From Autonomous to Nonautonomous Difference Equations: NEW CHALLENGES

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Autonomous Difference Equations/Maps

A given map $f : X \to X$, where X is a metric space or just $\mathbb{R}^n$, is associated with the difference equation

$$x_{n+1} = f(x_n). \tag{1}$$

For $\mathrm{x}_0 \in X$, $\mathrm{x}_n = f^n(x_0)$, $f^n = f \circ f^{n-1,}, ..., f^2 = f \circ f$.

So one speaks of the orbit of x_0 under the map f as the set

$$O(x_0) = \{x_0, f(x_0), ..., f^n(x_0), ...\},$$

or equivalently

$$O(x_0) = \{x_0, x_1, ..., x_n, ...\}.$$

A point x^* is fixed if $f(x^*) = x^*$; it is of period r if $f^r(x^*) = x^*$.

Similarly, a map $f : X \times X \to X$, gives rise to the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \tag{2}$$

and so on.

The Fundamental Theorem of First Order Difference Equation (Sharkovsky Theorem)

Let us order the set of natural numbers $\mathbb{N} \cup \{2^\infty\} = \mathbb{N}^*$ as follows.

$$\begin{gathered}
3 \rhd 5 \rhd 7 \rhd \dots \\
2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd \dots \\
\vdots \\
2^n \times 3 \rhd 2^n \times 5 \rhd 2^n \times 7 \rhd \dots \\
\vdots \\
2^\infty \rhd \dots \rhd 2^n \rhd \dots 2 \rhd 1
\end{gathered}$$

Theorem 1 (Sharkovsky) *Let $f : I \to I$ be a continuous map on a closed interval $I \subseteq \mathbb{R}$. If the equation $x_{n+1} = f(x_n)$ has a cycle of minimal period r, then it has also cycles of the minimal period s for all $r \rhd s$.*

Open Problem 1 Extend Sharkovsky's Theorem to the equation

$$x_n = f(x_{n-2})$$

and more generally to

$$x_n = f(x_{n-k}).$$

Let $q \in \mathbb{N}^*$. Define

$$S_k(q) = \begin{cases} \{1\} & \text{if } q = 1, \\ \{l.q : l|k \text{ and } \gcd\left(\frac{k}{l}, q\right) = 1\} & \text{if } q \in \mathbb{N} \backslash \{1\}, \\ \{l : l \geq 2 \text{ and } l|k\} & \text{if } q = 2^\infty. \end{cases}$$

Definition 2 *Let $k \in \mathbb{N}$. The k-Sharkovsky order is given by*

$$\begin{gathered} S_k(3) \rhd S_k(5) \rhd \cdots \rhd S_k(2 \cdot 3) \rhd \ldots \\ \rhd \cdots \rhd S_k(2^2) \rhd S_k(2) \rhd S_k(1). \end{gathered}$$

Example 3 *Let $k = 2$. Then*

$$\begin{aligned} S_2(3) &= \{3, 2 \cdot 3\} \\ S_2(2^m \cdot 3) &= \{2^{m+1} \cdot 3\} \\ \{3, 2 \cdot 3\} &\rhd \{5, 2 \cdot 5\} \rhd \ldots \\ 2^2 \cdot 3 &\rhd 2^2 \cdot 5 \rhd \ldots \\ &\vdots \\ &\ldots \rhd 2^{m+1} \rhd \cdots \rhd 2^2 \rhd 1. \end{aligned}$$

Open Problem 2 Let $f : I \to I$ be a continuous function on I. Prove that if the difference equation

$$x_n = f(x_{n-1}) \tag{3}$$

has a cycle of minimal period r, then the difference equation

$$x_n = f(x_{n-k}) \tag{4}$$

has cycles of period q for all numbers $q \in S_k(m)$, whenever $S_k(r) \trianglerighteq S_k(q)$. Moreover, if f has more than one fixed point, then equation (4) has cycles of minimal periods q for all $q \in S_k(2^\infty)$.

$*$ A nice trick.

