





# An Introduction to Special Functions

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# Chapter 1

## Classical Special Functions

Our main objective in this chapter is to present in a compact form main facts about the classical special functions of hypergeometric type, i.e., classical orthogonal polynomials (Jacobi, Laguerre and Hermite) and functions of the second kind, hypergeometric functions, confluent hypergeometric functions and Bessel functions, based on the second order differential equation they satisfy. This gives a foundation for our study of difference hypergeometric functions,  $q$ -orthogonal polynomials,  $q$ -beta integrals and biorthogonal rational functions in the subsequent chapters.

### 1.1 Classical Hypergeometric Functions

Classical orthogonal polynomials, hypergeometric functions and Bessel functions are particular solutions of the differential equation

$$(1.1.1) \quad \sigma(x)y'' + \tau(x)y' + \lambda y = 0,$$

where  $\sigma(x)$  and  $\tau(x)$  are polynomials of respective degrees at most two and one, and  $\lambda$  is a constant. Equation (1.1.1) can also be rewritten in the self-adjoint form

$$(1.1.2) \quad (\sigma\rho y')' + \lambda\rho y = 0, \quad (\sigma\rho)' = \tau\rho.$$

We shall refer to (1.1.1) as an *equation of hypergeometric type*, and its solutions as *functions of hypergeometric type*. Generally speaking, these functions can be studied in a domain of the complex plane. In this case we shall usually use the complex variable  $z$  instead of  $x$ .

### 1.1.1 Power series method

It is convenient to construct particular solutions of equation (1.1.1) by using the power series method (see, for example, the classical work of Boole [1959]).

**Theorem 1.1** *Let  $a$  be a root of the equation  $\sigma(x) = 0$ . Then (1.1.1) has a particular solution of the form*

$$(1.1.3) \quad y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where

$$(1.1.4) \quad \frac{c_{n+1}}{c_n} = -\frac{\lambda + n(\tau' + (n-1)\sigma''/2)}{(n+1)(\tau(a) + n\sigma'(a))},$$

if:

- (i)  $\lim_{m \rightarrow \infty} \frac{d^k}{dx^k} y_m(x) = \frac{d^k}{dx^k} y(x)$  with  $k = 0, 1, 2$ ;
- (ii)  $\lim_{m \rightarrow \infty} (\lambda - \lambda_m) c_m (x-a)^m = 0$ .

(Here  $y_m(x) = \sum_{n=0}^m c_n (x-a)^n$  and  $\lambda_m = -m\tau' - \frac{1}{2}m(m-1)\sigma''$ .)

In the case  $\sigma(x) = \text{constant} \neq 0$  series (1.1.3) satisfies (1.1.1) when  $a$  is a root of the equation  $\tau(x) = 0$ ,

$$(1.1.5) \quad \frac{c_{n+2}}{c_n} = -\frac{\lambda + n\tau'}{(n+1)(n+2)\sigma}$$

and convergence conditions (i)–(ii) are valid.

The **proof** of Theorem 1.1 follows from the identity

$$(1.1.6) \quad \rho^{-1} \frac{d}{dx} \left[ \sigma \rho \frac{d}{dx} (x - \xi)^n \right] = n(n-1)\sigma(\xi)(x - \xi)^{n-2} \\ + n\tau_{n-1}(\xi)(x - \xi)^{n-1} - \lambda_n(x - \xi)^n,$$

where  $\tau_m(\xi) = \tau(\xi) + m\sigma'(\xi)$  and  $\lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''$ , which can be easily verified (see Ex. 1.1).

In fact, for a partial sum of the series (1.1.3) we can write

$$\left[ \sigma(x) \frac{d^2}{dx^2} + \tau(x) \frac{d}{dx} + \lambda \right] y_m(x) \\ = \sigma(a) \sum_{n=0}^m c_n n(n-1)(x-a)^{n-2} \\ + \sum_{n=0}^m c_n n\tau_{n-1}(a)(x-a)^{n-1} \\ + \sum_{n=0}^m c_n (\lambda - \lambda_n)(x-a)^n.$$

By the hypothesis  $\sigma(a) = 0$ , the first term in the right side is equal to zero. Equating coefficients in the next two terms with the aid of

$$\frac{c_{n+1}}{c_n} = \frac{\lambda_n - \lambda}{(n+1)\tau_n(a)},$$

which is equivalent to (1.1.4), we get

$$(1.1.7) \quad \left[ \sigma(x) \frac{d^2}{dx^2} + \tau(x) \frac{d}{dx} + \lambda \right] y_m(x) = c_m (\lambda - \lambda_m) (x-a)^m.$$

Taking the limit  $m \rightarrow \infty$  we prove the first part of the theorem under convergence conditions (i)–(ii).

