An Introduction to the Exponential Function on a Time Scale

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Remark: The unique solution of the IVP

$$x' = px, \quad x(0) = 1,$$

is the exponential function

$$x(t) = e^{pt}.$$

For $p \in \mathcal{R}$, let $e_p(t, t_0)$ be the unique solution of the IVP

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1.$$

Example 1. *Find* $e_p(t,0)$ *for* $\mathbb{T} = h\mathbb{Z}$ *and* $p(t) \equiv p$:

$$x^{\Delta}(t) = px(t), \quad x(0) = 1$$

$$\begin{aligned}
\frac{x(\sigma(t)) - x(t)}{\mu(t)} &= px(t) \\
\frac{x(t+h) - x(t)}{h} &= px(t) \\
x(t+h) &= (1+hp)x(t)
\end{aligned}$$

$$\begin{aligned}
x(h) &= (1+hp)x(0) = (1+hp) \\
x(2h) &= (1+hp)x(h) = (1+hp)^2 \\
x(3h) &= (1+hp)x(2h) = (1+hp)^3 \\
&\cdots
\end{aligned}$$

$$x(t) = e_p(t,0) = (1+hp)^{\frac{t}{h}}.$$

Example 2. *Find* $e_1(t,0)$ *if* $\mathbb{T} = \mathbb{N}_0^2 := \{t = n^2 : n \in \mathbb{N}_0\}$. *Here* $\sigma(t) = (n+1)^2$, *so* $\mu(t) = \sigma(t) - t = (n+1)^2 - n^2 = 2n+1 = 2\sqrt{t}+1$.

Now $y(t) := e_1(t,0)$ *solves the IVP*

$$y^\Delta = 1 \cdot y, \quad y(0) = 1.$$

$$\begin{aligned} y^\Delta(t) &= 1 \cdot y(t) \\ \frac{y(\sigma(t)) - y(t)}{\mu(t)} &= y(t) \\ y(\sigma(t)) - y(t) &= (1 + 2\sqrt{t})y(t) \\ y(\sigma(t)) &= 2(1+\sqrt{t})y(t). \end{aligned}$$

Interating

$$y(\sigma(t)) = 2(1+\sqrt{t})y(t)$$

we get

$$\begin{aligned} y(1^2) &= 2(1+\sqrt{0})y(0) = 2\cdot 1 \\ y(2^2) &= 2(1+\sqrt{1^2})y(1^2) = 2^2\cdot 2! \\ y(3^2) &= 2(1+\sqrt{2^2})y(2^2) = 2^3\cdot 3! \\ y(t) &= y(n^2) = 2^n\, n! \end{aligned}$$

Hence $\qquad e_1(t,0) = 2^{\sqrt{t}}(\sqrt{t})!.$

Consider for $p, q \in \mathcal{R}$,

$$\begin{aligned} e_p(t,0)e_q(t,0) &= (1+hp)^{\frac{t}{h}}(1+hq)^{\frac{t}{h}} \\ &= \{(1+hp)(1+hq)\}^{\frac{t}{h}} \\ &= (1+hp+hq+h^2pq)^{\frac{t}{h}} \\ &= \{1+h(p+q+hpq)\}^{\frac{t}{h}} \\ &= \{1+h(p\oplus q)\}^{\frac{t}{h}} \\ &= e_{p\oplus q}(t,0), \end{aligned}$$

where

$$p \oplus q := p + q + hpq.$$

We say $p : \mathbb{T} \to \mathbb{R}$ is **regressive** provided

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}.$$

Define the set of regressive functions by

$$\mathcal{R} := \{\text{regressive, right-dense continuous functions}\}$$

and the set of all positively regressive functions by

$$\mathcal{R}^+ := \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0\}.$$

Theorem 1. *For $p, q \in \mathcal{R}$, define*

$$p \oplus q := p + q + \mu pq,$$

then $(\mathcal{R}, \oplus)$ is an Abelian group.

$p(t) \equiv 0$ is the additive identity

Additive Inverse: Want

$$p \oplus q = 0$$

$$p + q + \mu pq = 0$$

$$(1 + \mu p)q = -p$$

$$\ominus p := \frac{-p}{1 + \mu p}.$$

Theorem 2. *Assume* $p, q \in \mathcal{R}$, *then*

(i) $e_0(t,s) \equiv 1,\ e_p(t,t) \equiv 1;$

(ii) $e_p^{\Delta}(t,s) = p(t)e_p(t,s);$

(iii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s);$

(iv) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s);$

(v) $e_p(t,s)e_p(s,r) = e_p(t,r);$

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s);$

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s).$

Product Rule:

$$(f(t)g(t))^{\Delta} = f(\sigma(t))g^{\Delta}(t) + f^{\Delta}(t)g(t) :$$

Quotient Rule:

$$\left(\frac{f(t)}{g(t)}\right)^{\Delta} = \frac{g(t)f^{\Delta}(t) - g^{\Delta}(t)f(t)}{g(t)g(\sigma(t))}.$$