Let

$$y_1(n) = x_{nk-1}, y_2(n) = x_{nk-2}, \ldots, y_k(n) = x_{nk-k}.$$

Then

$$y_1(n+1) = f(y_1(n)), y_2(n+1) = f(y_2(n)),$$
$$\ldots, y_k(n+1) = f(y_k(n)).$$

Open Problem 3 (Ladas) Let $f : I \to I$ be a continuous map and suppose that every orbit (solution) of the equation $x_{n+1} = f(x_n)$ is of minimal period r. Show that either $r = 1$ or $r = 2$.

Is this extendable to

- $x_{n+1} = f(x_n, x_{n-1})$?
- $x_{n+1} = f(x_{n-k})$?

Nonautonomous Difference Equations: The Periodic Case

Motivation: The following model (Beverton-Holt) is used to model fish such as salmon.

$$x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \tag{5}$$

where μ is the intrinsic growth rate of the population, and K is its carrying capacity. This is a simple monotonic model and it is known that if $K > 0$, $\mu > 1$, the equation has a positive fixed point $x^* = K$ which is globally asymptotically stable on $(0, \infty)$.

Cushing and Henson suggested nonautonomizing (5) by letting K to vary periodically, i.e, $K_{n+p} = K_n$, for all $n \in \mathbb{Z}^+$. Hence, we have the following nonautonomous difference equation.

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n} \tag{6}$$

Open Problem 4 Show that if $\mu > 1$ and $K_n > 0$, equation (6) has a periodic solution (of minimal period p) which is globally asymptotically stable on $(0, \infty)$.

Open Problem 5 Let $\{\bar{x}_0, \bar{x}_1, ..., \bar{x}_{p-1}\}$ be the periodic orbit of equation (6). Show that

$$\frac{1}{p}\sum_{i=0}^{p-1} x_i < \frac{1}{p}\sum_{i=0}^{p-1} K_i, \;\; p \geq 2.$$

Open Problem 6 Extend problems 4 and 5 to

(a) $x_{n+1} = \dfrac{\mu_n K_n x_n}{K_n + (\mu_n - 1)x_n},$

(b) $x_{n+1} = \dfrac{\mu K_n x_{n-k}}{K_n + (\mu - 1)x_{n-k}},$

(c) $x_{n+1} = \dfrac{\mu K_n x_{n-t}}{K_n + (\mu - 1)x_{n-s}},$

The setup: Why skew-product dynamical system?

What is a dynamical system?

Let X be a metric space ($\mathbb{R}^n$ as an example), and T a topological group (topological semigroup). Then $\pi : X \times T \to X$ is called a dynamical system or (X, T, π) is a dynamical system if

(i) π is continuous,

(ii) $\pi(x, 0) = x$ for all $x \in X$,

(iii) $\pi(\pi(x, s), t) = \pi(x, s + t)$ (the associative property or the group property).

Important Examples

1. $T = \mathbb{R}^+(\mathbb{R})$: differential equations
2. $T = \mathbb{Z}^+(\mathbb{Z})$: difference equations
3. $T = h\mathbb{Z}^+, \quad h \in \mathbb{R}$

Example 4 $T = \mathbb{Z}^+$. *We let* $\pi(x_0, n) = f^n(x_0)$. *Hence every difference equation*

$$x_{n+1} = f(x_n)$$

with continuous f *generates a dynamical system.*

Example 5 *Consider the nonautonomous difference equation*

$$x_{n+1} = (-1)^n \left(1 + \frac{1}{n+1}\right) x_n, \quad n \in \mathbb{Z}^+.$$

Then $x_n = (-1)^{\frac{n(n-1)}{2}}(n+1)x_0$. *If we let* $\pi(x_0, n) = x_n$. *Then*

$$\pi(\pi(x_0, s), t) = (-1)^{\frac{t(t-1)}{2}}(-1)^{\frac{s(s-1)}{2}}(t+1)(s+1)x_0$$