When  $\sigma = \text{constant}$  we can obtain in the same manner

$$(1.1.8) \quad \left[ \sigma(x) \frac{d^2}{dx^2} + \tau(x) \frac{d}{dx} + \lambda \right] y_m(x) \\ = \sigma \sum_{n=0}^m c_n n(n-1)(x-a)^{n-2} \\ + \sum_{n=0}^m c_n (\lambda - \lambda_n)(x-a)^n \\ = c_m (\lambda - \lambda_m) (x-a)^m,$$

which proves the second part of the theorem in the limit  $m \rightarrow \infty$ .  $\square$

**Corollary** Equation (1.1.1) has polynomial solutions  $y_m(x)$  corresponding to the eigenvalues  $\lambda = \lambda_m = -m\tau' - \frac{1}{2}m(m-1)\sigma''$ ,  $m = 0, 1, 2, \dots$

(This follows from (1.1.7) and (1.1.8).)

**Examples** With the aid of linear transformations of the independent variable, equation (1.1.1) for  $\tau' \neq 0$  can be reduced to one of the following *canonical forms*

$$\begin{aligned} x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y &= 0, \\ xy'' + (\gamma - x)y' - \alpha y &= 0, \\ y'' - 2xy' + 2\nu y &= 0. \end{aligned}$$

According to (1.1.3)–(1.1.5) the appropriate particular solutions are:

the *hypergeometric function*,

$$y(x) = {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} x^n,$$

the *confluent hypergeometric function*,

$$y(x) = {}_1F_1(\alpha; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} x^n,$$

and the *Hermite function*,

$$\begin{aligned} y(x) = H_\nu(x) &= \frac{2^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; x^2\right) \\ &+ \frac{2^\nu \Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)} x {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; x^2\right) \\ &= \frac{1}{2\Gamma(-\nu)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n-\nu}{2}\right) \frac{(-2x)^n}{n!}, \end{aligned}$$

respectively. Here  $(a)_n = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a)$  and  $\Gamma(a)$  is the gamma function of Euler.

Generally speaking, these solutions arise under some restrictions on the variable and parameters. They can be extended to wider domains by analytic continuation.

**Definition** All the hypergeometric series above are special cases of the (*generalized*) *hypergeometric series* with  $r$  numerator parameters  $\alpha_1, \dots, \alpha_r$  and  $s$  denominator parameters  $\beta_1, \dots, \beta_s$  defined by

$$(1.1.9) \quad {}_rF_s(\alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \dots, \beta_s; x) = {}_rF_s \left( \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_r \\ \beta_1, \dots, \beta_s \end{matrix} ; x \right) \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_r)_n}{n! (\beta_1)_n \dots (\beta_s)_n} x^n,$$

where  $(a)_n = a(a+1)\dots(a+n-1)$  and  $(a)_0 = 1$ .

By the ratio test, the  ${}_rF_s$  series converges absolutely for all complex values of  $x$  if  $r \leq s$ , and for  $|x| < 1$  if  $r = s + 1$ . By an extension of the ratio test (Bromwich [1959], p. 241), it converges absolutely for  $|x| = 1$  if  $r = s + 1$  and  $x \neq 0$  or  $r = s + 1$  and  $\text{Re} [\beta_1 + \dots + \beta_s - (\alpha_1 + \dots + \alpha_r)] > 0$ . If  $r > s + 1$  and  $x \neq 0$  or  $r = s + 1$  and  $|x| > 1$ , then this series diverges, unless it terminates.

**Extended power series method** The solution (1.1.3)–(1.1.4) can be rewritten in the following explicit form

$$(1.1.10) \quad y(x) = c_0 \sum_{n=0}^{\infty} \prod_{k=0}^{n-1} \frac{(\lambda - \lambda_k)(a - x)}{\tau_k(a)(k+1)},$$

where  $c_0$  is a constant.

