Time Scales Calculus and an Introduction to Sturmian Theory

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Introduction and preliminary results

A **time scale** $\mathbb{T}$ is a closed, nonempty subset of the real numbers.

The simplest examples are:

$\mathbb{R}$, $\mathbb{Z}$, $[a,b]$, $\mathbb{N}_0 =$ natural numbers

and the union or (nonempty) intersection of any of these:
(for example)

$\bigcup_i [a_i, b_i]$, $\quad \mathbb{Z} \cup [a,b]$

Some elementary concepts

Let $\mathbb{T}$ be a time scale. We define the *forward jump operator* $\sigma(t)$ at t , for $t \in \mathbb{T}$, by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the *backward jump* operator $\rho(t)$ at t, for $t \in \mathbb{T}$, by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

$\sigma(t) > t \Leftrightarrow t$ is *right-scattered*,

$\rho(t) < t \Leftrightarrow t$ is *left-scattered*,

$\sigma(t) = t \quad \Leftrightarrow t$ is *right-dense*,

$\rho(t) = t \quad \Leftrightarrow t$ is *left-dense*.

$\rho(t) < t < \sigma(t) \quad \Leftrightarrow t$ is *isolated*

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be **right-dense continuous** provided f is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite.

Definition: For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (if $t = \sup \mathbb{T}$, assume t is not left scattered), define the **Δ-derivative** $f^{\Delta}(t)$ of $f(t)$ to be the number, provided it exists, with the property that, for any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We say that f is Δ-differentiable on $\mathbb{T}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

It can be shown that if $f : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, the set of integers, then

$$f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t).$$

If $t \in \mathbb{T}$ is right dense and $f : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then

$$f^{\Delta}(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

A very important function which arises in the time-scale calculus is the **graininess function** defined by

$$\mu(t) := \sigma(t) - t$$

One easily has

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

It may also be shown that every right dense continuous function has an antiderivative:

Thus if f is rd-continuous then there exists F such that

$$F^{\Delta}(t) = f(t).$$

Some additional examples of time scales

(i) Let $q > 1$ and consider

$$q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$$

$$\overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$$

The time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ arises in the study of

“q-difference” equations and one obtains

$\sigma(t) = qt$ and $\rho(t) = t/q$ for all $t \in \mathbb{T}$

(ii) Let $\mathbb{N}_0$ denote the nonnegative integers and consider

$$\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}.$$

We have $\sigma(n^2) = (n+1)^2$ and $\mu(n^2) = 2n + 1$.

(iii) Let H_n be the so-called 'harmonic numbers' given by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k}$$

and consider

$$\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}.$$

One has $\sigma(H_n) = H_{n+1}$ and $\mu(H_{n+1}) = \frac{1}{n+1}$.

(iv) Other examples which arise involve the unitary linear lattices, which may be regarded as an extension of equidistant lattices of the form $h\mathbb{Z}$ and the q-linear grids $\overline{q^{\mathbb{Z}}}$, with $0 < q < 1$.

To illustrate, suppose $h > 0$ and $0 < q < 1$. One can define a timescale (unitary lattice) by $\mathbb{T} \equiv \mathbb{T}_{q,h,y}$ for $y > y_0 \equiv h(1-q)^{-1}$ in the following way:

$$\mathbb{T} = \overline{\{u_n(y) : n \in \mathbb{Z}\}}$$

where the generating function $u : [y_0, \infty) \to [y_0, \infty)$ is given by

$$u(x) = qx + h < x \text{ for } x > y_0, \; u(y_0) = y_0$$

One has for $y > y_0$

$$\lim_{n\to\infty} u^n(y) = y_0$$

and

$$\lim_{n\to\infty} u^{-n}(y) = \infty$$

That is, the function u is just the left-shift $\rho : \mathbb{T} \to \mathbb{T}$ on the lattice. Consequently, one can describe the time scale by