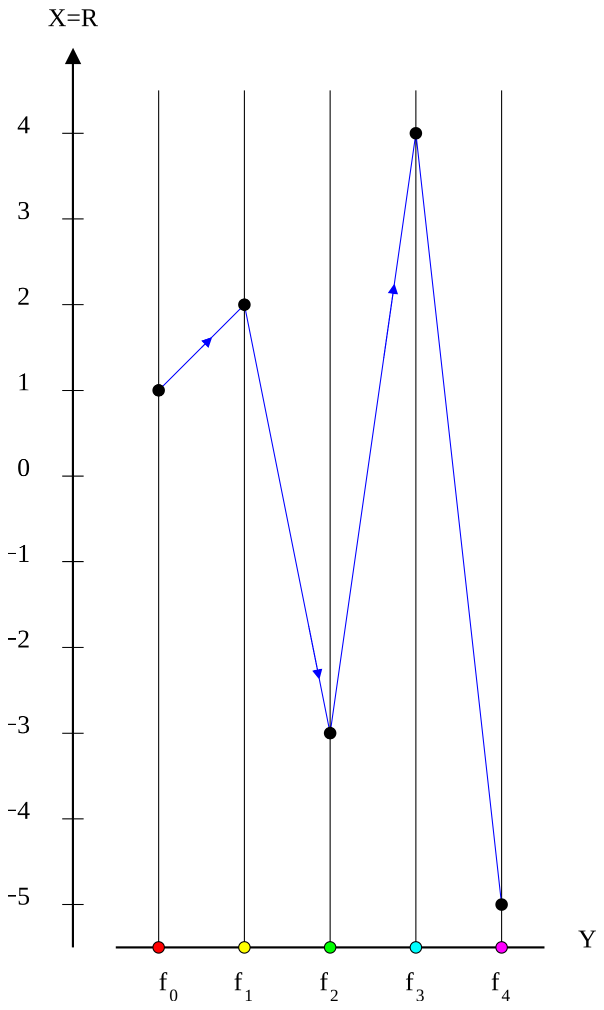
while

$$\begin{aligned}\pi(x_0, s+t) &= (-1)^{\frac{(s+t)(s+t-1)}{2}}(s+t+1)x_0 \\ &\neq \pi(\pi(x_0, s), t)\end{aligned}$$

To remedy this situation, let $Y = \{f_0, f_1, f_2, \dots\}$, *where* $f_i(x) = (-1)^i \left(1 + \frac{1}{i+1}\right) x$. *Let* $X = \mathbb{R} \times Y$ *and define* $\pi : X \times \mathbb{Z}^+ \to X$ *as* $\pi((x, f_i), n) = (f_{i+n-1} \cdots \circ f_{i+1} \circ f_i(x), f_{i+n})$. *Then* $\pi(\pi((x, f_i), s), t) = \pi((x, f_i), s+t)$ *and* π *is a dynamical system.*

$$\begin{tikzcd}
\mathbb{R} \times Y \times \mathbb{Z}^+ \arrow[r, "\pi"] \arrow[d, "p \times id"] & \mathbb{R} \times Y \arrow[d, "\phi p(x, f_i) = f_i"] \\
Y \times \mathbb{Z}^+ \arrow[r, "\sigma"] & Y
\end{tikzcd}$$

$$\sigma(f_i, n) = f_{i+n}$$


X=R
4
3
2
1
0
-1
-2
-3
-4
-5
Y
f_0
f_1
f_2
f_3
f_4

$$x_{n+1} = f_n(x_n), \quad f_{n+p} = f_n \quad \text{for all } n \in \mathbb{Z}^+$$

A geometric r-cycle is a set of points

$$c_r = \{\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{r-1}\}$$

such that

$$f_{(i+nr) \bmod p}(\bar{x}_i) = \bar{x}_{(i+1) \bmod r}.$$

A complete r-cycle in the skew-product system generated by c_r is the set

$$c_r = \{(x_0, f_0), (x_1, f_1), \dots, (x_{s \bmod r}, f_{s \bmod p})\}$$

where $s = \operatorname{lcm}[r, p]$.

For $r = 4$, $p = 6$ we have the following diagram.