Using the expansion

$$y(x) = \sum_n c_n (x - \xi)^{\alpha+n}, \quad \frac{c_{n+1}}{c_n} = \frac{\lambda_{\alpha+n} - \lambda}{(\alpha + n + 1)\tau_{\alpha+n}(a)}$$

in the same manner one can find solutions of the more general form

$$(1.1.11) \quad y(x) = c_0 (x - a)^\alpha \sum_{n=0}^{\infty} \prod_{k=0}^{n-1} \frac{(\lambda - \lambda_{\alpha+k})(a - x)}{\tau_{\alpha+k}(a)(\alpha + k + 1)},$$

provided that  $\sigma(a) = 0$  and  $\alpha \tau_{\alpha-1}(a) = 0$  (in particular, putting  $\alpha = 0$  we recover (1.1.10)).

We can also satisfy (1.1.1) by using the series of the form

$$y(x) = \sum_n \frac{c_n}{(x-\xi)^{\alpha+n}}, \quad \frac{c_{n+1}}{c_n} = \frac{(\alpha+n)\tau_{-\alpha-n-1}(a)}{\lambda - \lambda_{-\alpha-n-1}},$$

if  $\sigma(a) = 0$  and  $\lambda = \lambda_{-\alpha}$ . Hence

$$(1.1.12) \quad y(x) = \frac{c_0}{(x-a)^\alpha} \sum_{n=0}^{\infty} \prod_{k=0}^{n-1} \frac{(\alpha+k)\tau_{-\alpha-k-1}(a)}{(\lambda - \lambda_{-\alpha-k-1})(x-a)}.$$

When  $\sigma = \text{constant} \neq 0$  we can write the solution as

$$(1.1.13) \quad y(x) = \sum_n \frac{c_n}{(x-a)^{\alpha+n}}, \quad \frac{c_{n+2}}{c_n} = -\frac{(\alpha+n)(\alpha+n+1)\sigma}{\lambda - \lambda_{-\alpha-n-2}},$$

if  $\tau(a) = 0$  and  $\lambda = \lambda_{-\alpha}$  (for even integer values of  $n$ ) or  $\lambda = \lambda_{-\alpha-1}$  (for odd integer values of  $n$ ).

## 1.1.2 Some solutions of hypergeometric equations

Let us apply the extended power series method to the main equations of hypergeometric type.

(1) Consider particular solutions of the *hypergeometric equation*,

$$(1.1.14) \quad x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0,$$

when  $\sigma(x) = x(1-x)$ ,  $\tau(x) = c - (a+b+1)x$  and  $\lambda = -ab$ . Here  $\tau_\mu(\xi) = c - (a+b+1)\xi + \mu(1-2\xi)$ ,  $\lambda_\mu = \mu(a+b+\mu)$  and  $\lambda_\mu - \lambda = (a+\mu)(b+\mu)$ . Equation  $\sigma(\xi) = \xi(1-\xi)$  has two roots.

Solution (1.1.10) for  $\sigma(0) = 0$  is the hypergeometric function,

$$(1.1.15) \quad y_1(x) = {}_2F_1(a, b; c; x), \quad |x| < 1;$$

for  $\sigma(1) = 0$  we get

$$(1.1.16) \quad y_2(x) = {}_2F_1(a, b; a + b - c + 1; 1 - x), \quad |1 - x| < 1.$$

The corresponding solutions (1.1.11) are

$$(1.1.17) \quad y_3(x) = x^{1-c} {}_2F_1(1 + a - c, 1 + b - c; 2 - c; x)$$

and

$$(1.1.18) \quad y_4(x) = (1 - x)^{c-a-b} {}_2F_1(c - b, c - a; c - a - b + 1; 1 - x).$$

Solutions (1.1.12) take the form

$$(1.1.19) \quad y_5(x) = x^{-a} {}_2F_1(a, a - c + 1; a - b + 1; x^{-1}), \quad |x| > 1,$$

$$(1.1.20) \quad y_6(x) = y_5(x)|_{a \leftrightarrow b};$$

and

$$(1.1.21) \quad y_7(x) = (x - 1)^{-a} {}_2F_1\left(a, c - b; a - b + 1; \frac{1}{1 - x}\right), \quad |1 - x| > 1,$$

$$(1.1.22) \quad y_8(x) = y_7(x)|_{a \leftrightarrow b}.$$

Any three of these solutions are linearly dependent. For example,

$$(1.1.23) \quad {}_2F_1(a, b; c; x) = (1 - x)^{c-a-b} {}_2F_1(c - a, c - b; c; x)$$

and

$$(1.1.24) \quad \begin{aligned} & {}_2F_1(a, b; a + b - c + 1; 1 - x) \\ &= \frac{\Gamma(a + b - c + 1)\Gamma(1 - c)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} {}_2F_1(a, b; c; x) \\ &+ \frac{\Gamma(a + b - c + 1)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} x^{1-c} {}_2F_1(1 + a - c, 1 + b - c; 2 - c; x). \end{aligned}$$