$$\mathbb{T} = \{y_0\} \cup \{\rho^n(y) : n \in \mathbb{Z}\}$$

For this time scale, one has

$$\mu(x) = q^{-1}[(1-q)x - h]$$

and the left-graininess

$$\nu(x) := x - \rho(x) = (1 - qx) - h$$

for $x \in \mathbb{T}, x \neq y_0$, and so one has

$$\frac{\mu(x)}{\nu(x)} = q^{-1} \text{ for } x \in \mathbb{T}.$$

Product Rule: From the formula

$$w(\sigma(t)) = w(t) + \mu(t)w^{\Delta}(t)$$

one easily obtains (with $w = fg$)

$$(fg)^{\Delta}(t) = f(\sigma(t))g^{\Delta}(t) + f^{\Delta}(t)g(t) :$$

and also, by symmetry, one has

$$(fg)^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)) :$$

Quotient Rule:

$$\left(\frac{f(t)}{g(t)}\right)^{\Delta} = \frac{g(t)f^{\Delta}(t) - g^{\Delta}(t)f(t)}{g(t)g(\sigma(t))}.$$

There exist several versions of a chain rule for dynamic equations. We note also that one may also obtain analogs of the product formula, quotient rule, and a chain rule, involving the backward jump operator, but for simplicity, we will consider only the delta case, which generalizes the usual forward difference operator.

Recall that if $f, g : \mathbb{R} \to \mathbb{R}$ are such that g is differentiable at t and f is differentiable at $g(t)$, then the composition $f \circ g$ is differentiable and one has

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

This result is not valid in general for dynamic equations, as is easily seen by means of some simple examples: One can easily show that if $\mathbb{T} = \mathbb{Z}$ and $f(t) = g(t) = t^2$, then for all $t \neq 0$

$$(f \circ g)^{\Delta}(t) \neq f^{\Delta}(t)g^{\Delta}(t)$$

Two Chain Rules

Chain Rule (Version I): Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

Chain Rule (Version II): Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\}.$$

Applications to second order Sturmian theory

The general second order linear differential equation

$$y''(x) + p(x)y' + q(x)y = 0,$$

arises in many different areas of mathematics and physics. (Mechanical vibrations, electric circuits, Newton's law of motion, and in the solution of partial differential equations for the heat equation, potential equation, and the wave equation.) (We assume above that $p(x)$ and $q(x)$ are continuous real valued functions defined on some interval I of $\mathbb{R}$.)

In practice, one is unable to explicitly solve most equations of the second order, but one can often deduce the qualitative nature of the solutions.

It is often important to be able to describe the distribution of zeros for solutions.

Consider the second order linear dynamic equation, which we write in the form

$$(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0. \tag{1}$$

We assume $p, q \in C_{rd}$, $p > 0$ and $\sup \mathbb{T} = \infty$,

with no (explicit) sign assumptions on q .

Because the equation is linear, solutions will exist on any interval in which the coefficients are right–dense continuous.

Definition 1. We say that a solution x of (1) has a *generalized zero* at t in case $x(t) = 0$. We say x has a *generalized zero* in $(t, \sigma(t))$ in case $x(t)x(\sigma(t)) < 0$ and $\mu(t) > 0$. We say that (1) is *disconjugate* on the interval $[c, d]$, if there is no nontrivial solution of (1) with two (or more) generalized zeros in $[c, d]$.

Definition 2. Equation (1) is said to be $nonoscillatory$ on $[\tau, \infty)$ if there exists $c \in [\tau, \infty)$ such that this equation is disconjugate on $[c, d]$ for every $d > c$. In the opposite case (1) is said to be $oscillatory$ on $[\tau, \infty)$. Oscillation of (1) may equivalently be defined as follows. A nontrivial solution y of (1) is called $oscillatory$ if it has infinitely many (isolated) generalized zeros in $[\tau, \infty)$. By the Sturm type separation theorem, one solution of (1) is (non)oscillatory iff every solution of (1) is (non)oscillatory. Hence we can speak about $oscillation$ or $nonoscillation$ of equation (1).

Basic oscillatory properties of (1) are described by the so-called Reid Roundabout Theorem.

Proposition 1.[**Reid Roundabout Theorem**] The following statements are equivalent:

(i) Equation (1) is disconjugate on $[c, d]$.

(ii) Equation (1) has a solution without generalized zeros on $[c, d]$.