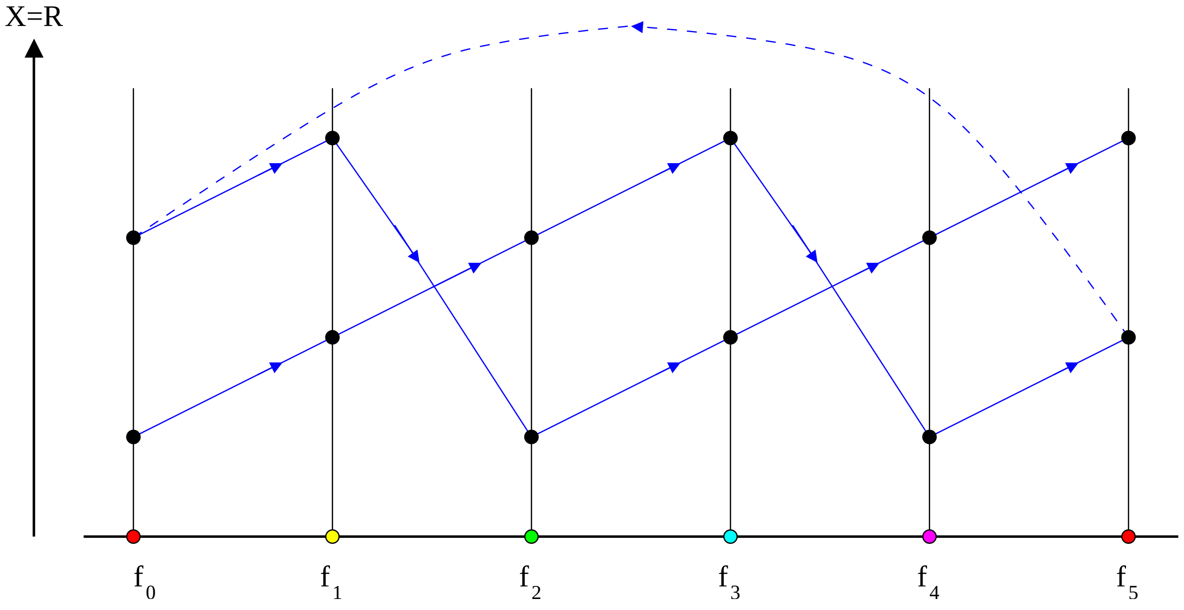


Figure 1: A geometric 4-cycle and a complete 12-cycle.

Remarks 6 *Prove the following statements.*

(i) *The complete cycle has period* $s = \operatorname{lcm}[r, p]$.

(ii) *Each fiber* $\mathcal{F}_i = p^{-1}(f_i)$, $1 \leq i \leq p$, *contains* $l = \dfrac{s}{p}$ *elements.*

(iii) *The maps* $f_0, f_d, f_{2d}, \ldots, f_{(m-1)d}$ *intersect at the points* $(\bar{x}_0, \bar{x}_1)$, $(\bar{x}_d, \bar{x}_{d+1})$, $\ldots$, $\big(\bar{x}_{(m-1)d \bmod r}, \bar{x}_{(m-1)d+1) \bmod r}\big)$.

(iv) *Use (iii) to show that it is possible that the p-periodic Beverton Hold equation*

$$x_{n+1} = \frac{\mu_n K_n x_n}{K_n + (\mu_n - 1)x_n}, \quad \mu_n > 1, \quad K_n > 0$$

may have periodic solutions (orbits) of period $r < p$.

Open Problem 7 Let $f_n : X \to X$, $f_{n+p} = f_n$ for all $n \in \mathbb{Z}^+$, be continuous maps on a connected metric space X. Show that if $c_r = \{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{r-1}\}$ is globally asymptotically stable geometric cycle of $x_{n+1} = f_n(x_n)$, then r must divide p.

Corollary 7 *If* $f_n = f$ *for all* n, *that is* $p = 1$, *then* $r = 1$. *In other words, there are no globally asymptotically periodic cycles of period* $r > 1$ *in a connected metric space.*

Open Problem 8

(i) Prove Corollary 7.

(ii) Suppose that $f : X \to X$, $X = \overset{k}{\underset{i=1}{\cup}} M_i$, where each M_i is a connected component in the metric space X. If a periodic orbit of period r is globally asymptotically stable, prove the $r \leq k$.

Open Problem 9 Extend to difference equations with delay in the form

(a) $x_{n+1} = f_{n-1}(x_{n-k})$

(b) $x_{n+1} = f_n(x_n, x_{n-1})$

(c) $x_{n+1} = f_n(x_n, x_{n-k})$, $k \neq 1$

(d) $x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_0)$