(2) For the *confluent hypergeometric equation*,

$$(1.1.25) \quad xy'' + (c - x)y' - ay = 0,$$

we have  $\sigma(x) = x$ ,  $\tau(x) = c - x$ ,  $\tau_\mu(x) = c + \mu - x$  and  $\lambda = -a$ . Particular solutions are

$$(1.1.26) \quad y_1(x) = {}_1F_1(a; c; x),$$

$$(1.1.27) \quad y_2(x) = x^{1-c} {}_1F_1(1 + a - c; 2 - c; x)$$

and

$$(1.1.28) \quad y_3(x) = x^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1 + a - c)_n}{n! (-x)^n}$$

(this formal series does not converge unless it terminates).

(3) In the case of the *Hermite equation*,

$$(1.1.29) \quad y'' - 2xy' + 2\nu y = 0,$$

when  $\sigma(x) = 1$ ,  $\tau(x) = -2x$  and  $\lambda = 2\nu$ , particular solutions have the forms

$$(1.1.30) \quad y_1(x) = H_\nu(x) = \frac{2^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; x^2\right) \\ + \frac{2^\nu \Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)} x {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; x^2\right)$$

and

$$(1.1.31) \quad y_2(x) = x^\nu \sum_{n=0}^{\infty} \frac{\left(-\frac{\nu}{2}\right)_n \left(\frac{1-\nu}{2}\right)_n}{n! (-x^2)^n}$$

(this  ${}_2F_0$  series diverges, unless it terminates).

(4) Finally, let us consider the equation

$$(1.1.32) \quad xy'' + cy' + \lambda y = 0$$

with  $\sigma(x) = x$ ,  $\tau(x) = c$  and  $\tau_\mu = c + \mu$ . Solutions (1.1.10) and (1.1.11) take the form

$$(1.1.33) \quad y_1(x) = {}_0F_1(-, c; -\lambda x) = \sum_{n=0}^{\infty} \frac{(-\lambda x)^n}{(c)_n n!}$$

and

$$(1.1.34) \quad y_2(x) = x^{1-c} {}_0F_1(-; 2 - c; -\lambda x),$$

respectively. These functions are closely related to the *Bessel function*

$$(1.1.35) \quad J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1)} {}_0F_1(-; \nu + 1; -x^2/4).$$

## 1.2 Integral Representations

The special functions of hypergeometric type are also easily studied by means of integral representations, which hold for solutions of the differential equation (1.1.1).

### 1.2.1 Main theorem

**Theorem 1.2** *Let  $\rho(z)$  satisfy the equation*

$$(1.2.1) \quad [\sigma(z)\rho(z)]' = \tau(z)\rho(z)$$

*and let  $\nu$  be a root of the equation*

$$(1.2.2) \quad \lambda + \nu\tau' + \frac{1}{2}\nu(\nu - 1)\sigma'' = 0.$$

*Then (1.1.1) has particular solution of the form*

$$(1.2.3) \quad y = y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\rho(s)}{(s - z)^{\nu+1}} ds,$$

*where  $C_\nu$  is a constant and  $C$  is a contour in the complex  $s$ -plane, if:*

(i) *the derivative of the integral*

$$\varphi_{\nu\mu}(z) = \int_C \frac{\rho_\nu(s)}{(s - z)^{\mu+1}} ds \quad \text{with} \quad \rho_\nu(s) = \sigma^\nu(s)\rho(s)$$

can be evaluated for  $\mu = \nu - 1$  and  $\mu = \nu$  by using the formula

$$\varphi'_{\nu\mu}(z) = (\mu + 1)\varphi_{\nu, \mu+1}(z);$$

(ii) the contour  $C$  is chosen so that the equality

$$\frac{\sigma(s)\rho_\nu(s)}{(s-z)^{\nu+1}} \Big|_{s_1}^{s_2} = 0$$

holds, where  $s_1$  and  $s_2$  are the end points of the contour  $C$ .