(iii) The Riccati dynamic equation

$$u^{\Delta}(t) + q(t) + \frac{u^2(t)}{p(t) + \mu(t)u(t)} = 0 \qquad (2)$$

has a solution u with $p(t) + \mu(t)u(t) > 0$ for $t \in [c, d]^{\kappa}$ (except for the left-dense right-scattered d at which $p + \mu u$ may be nonpositive).

(iv) The quadratic functional

$$\mathcal{F}(\xi; c, d) = \int_c^d \left\{ p(t) \left(\xi^{\Delta}(t) \right)^2 - q(t)(\xi^{\sigma}(t))^2 \right\} \Delta t$$

is positive definite for $\xi \in U(c, d)$, where

$$U(c, d) = \{\xi \in C_p^1[c, d] \, : \, \xi(c) = \xi(d) = 0\}.$$

The above proposition makes it therefore clear that there are at least two methods of investigation of (non)oscillation of (1). The first one – the *variational method* – is based on the equivalence of (i) and (iv) and its basic statement can be reformulated as follows:

Lemma 1. [**Variational method**] If for any $T \in [\tau, \infty)$ there exists $0 \not\equiv \xi \in U(T)$, where

$$U(T) = \{\xi \in C_p^1[T, \infty) \,:\, \xi(t) = 0 \text{ for } t \in [\tau, T] \text{ and } \exists \hat{T}, \hat{T} >$$
$$\text{such that } \xi(t) = 0 \text{ for } t \in [\sigma(\hat{T}), \infty)\},$$

such that $\mathcal{F}(\xi; T, \infty) = \mathcal{F}(\xi, T, \sigma(\hat{T})) \leq 0$, then (1) is oscillatory.

Another method of investigation for the oscillation theory of (1) is based on the equivalence of (i) and (iii) in Proposition . This is usually referred to as the *Riccati technique* and by virtue of the Sturm Comparison Theorem implies that for nonoscillation of

(1), it is sufficient to find a solution of the Riccati-type inequality as given in the next lemma.

Lemma 2. [**Riccati technique**] Equation (1) is nonoscillatory if and only if there exists $T \in [\tau, \infty)$ and a function u satisfying the Riccati dynamic inequality

$$u^{\Delta}(t) + q(t) + \frac{u^2(t)}{p(t) + \mu(t)u(t)} \leq 0$$

with $p(t) + \mu(t)u(t) > 0$ for $t \in [T, \infty)$.

For completeness, we recall the following

Lemma 3. [**Sturm-Picone Comparison Theorem**] Consider the equation

$$[\tilde{p}(t)x^{\Delta}(t)]^{\Delta} + \tilde{q}(t)x^{\sigma}(t) = 0, \tag{3}$$

where $\tilde{p}$ and $\tilde{q}$ satisfy the same assumptions as p and q. Suppose that $\tilde{p}(t) \geq p(t)$ and $q(t) \leq \tilde{q}(t)$ on $[T, \infty)$ for all large T. Then (3) is nonoscillatory on $[\tau, \infty)$ implies (1) is nonoscillatory on $[\tau, \infty)$.

In order to discuss some of these ideas more fully we focus first on a comparison of

the two second order linear dynamic equations

$$L_1 x = (r_1(t)x^{\Delta})^{\Delta} + q_1(t)x^{\sigma} = 0, \tag{4}$$

$$L_2 x = (r_2(t)x^{\Delta})^{\Delta} + q_2(t)x^{\sigma} = 0, \tag{5}$$

where r_i and q_i, $i = 1, 2$ are real–valued, right–dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$.

We assume throughout that $0 < r_1(t) \leq r_2(t)$, $t \in \mathbb{T}$.

If $r_i(t) > 0$, $q_i(t)$ are real-valued continuous functions satisfying

$$0 < r_1(t) \leq r_2(t), \quad q_2(t) \leq q_1(t)$$

for t in the real interval $[a, \infty)$, then the Sturm comparison theorem says that if

$$(r_1(t)x')' + q_1(t)x = 0$$

is nonoscillatory on $[a, \infty)$ (i.e. all nontrivial solutions have finitely many zeros in $[a, \infty)$), then

$$(r_2(t)x')' + q_2(t)x = 0$$

is also nonoscillatory on $[a, \infty)$.