Proof of

$$e_p(t,t_0)e_q(t,t_0) = e_{p\oplus q}(t,t_0):$$

Let

$$x(t) = e_p(t,t_0)e_q(t,t_0).$$

Then

$$\begin{aligned}
x^{\Delta}(t) &= e_p(\sigma(t),t_0)e_q^{\Delta}(t,t_0) + e_p^{\Delta}(t,t_0)e_q(t,t_0) \\
&= (1+\mu(t)p(t)e_p(t,t_0))q(t)e_q(t,t_0) \\
&+ p(t)e_p(t,t_0)e_q(t,t_0) \\
&= (p(t)+q(t)+\mu(t)p(t)q(t))e_p(t,t_0)e_q(t,t_0) \\
&= (p\oplus q)(t)x(t).
\end{aligned}$$

Also $x(t_0) = 1 \cdot 1 = 1$.

Proof of

$$\frac{1}{e_p(t,t_0)} = e_{\ominus p}(t,t_0).$$

Let

$$x(t) = \frac{1}{e_p(t,t_0)}.$$

Then

$$\begin{aligned} x^\Delta(t) &= \frac{-e_p^\Delta(t,t_0)}{e_p(\sigma(t),t_0)e_p(t,t_0)} \\ &= \frac{-p(t)e_p(t,t_0)}{(1+\mu(t)p(t))e_p(t,t_0)e_p(t,t_0)} \\ &+ \frac{-p(t)}{(1+\mu(t)p(t))e_p(t,t_0)} \\ &= (\ominus p)(t)x(t). \end{aligned}$$

Also $x(t_0) = 1$.

Theorem 3 (Keller's Chain Rule). *Assume* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}$*. Assume further that* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable and satisfies*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t).$$

Example 3.

$$f(x) = e^{2x},\ g(t) = t^2,\ \mathbb{T} = \mathbb{Z}.$$

$$\begin{aligned}
(f \circ g)^{\Delta}(t) &= \int_0^1 2e^{2(g(t)+h\mu(t)g^{\Delta}(t))}\, dh\, g^{\Delta}(t) \\
&= \int_0^1 2e^{2(t^2+h(2t+1))}\, dh\, (2t+1) \\
&= e^{2t^2} \int_0^1 e^{2(2t+1)h} 2(2t+1)\, dh \\
&= e^{2t^2} \left[e^{2(2t+1)h} \right]_0^1 \\
&= (e^{2(t+1)^2} - e^{2t^2}) \\
&= (e^{2t^2})^{\Delta}.
\end{aligned}$$

$$
\begin{aligned}
(x^{\alpha}(t))^{\Delta} &= \int_0^1 \alpha(x(t)+h\mu(t)x^{\Delta}(t))^{\alpha-1}dh\ x^{\Delta}(t) \\
&= \alpha\frac{x^{\Delta}}{x}(t)\int_0^1 (1+\mu(t)\frac{x^{\Delta}}{x}(t)h)^{\alpha-1}dh\ x^{\alpha}(t)
\end{aligned}
$$

Definition 1. *For $\alpha \in \mathbb{R}$ and $p \in R^{+}$*

$$
(\alpha \odot p)(t) := \alpha p(t)\int_0^1 (1+\mu(t)p(t)h)^{\alpha-1}dh
$$

Remark 1. *If $\mathbb{T} = \mathbb{R}$, then $\alpha \odot p = \alpha p$.*

Power Rule:

$$(x^\alpha)^\Delta(t) = \left(\alpha \odot \frac{x^\Delta}{x}\right)(t)\, x^\alpha(t)$$

Theorem 4. *Assume* $\alpha \in \mathbb{R}$ *and* $p \in R^+$, *then*

$$e_p^{\alpha}(t, t_0) = e_{\alpha \odot p}(t, t_0).$$

Proof

Let $y(t) := e_p^\alpha(t, t_0)$, then

$$\begin{aligned} y^\Delta(t) &= \left(\alpha \odot \frac{e_p^\Delta}{e_p}\right)(t)\, e_p^\alpha(t, t_0) \\ &= (\alpha \odot p)(t) y(t) \end{aligned}$$

Also $y(t_0) = 1$ and consequently

$$e_p^\alpha(t, t_0) = e_{\alpha \odot p}(t, t_0).$$

Theorem 5. $(R^+, \oplus, \odot)$ *is a vector space.*

Claim $\alpha \odot (\beta \odot p) = (\alpha\beta) \odot p$:

Proof:

$$\begin{aligned}
e_{\alpha\odot(\beta\odot p)}(t,t_0) &= e^{\alpha}_{\beta\odot p}(t,t_0)\\
&= (e^{\beta}_{p}(t,t_0))^{\alpha}\\
&= e^{\alpha\beta}_{p}(t,t_0)\\
&= e_{(\alpha\beta)\odot p}(t,t_0).
\end{aligned}$$

Hence

$$\alpha \odot (\beta \odot p) = (\alpha\beta) \odot p.$$

Claim $\alpha \odot (p \oplus q) = (\alpha \odot p) \oplus (\beta \odot q)$:

Proof:

$$\begin{aligned}
e_{\alpha\odot(p\oplus q)}(t,t_0) &= e^{\alpha}_{p\oplus q}(t,t_0) \\
&= (e_p(t,t_0)e_q(t,t_0))^{\alpha} \\
&= e^{\alpha}_{p}(t,t_0)e^{\alpha}_{q}(t,t_0) \\
&= e_{\alpha\odot p}(t,t_0)e_{\alpha\odot q}(t,t_0) \\
&= e_{(\alpha\odot p)\oplus(\alpha\odot q}(t,t_0).
\end{aligned}$$

Hence

$$\alpha \odot (p \oplus q) = (\alpha \odot p) \oplus (\beta \odot q).$$

Similarly

$$1 \odot p = p$$

and

$$(\alpha + \beta) \odot p = (\alpha \odot p) \oplus (\beta \odot p).$$

Look for solutions of

$$x^{\Delta\Delta} + ax^{\Delta} + bx = 0$$

of the form

$$x(t) = e_{\lambda}(t, t_0).$$

$$x^{\Delta\Delta}(t) + ax^{\Delta}(t) + bx(t) = (\lambda^2 + a\lambda + b)e_{\lambda}(t, t_0).$$

Want

$$\lambda^2 + a\lambda + b = 0.$$

Assume the regressivity condition

$$1 - a\mu(t) + b\mu^2(t) \neq 0.$$

If this is true, then $\lambda_1, \lambda_2 \in \mathcal{R}$.

Case 1: $\lambda_1 \neq \lambda_2$, both real.

$$x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0).$$

Case 2: $\lambda_1 = \lambda_2$.

$$x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_1}(t, t_0) \int_{t_0}^{t} \frac{1}{1 + \mu(\tau)\lambda_1} \Delta\tau.$$

If $\mu p^2 \in \mathcal{R}$ and $p \in C_{rd}$, then

$$\cos_p(t, t_0) := \frac{e_{ip}(t, t_0) + e_{-ip}(t, t_0)}{2},$$

and

$$\sin_p(t, t_0) := \frac{e_{ip}(t, t_0) - e_{-ip}(t, t_0)}{2i}.$$

Properties of trig functions:

(i) $\cos_p^\Delta(t, t_0) = -p(t)\sin_p(t, t_0)$;

(ii) $\sin_p^\Delta(t, t_0) = p(t)\cos_p(t, t_0)$;

(iii) $\sin_p^2(t, t_0) + \cos_p^2(t, t_0) = e_{\mu p^2}(t, t_0)$.

Case 3: $\lambda_{1,2} = p \pm iq$.

$$x(t) = c_1 e_p(t,t_0) \cos_{\frac{q}{1+\mu p}}(t,t_0) + c_2 e_p(t,t_0) \sin_{\frac{q}{1+\mu p}}(t,t_0).$$

Laplace Transforms

Assume $\mathbb{T}_0 \subset \mathbb{R}^+$ and $0 \in \mathbb{T}_0$.

$$L\{x\}(z) = \int_0^\infty x(t) e_{\ominus z}(\sigma(t), 0) \Delta t.$$

Example 4. *Use Laplace transforms to solve the IVP*

$$x^{\Delta\Delta} - 4x^{\Delta} + 13x = 0, \quad x(0) = 1, \quad x^{\Delta}(0) = 1. \quad (1)$$

Assuming x is the solution of (1) *we obtain after taking the Laplace transform of both sides of the dynamic equation in* (1)

$$z^2\mathcal{L}\{x\}(z) - zx_0 - x_0^{\Delta} - 4[z\mathcal{L}\{x\}(z) - x_0] + 13\mathcal{L}\{x\}(z) = 0.$$

Using the initial conditions we get

$$z^2\mathcal{L}\{x\}(z) - z - 1 - 4[z\mathcal{L}\{x\}(z) - 1] + 13\mathcal{L}\{x\}(z) = 0.$$