**Proof.** The function  $\rho_\nu(s) = \sigma^\nu(s)\rho(s)$  satisfies the equation

$$[\sigma(s)\rho_\nu(s)]' = \tau_\nu(s)\rho_\nu(s),$$

where  $\tau_\nu(s) = \tau(s) + \nu\sigma'(s)$ . We multiply both sides of this equality by  $(s-z)^{-\nu-1}$  and integrate over the contour  $C$ . Upon integrating by parts we obtain

$$(1.2.4) \quad \frac{\sigma(s)\rho_\nu(s)}{(s-z)^{\nu+1}} \Big|_{s_1}^{s_2} + (\nu+1) \int_C \frac{\sigma(s)\rho_\nu(s)}{(s-z)^{\nu+2}} ds = \int_C \frac{\tau_\nu(s)\rho_\nu(s)}{(s-z)^{\nu+1}} ds.$$

By hypothesis, the first term is equal to zero. We expand the polynomials  $\sigma(s)$  and  $\tau_\nu(s)$  in powers of  $s-z$ :

$$\begin{aligned} \sigma(s) &= \sigma(z) + \sigma'(z)(s-z) + \frac{1}{2}\sigma''(s-z)^2, \\ \tau_\nu(s) &= \tau_\nu(z) + \tau'_\nu(s-z). \end{aligned}$$

Taking into account the integral formulas for the functions  $\varphi_{\nu, \nu-1}$ ,  $\varphi_{\nu\nu}$  and  $\varphi_{\nu, \nu+1}$ , we arrive at the relation

$$(\nu+1) \left[ \sigma(z)\varphi_{\nu, \nu+1} + \sigma'(z)\varphi_{\nu\nu} + \frac{1}{2}\sigma''\varphi_{\nu, \nu-1} \right] = \tau_\nu(z)\varphi_{\nu\nu} + \tau'_\nu\varphi_{\nu, \nu-1}.$$

Upon substituting  $\tau_\nu = \tau + \nu\sigma'$  and using the formula  $\varphi'_{\nu\nu} = (\nu+1)\varphi_{\nu, \nu+1}$  we get

$$(1.2.5) \quad \sigma\varphi'_{\nu\nu} + (\sigma' - \tau)\varphi_{\nu\nu} = \left( \tau' + \frac{1}{2}(\nu-1)\sigma'' \right) \varphi_{\nu, \nu-1}.$$

At the same time, by differentiating the relation  $\sigma\rho y' = C_\nu\sigma\varphi_{\nu\nu}$  we find that

$$(1.2.6) \quad \frac{1}{C_\nu} \sigma \rho y' = \sigma \varphi'_{\nu\nu} + (\sigma' - \tau) \varphi_{\nu\nu}.$$

Comparing (1.2.5) and (1.2.6) we obtain

$$(1.2.7) \quad \sigma \rho y' = \kappa_\nu C_\nu \varphi_{\nu, \nu-1},$$

where  $\kappa_\nu = \tau' + (\nu - 1) \sigma''/2$ . Upon differentiating (1.2.7) we arrive at (1.1.1) in the self-adjoint form

$$(\sigma \rho y')' + \lambda \rho y = 0,$$

where  $\lambda = -\nu \kappa_\nu = -\nu \tau' - \nu(\nu - 1) \sigma''/2$ . This proves the theorem.  $\square$

In the proof of Theorem 1.3 we have, en route, deduced the formula (1.2.7), which is a simple *integral representation* for the first derivative of the function of hypergeometric type:

$$(1.2.8) \quad y'_\nu(z) = \frac{C_\nu^{(1)}}{\sigma(z)\rho(z)} \int_C \frac{\rho_\nu(s)}{(s-z)^\nu} ds,$$

where  $C_\nu^{(1)} = \kappa_\nu C_\nu = \left( \tau' + \frac{1}{2}(\nu - 1)\sigma'' \right) C_\nu$ . Hence

$$(1.2.9) \quad y_\nu^{(k)}(z) = \frac{C_\nu^{(k)}}{\rho_k(z)} \varphi_{\nu, \nu-k}(z) = \frac{C_\nu^{(k)}}{\sigma^k(z)\rho(z)} \int_C \frac{\rho_\nu(s)}{(s-z)^{\nu-k+1}} ds,$$

where  $C_\nu^{(k)} = \prod_{p=0}^{k-1} \left( \tau' + \frac{\nu + p - 1}{2} \sigma'' \right) C_\nu$ .