The Hille–Wintner theorem replaces the point-wise comparisons on the coefficient functions by integral comparisons and aside from the Sturm Comparison therorem, is one of the most useful results for comparing the distribution of zeros of two related equations.

Theorem 1 (Hille–Wintner Theorem). *Assume that* $0 < r_1(t) \leq r_2(t)$, $t \in \mathbb{T}$ *and*

$$0 \leq \int_t^\infty q_2(s)\Delta s \leq \int_t^\infty q_1(s)\Delta s \qquad (6)$$

for all large t. *Let*

$$\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$$

and let χ *denote the characteristic function of* $\hat{\mathbb{T}}$. *Assume further that there is an* $M > 0$ *such that*

$$r_1(t)\chi(t) \leq M\mu(t), \quad t \in \mathbb{T}. \qquad (7)$$

Then $L_1 x = 0$ *nonoscillatory on* $[a, \infty)$ *implies*

$L_2 x = 0$ *is nonoscillatory on* $[a, \infty)$.

Of fundamental importance in Sturmian theory, is the relation between solutions of the linear equation and a related nonlinear equation. Thus, if $L_1 x = 0$ is nonoscillatory on $[a, \infty)$, there is a $T \in [a, \infty)$ and a solution x with $x(t) > 0$ on $[T, \infty)$.
If we make the Riccati substitution

$$z(t) := \frac{r_1(t) x^{\Delta}(t)}{x(t)}, \quad t \geq T,$$

then z is a solution of the Riccati equation

$$R_1 z = z^{\Delta} + q_1(t) + \frac{z^2}{r_1(t) + \mu(t) z} = 0 \tag{8}$$

on $[T, \infty)$ and satisfies

$$r_1(t) + \mu(t) z(t) > 0 \tag{9}$$

on $[T, \infty)$.

Now let us briefly indicate some of the main points in the proof of the Hille-Wintner Theorem. Suppose then that z is a solution of the Riccati equation

$$R_1 z = z^{\Delta} + q_1(t) + \frac{z^2}{r_1(t) + \mu(t)z} = 0 \qquad (10)$$

on $[T, \infty)$, and denote

$$F(t) := \frac{z^2(t)}{r_1(t) + \mu(t)z(t)} \geq 0.$$

Then integrating the Riccati equation gives

$$z(t) + \int_T^t q_1(s)\Delta s + \int_T^t F(s)\Delta s = z(T). \qquad (11)$$

One can easily show that $z(t) > 0$ for all large $t \geq T$. Since the first integral in (11) converges, it follows that

$\int_T^\infty F(s)\Delta s < \infty.$

Therefore from (11) we get that

$\lim_{t\to\infty} z(t)$ exists.

We may now show that

$\lim_{t\to\infty} z(t) = 0.$

Let $\{t_k\} \subset \hat{\mathbb{T}}$ with $\lim_{k\to\infty} t_k = \infty$. From (11) and $z(t) > 0$ we have

$$\int_T^{t_k} q_1(s)\Delta s + \int_T^{t_k} F(s)\Delta s \le z(T).$$

We recall that $\int_t^{\sigma(t)} g(s)\Delta s = \mu(t)g(t)$ and so we have, for n_0 sufficiently large,

$$\sum_{k=n_0}^{\infty} \mu(t_k)F(t_k) = \sum_{k=n_0}^{\infty} \int_{t_k}^{\sigma(t_k)} F(t)\Delta t \leq \int_T^{\infty} F(s)\Delta s < \infty.$$