Solving for $\mathcal{L}\{x\}(z)$ we obtain

$$\mathcal{L}\{x\}(z) = \frac{z-3}{z^2 - 4z + 9} = \frac{z-2}{(z-2)^2 + 9} - \frac{1}{3}\frac{3}{(z-2)^2 + 9}.$$

Using the Shifting Theorem we get that the solution of (1) *is*

$$x(t) = e_2(t,0)\cos_{\frac{3}{1+2\mu}}(t,0) - \frac{1}{3}e_{\frac{3}{1+2\mu}}(t,0)\sin_3(t,0).$$

Example 5. *Use Laplace transforms to solve the IVP*

$$x^{\Delta\Delta} + 4x = e_3(t,0), \quad x(0) = x^{\Delta}(0) = 0.$$

Let x be the solution of the given IVP. Then taking the Laplace transform of both sides of the given dynamic equation gives

$$z^2\mathcal{L}\{x\}(z) - zx_0 - x_0^{\Delta} + 4\mathcal{L}\{x\}(z) = \frac{1}{z-3}.$$

Hence

$$\mathcal{L}\{x\}(z) = \frac{1}{z-3}\frac{1}{z^2+4} = \frac{1}{2}\frac{1}{z-3}\frac{2}{z^2+4}.$$

From the convolution theorem,

$$x(t) = \frac{1}{2}(e_3(\cdot,0) * \sin_2(\cdot,0))(t).$$

It follows that

$$x(t) = \frac{1}{13}e_3(t,0) - \frac{3}{26}\sin_2(t,0) - \frac{1}{13}\cos_2(t,0).$$

Definition 2. *For* $h > 0$*, we define the cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h}\,\mathrm{Log}(1 + zh),$$

where Log *is the principal logarithm function. For* $h = 0$*, we define* $\xi_0(z) = z$ *for all* $z \in \mathbb{C}$.

Definition 3. *If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \qquad \textit{for} \qquad s,t \in \mathbb{T}, \tag{2}$$

where $\xi_h(z)$ is the cylinder transformation.

Theorem 6. *If* $p \in \mathcal{R}$, *then the semigroup property*

$$e_p(t,r)e_p(r,s) = e_p(t,s) \quad \textit{for all} \quad r,s,t \in \mathbb{T}$$

is satisfied.

Theorem 7. *If* $p \in \mathcal{R}^+$, *then*

$$e_p(t, t_0) > 0.$$

Theorem 8. *Assume* $p \in \mathcal{R}$ *and* $t_0 \in \mathbb{T}$. *If* $1+\mu p < 0$ *on* $\mathbb{T}^\kappa$, *then*

$$e_p(t, t_0) = \alpha(t, t_0)(-1)^{n_t}$$

for all $t \in \mathbb{T}$, *where*

$$\alpha(t, t_0) := \exp\left(\int_{t_0}^{t} \frac{\log|1 + \mu(\tau)p(\tau)|}{\mu(\tau)} \Delta\tau\right) > 0$$

and

$$n_t = \begin{cases} |[t_0, t)| & \text{if} \quad t \geq t_0 \\ |[t, t_0)| & \text{if} \quad t < t_0. \end{cases}$$

Proof. Since $1 + \mu(t)p(t) < 0$, we have

$$\mathrm{Log}[1+\mu(t)p(t)] = \log|1+\mu(t)p(t)|+i\pi \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

Note also that $\mu(t)$ never vanishes in this case and that

$$n_t < \infty$$

Then

$$
\begin{aligned}
e_p(t,t_0) &= \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\
&= \exp\left(\int_{t_0}^{t} \frac{\operatorname{Log}(1+\mu(\tau)p(\tau))}{\mu(\tau)}\Delta\tau\right) \\
&= \exp\left(\int_{t_0}^{t} \frac{\log|1+\mu(\tau)p(\tau)|+i\pi}{\mu(\tau)}\Delta\tau\right) \\
&= \exp\left(\int_{t_0}^{t} \left\{\frac{\log|1+\mu(\tau)p(\tau)|}{\mu(\tau)}+\frac{i\pi}{\mu(\tau)}\right\}\Delta\tau\right) \\
&= \alpha(t,t_0)\exp\left(i\pi\int_{t_0}^{t}\frac{\Delta\tau}{\mu(\tau)}\right).
\end{aligned}
$$

By *Euler's formula*,

$$\begin{aligned}\exp\left(i\pi\int_{t_0}^{t}\frac{\Delta\tau}{\mu(\tau)}\right) &= \cos\left(\pi\int_{t_0}^{t}\frac{\Delta\tau}{\mu(\tau)}\right)\\ &= \cos(n_t\pi)\\ &= (-1)^{n_t}.\end{aligned}$$

$\square$