\
&\ldots \rhd \mathcal{A}_{2^n} \rhd ... \rhd \mathcal{A}_{2^2} \rhd \mathcal{A}_2 \rhd \mathcal{A}_1.
\end{aligned}
$$

In this ordering, we mean by $\mathcal{A}_{q_1} \rhd \mathcal{A}_{q_2}$, $q_1 \prec q_2$ in the original Sharkovsky's ordering and each element of $\mathcal{A}_{q_1}$ precedes each element of $\mathcal{A}_{q_2}$ in the $p$-Sharkovsky's ordering.

**(i)** If $p = 1$ then the 1-Sharkovsky's ordering reduces to the original Sharkovsky's ordering.

**(ii)** If $p = 2^m$ for some positive integer $m$ then the $2^m$-Sharkovsky's ordering simplifies to

$$
\begin{aligned}
&3, 3\cdot 2, ..., 3\cdot 2^m \prec 5, 5\cdot 2, ..., 5\cdot 2^m \prec 7, 7\cdot 2, ..., 7\cdot 2^m \prec ...\\
&3\cdot 2^{m+1} \prec 5\cdot 2^{m+1} \prec 7\cdot 2^{m+1} \prec ...\\
&\vdots\\
&3\cdot 2^{m+n} \prec 5\cdot 2^{m+n} \prec 7\cdot 2^{m+n} \prec ...\\
&\vdots\\
&\ldots \prec 2^{m+n} \prec ... \prec 2^{m+2} \prec 2^{m+1} \prec 2^m, 2^{m-1}, ..., 2^2, 2, 1
\end{aligned}
\tag{2}
$$

Prove the following result which extends Sharkovsky's Theorem to periodic difference equations.
Suppose the $p$-periodic difference equation $x_{n+1} = f_n(x_n)$ has an $r$-cycle, and let $\ell := \frac{lcm(p,r)}{p}$. Then each set $\mathcal{A}_q$ such that $\mathcal{A}_\ell \rhd \mathcal{A}_q$ contains a period of a cycle. Develop a converse of Sharkovsky's theorem for the above $p$-periodic difference equation.

## 0.2 Basin of attraction and stability

It is customary to call an asymptotically stable fixed point or a cycle an attractor. This name makes sense since in this case all nearby points tend to the attractor. The maximal set that is attracted to an attractor $M$ is called the **basin of attraction** of $M$. Our analysis applies to cycles of any period.

**Definition 0.2.1.** *Let $x^*$ be a fixed point of map $f$. Then the basin of attraction (or the stable set) $W^s(x^*)$ of $x^*$ is defined as*

$$W^s(x^*) = \{x : \lim_{n\to\infty} f^n(x) = x^*\}$$

*In other words, $W^s(x^*)$ consists of all points that are forward asymptotic to $x^*$.*

Observe that if $x^*$ is an attracting fixed point, $W^s(x^*)$ contains an open interval around $x^*$. The maximal interval in $W^s(x^*)$ that contains $x^*$ is called the immediate basin of attraction and is denoted by $\mathcal{B}^s(x^*)$.

**Example 0.2.2.** *The map $f(x) = x^2$ has one attracting fixed point $x^* = 0$. Its basin of attraction $W^s(0) = (-1, 1)$. Note that 1 is an unstable fixed point and -1 is an eventually fixed point that goes to 1.*

**Example 0.2.3.** *Let us now modify the map $f$. Consider the map $g : [-2, 4] \to [-2, 4]$ defined as*

$$g(x) = \begin{cases} x^2 & \text{if} \quad -2 \le x \le 1 \\ 3\sqrt{x} - 2 & \text{if} \quad 1 < x \le 4 \end{cases}$$

*The map $g$ has three fixed points $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 4$. The basin of attraction of $x_1^* = 0$, $W^s(0) = (-1, 1)$ while the basin of attraction of $x_3^* = 4$, $W^s(4) = [-1, -1) \bigcup (1, 4]$. Moreover, the immediate basin of attractions of $x_1^* = 0$, and $x_3^* = 4$ are $\mathcal{B}^s(0) = W^s(0)$ and $\mathcal{B}^s(4) = (1, 4]$.*

**Remark:** Observe that in the preceding example, the basins of attraction of the two fixed points $x_1^* = 0$ and $x_3^* = 4$ are disjoint. This is no accident and in fact is generally true. This is due to the uniqueness of a limit of a sequence. In other words, if the $\lim_{n\to\infty} f^n(x) = L_1$ and $\lim_{n\to\infty} f^n(x) = L_2$, then certainly $L_1 = L_2$.

It is worth noting here that finding the basin of attraction of a fixed point is in general a difficult task. But even more difficult is providing a rigorous proof. The most efficient method to determining the basin of attraction is the method of Liapunov functions.In this section, we will develop some of the basic topological properties of basin of attractions. Henceforth, all our maps are assumed to be continuous. We begin our exposition by defining the important notion of invariance.

**Definition 0.2.4.** *A set $M$ is invariant under a map $f$ if $f(M) \subseteq M$. In other words, for every $x \in M$, the orbit of $x$, $O(x) \subseteq M$.*

Clearly an orbit of a point is invariant.
Next we show that the basin of attraction of an attracting fixed point is invariant and open.