Therefore it follows that

$$0 = \lim_{k\to\infty} \mu(t_k)F(t_k)$$

$$= \lim_{k\to\infty} \frac{\mu(t_k)z^2(t_k)}{r_1(t_k)+\mu(t_k)z(t_k)}$$

$= \lim_{k\to\infty} \frac{z^2(t_k)}{\frac{r_1(t_k)}{\mu(t_k)}+z(t_k)}$. Hence given an $\varepsilon > 0$ there is a positive integer k_0 such that

$$0 < \frac{z^2(t_k)}{\frac{r_1(t_k)}{\mu(t_k)} + z(t_k)} < \varepsilon$$

for $k \geq k_0$. This implies that

$$z^2(t_k) < \varepsilon \left(\frac{r_1(t_k)}{\mu(t_k)} + z(t_k) \right),$$

which implies that

$$\left(z(t_k) - \frac{\varepsilon}{2}\right)^2 < \frac{\varepsilon^2}{4} + \varepsilon \frac{r_1(t_k)}{\mu(t_k)} \leq \frac{\varepsilon^2}{4} + \varepsilon M.$$

Therefore

$$\left|z(t_k) - \frac{\varepsilon}{2}\right| < \frac{\varepsilon}{2} + \sqrt{\varepsilon M}$$

and consequently

$$|z(t_k)| < \varepsilon + \sqrt{\varepsilon M}.$$

Since $\varepsilon > 0$ is arbitrary, we get that

$$\lim_{k\to\infty} z(t_k) = 0.$$

But from (11) we know that $\lim_{t\to\infty} z(t) = z_0 \geq 0$ exists, and hence we get the desired result

$$\lim_{t\to\infty} z(t) = 0.$$

Now letting $t \to \infty$ in (11) we get

$$\int_T^\infty q_1(s)\Delta s + \int_T^\infty F(s)\Delta s = z(T).$$

Define, for large t,

$$v(t) := \int_t^\infty q_2(s)\Delta s + \int_t^\infty F(s)\Delta s,$$

so it follows that

$$0 \le v(t) \le \int_t^\infty q_1(s)\Delta s + \int_t^\infty F(s)\Delta s = z(t).$$

Note that

$$v^\Delta(t) = -q_2(t) - F(t) = -q_2(t) - \frac{z^2(t)}{r_1(t) + \mu(t)z(t)}.$$

We now claim that

$$\frac{z^2(t)}{r_1(t) + \mu(t)z(t)} \ge \frac{v^2(t)}{r_1(t) + \mu(t)v(t)}, \quad t \ge T.$$

This follows from the fact that for each fixed t,

$$H(w) := \frac{w^2}{r_1(t) + \mu(t)w}$$

is strictly increasing for $w \geq 0$ and the fact that $v(t) \geq 0$.
But this implies that the Riccati dynamic inequality

$$v^{\Delta} + q_2(t) + \frac{v^2}{r_1(t) + \mu(t)v} \leq 0$$

has a solution on $[T, \infty)$ with $r_1(t) + \mu(t)v(t) > 0$, and this means that

the Riccati equation

$$v^{\Delta} + q_2(t) + \frac{v^2}{r_1(t) + \mu(t)v} = 0$$

has a solution on $[T, \infty)$ with $r_1(t) + \mu(t)v(t) > 0$

and this means, in turn, that the equation

$$(r_1(t)x^{\Delta})^{\Delta} + q_2(t)x^{\sigma} = 0$$

is nonoscillatory on $[a, \infty)$. Since $0 < r_1(t) \leq r_2(t)$ we have by the Sturm comparison theorem that $L_2 x = 0$ is nonoscillatory on $[a, \infty)$.

Remark. If $\mathbb{T} = \mathbb{R}$, $r_i(t) \equiv 1$, $i = 1, 2$, and

$$0 \leq \int_t^\infty q_2(s)\Delta s \leq \int_t^\infty q_1(s)\Delta s \tag{12}$$

holds, then the above result was first obtained by Hille with the additional assumption that the $q_i(t)$, $i = 1, 2$, are positive.

Wintner showed later that (12) is sufficient for the conclusion to hold without the assumption of positivity.

Taam showed finally that the conclusion of the theorem holds with

$$\left|\int_t^\infty q_2(s)\Delta s\right| \leq \int_t^\infty q_1(s)\Delta s \tag{13}$$

replacing (12).
However, for the general case of time scales, additional assumptions are needed to obtain the analogous result when (13) replaces (12).

Examples

Example 1. Consider the 'Euler–Cauchy' like equation

$$x^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)} x^{\sigma} = 0. \tag{14}$$

For $\gamma < \frac{1}{4}$ it is known (Řehák), that (14) is nonoscillatory near infinity provided

$$\frac{\mu(t)}{t} \to 0 \quad \text{as } t \to \infty. \tag{15}$$

Since $\int_t^{\infty} \frac{\gamma}{s\sigma(s)} \Delta s = \frac{\gamma}{t}$, we get from Theorem 1 that if (15) holds, if $\mu(t)$ is bounded below by a positive number for $t \in \hat{\mathbb{T}}$, and if

$$0 \leq \limsup_{t\to\infty} \, t \int_t^{\infty} q(s)\Delta s < \frac{1}{4},$$

then $x^{\Delta\Delta} + q(t)x^{\sigma} = 0$ is nonoscillatory on $[a, \infty)$.

To see that the above result is sharp we note the following example.

Example 2. If

$$\liminf_{t\to\infty} t\int_t^\infty q(s)\Delta s > \frac{1}{4},$$

then $x^{\Delta\Delta} + q(t)x^\sigma = 0$ is oscillatory on $[a, \infty)$. One may show that this follows from a linear version of a result of Erbe, Peterson, and Saker.

In Example 1 we assumed that $\mathbb{T}$ was a time scale satisfying (15). A time scale that is important in the theory of orthogonal polynomials and quantum theory is $\mathbb{T} = q^{\mathbb{N}_0} := \{1, q, q^2, q^3, \cdots\}$. For this time scale $\mu(t) = (q-1)t$ so (15) is not satisfied. We now give an application of Theorem 1 for this time scale.

Example 3. It is not difficult to show that $x(t) = t^\alpha$ is a solution of the dynamic equation

$$x^{\Delta\Delta} + \frac{C_\alpha}{t\sigma(t)}x^\sigma = 0, \tag{16}$$

where $C_\alpha := \frac{(q^\alpha-1)(q^{1-\alpha}-1)}{(q-1)^2}$, for $t \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. If we let $0 < \alpha < 1$, then $C_\alpha > 0$. We have with $t = q^n$,

$$\int_t^\infty \frac{C_\alpha}{s\sigma(s)} \Delta s = \frac{C_\alpha}{t} = \frac{(q^\alpha - 1)(q^{1-\alpha} - 1)}{(q-1)^2 q^n}$$

and so if $Q(t)$ is defined on $\mathbb{T}$, then

$$\int_t^\infty Q(s)\Delta s = (q-1) \sum_{k=n}^{\infty} q^k Q(q^k).$$

Hence, if

$$0 \leq \sum_{k=n}^{\infty} q^k Q(q^k) \leq \frac{(q^\alpha - 1)(q^{1-\alpha} - 1)}{(q-1)^3 q^n},$$

for all large n, then by Theorem 1, $x^{\Delta\Delta} + Q(t)x^\sigma = 0$, is nonoscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$.

Remark 2. It is known (see Example 4.48 of Bohner-Peterson) that $x^{\Delta\Delta} + \frac{c}{(q-1)t\sigma(t)} x^\sigma = 0$ is oscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, if $c > 1$. We can use the result of Example 3 to show that this is sharp. To see this, notice that for fixed $0 < \alpha < 1$, $h(x) := \frac{(x^{\alpha}-1)(x^{1-\alpha}-1)}{x-1}$ satisfies $h'(x) > 0$, $x > 1$ and $\lim_{x\to\infty} h(x) = 1$. Hence given any $c_0 < 1$, we can chose q sufficiently large so that $h(q) = (q-1)C_\alpha > c_0$ and so since (16) has the nonoscillatory solution $x(t) = t^\alpha$ on $\mathbb{T} = q^{\mathbb{N}_0}$, it follows by the Sturm comparison theorem that $x^{\Delta\Delta} + \frac{c_0}{(q-1)t\sigma(t)} x^\sigma = 0$ is nonoscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$. We note that the results in the above examples may not be established by any other criteria known to the authors. More sophisticated examples are also easily given.

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