**Theorem 0.2.5.** *Let $f : I \to I$, $I = [a, b]$, be a continuous map. Then the following statements hold true.*

*(a) The immediate basin of attraction $\mathcal{B}(x^*)$ is an interval containing $x^*$ which is either an open interval $(c, d)$ or of the form $[a, c)(c, b]$. Moreover, $\mathcal{B}(x^*)$ is (positively) invariant.*

*(b) $W^s(x^*)$ is (positively) invariant. Furthermore, $W^s(x^*)$ is the union (may be an infinite union) of intervals that are either open intervals or of the form $[a, u)$ or $(v, b]$.*

We now turn our attention to periodic points. If $\bar{x}$ is a periodic point of period $k$ under the map $f$, then its basin of attraction $W^s(\bar{x})$ is its basin of attraction as a fixed point under the map $f^k$. Hence $W^s(\bar{x}) = \{x : \lim_{n\to\infty} (f^k)^n(x) = \lim_{n\to\infty} f^{kn}(x) = \bar{x}\}$. Let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ be a $k$-cycle of a map $f$. Then clearly for $i \neq j$, $W^s(\bar{x}_i) \bigcap W^s(\bar{x}_j) = \varnothing$. (Why?) More generally, if $x$ is a periodic point of period $r$ and $y \neq x$ is a periodic point of period $s$ then $W^s(x) \bigcap W^s(y) = \varnothing$

**Example 0.2.6.** *Consider the function $f(x) = -x^{\frac{1}{3}}$. Then $x^* = 0$ is the only fixed point. There is a 2-cycle $\{-1, 1\}$ with $f(-1) = 1$, $f^2(-1) = -1$. The cobweb diagram (Fig ...) shows that $W^s(1) = (0, \infty)$, $W^s(-1) = (-\infty, 0)$.*

**Problem 4**

Consider the logistic map $F_\mu(x) = \mu x(1-x)$ (a) Prove that for $1 < \mu < 3$ the basin of attraction of the positive fixed point is the open interval $(0, 1)$ (b) Prove that for $3 < \mu < 1 + \sqrt{6}$ the basin of attraction of the attracting periodic cycle $c = \{\bar{x}_1, \bar{x}_2\}$ is given by $W^s(c) = W^s(\bar{x}_1) \bigcup W^s(\bar{x}_2)$ is all the points in $(0, 1)$ except the set of eventually fixed points (including the fixed point $\frac{\mu-1}{\mu}$).
(c) Extend (b) to attracting periodic orbits of periods $2^n$.

**• The Cushing-Henson Conjectures**

Cushing and Henson investigated the
Beverton-Holt equation

$$x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \quad x_0 \geq 0 \tag{3}$$

**Problem 5** Prove that for $\mu > 1$, $K > 0$, all nonzero-solutions converge to the equilibrium point $x^* = K$.
A modification of this equation that arises in the study of populations living in a periodically (seasonally) fluctuating environment replaces the constant carrying capacity $K$ with a periodic sequence $\{K_n\}$ of positive carrying capacities

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n} = f_n(x_n)$$

with

$$K_{n+p} = K_n \text{ for all } n \in \mathbb{Z}^+, \quad r \geq 2, \quad \mu > 1.$$

**Problem 6**

Cushing and Henson conjectured that for the periodic $p$-Beverton-Holt equation, $p \geq 2$
There is a positive $p$-periodic solution $\{\bar{x}_0. \ldots, \bar{x}_{p-1}\}$ and it globally attracts all positive solutions.
**Problem 7**

The average over $p$ values, $p \geq 2$,

$$\{x_0, x_1, \ldots, x_{p-1}\}, av(x_n) = \frac{1}{p}\sum_{i=0}^{p-1} x_i$$

satisfies

$$av(\bar{x}_n) < av(K_n).$$

**Problem 8**
Extend the results of Problems 6 and 7 to the Beverton-Holt equation with delay $k$

$$x_{n+1} = \frac{\mu K_n x_{n-k}}{K_n + (\mu - 1)x_{n-k}} = f_n(x_{n-k})$$

with

$$K_{n+p} = K_n \text{ for all } n \in \mathbb{Z}^+, \quad r \geq 2, \quad \mu > 1.$$

**Problem 9**
Extend the results of Problems 6 and 7 to the Beverton-Holt equation where both $\mu_n$ and $k_n$ are periodic of period $p$.

$$x_{n+1} = \frac{\mu_n K_n x_n}{K_n + (\mu_n - 1)x_n}$$

**Problem 10**

Let $f : X \to X$ be a continuous map on a connected metric space $X$ (or a subset of $\mathbb{R}^k$). Show that if a periodic orbit of period $r$ is globally asymptotically stable, then $r = 1$.

Let $f : X \to X$ be a continuous map on a metric space $X$ (or a subset of $\mathbb{R}^k$) which is th union of $m$ components. Show that if a periodic orbit of period $r$ is globally asymptotically stable, then $r \leq m$.

**Problem 11** Extend the results of Problem 10 to the difference equation with delay k.

$$x_{n+1} = f(x_{n-k})$$

**Problem 12** Suppose the $p$-periodic difference equation $x_{n+1} = f_n(x_n)$ has a globally asymptotically stable $r$-cycle. Then $r$ must divide $p$.

**Problem 13** Extend the resultw of Problem 12 to the equation

$$x_{n+1} = f_n(x_{n-k})$$