## 1.2.2 Integrals for hypergeometric and Bessel functions

Using Theorem 1.2 we can obtain integral representations for all the most commonly used special functions of hypergeometric type, in particular, for the hypergeometric functions:

$$(1.2.10) \quad {}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-zt)^{-\beta} dt,$$

$$(1.2.11) \quad {}_1F_1(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} e^{zt} dt,$$

$$(1.2.12) \quad H_\nu(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty e^{-t^2-2zt} t^{-\nu-1} dt.$$

Here  $\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$  and  $\operatorname{Re}(-\nu) > 0$ .

Let us mention also some solutions of the *Bessel equation*,

$$(1.2.13) \quad z^2 u'' + zu' + (z^2 - \nu^2) u = 0.$$

With the aid of the change of the function  $u = \varphi(z)y$  when  $\varphi(z) = z^\nu e^{iz}$  this equation can be reduced to the hypergeometric form

$$(1.2.14) \quad zy'' + (2iz + 2\nu + 1)y' + i(2\nu + 1)y = 0$$

and based on Theorem 1.2 one can obtain the *Poisson integral representations* for the Bessel function of the first kind,  $J_\nu(z)$ , and the Hankel functions of the first and second kind,  $H_\nu^{(1)}(z)$  and  $H_\nu^{(2)}(z)$ :

$$(1.2.15) \quad J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_{-1}^1 (1-t^2)^{\nu-1/2} \cos zt dt,$$

$$(1.2.16) \quad H_\nu^{(1,2)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{\pm i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})}}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-t} t^{\nu-1/2} \left(1 \pm \frac{it}{2z}\right)^{\nu-1/2} dt,$$

where  $\operatorname{Re} \nu > -1/2$ . It is then possible to deduce from these integral representations all the remaining properties of these functions. For details, see (Nikiforov and Uvarov [1988]).

### 1.3 Classical Orthogonal Polynomials

The Jacobi, Laguerre and Hermite polynomials are the simplest solutions of the equation (1.1.1).

### 1.3.1 Main property

By differentiating (1.1.1) we can easily verify that the function  $v_1(x) = y'(x)$  satisfies the equation of the same type

$$(1.3.1) \quad \sigma(x)v_1'' + \tau_1(x)v_1' + \mu_1v_1 = 0,$$

where  $\tau_1(x) = \tau(x) + \sigma'(x)$  is a polynomial of degree at most one and  $\mu_1 = \lambda + \tau'(x)$  is a constant.

The converse is also true: any solution of (1.3.1) is the derivative of a solution of (1.1.1) if  $\lambda = \mu_1 - \tau' \neq 0$ . Let  $v_1(x)$  be a solution of (1.3.1) and define the function

$$y(x) = -\frac{1}{\lambda} (\sigma(x)v_1' + \tau(x)v_1).$$

We have

$$\lambda y' = -(\sigma v_1'' + \tau_1 v_1' + \tau' v_1) = \lambda v_1$$

or  $v_1 = y'(x)$  and, therefore,  $y(x)$  satisfies (1.1.1).

### 1.3.2 Rodrigues formula

By differentiating (1.1.1)  $n$  times we obtain an equation of hypergeometric type for the function  $v_n(x) = y^{(n)}(x)$ ,

$$(1.3.2) \quad \sigma(x)v_n'' + \tau_n(x)v_n' + \mu_n v_n = 0,$$

where

$$(1.3.3) \quad \tau_n(x) = \tau(x) + n\sigma'(x),$$

$$(1.3.4) \quad \mu_n = \lambda + n\tau' + \frac{1}{2}n(n-1)\sigma''.$$

This property lets us construct the simplest solutions of (1.1.1) corresponding to some values of  $\lambda$ . Indeed, when  $\mu_n = 0$  equation (1.3.2) has the solution  $v_n = \text{constant}$ . Since  $v_n(x) = y^{(n)}(x)$ , the equation (1.1.1) has a particular solution  $y = y_n(x)$  which is a polynomial of degree  $n$  if

$$(1.3.5) \quad \lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma'' \quad (n = 0, 1, 2, \dots).$$

To find these polynomials explicitly let us rewrite equations (1.1.1) and (1.3.2) in the self-adjoint forms

$$(1.3.6) \quad (\sigma\rho y')' + \lambda\rho y = 0,$$

$$(1.3.7) \quad (\sigma\rho_n v_n')' + \mu_n\rho_n v_n = 0.$$

Functions  $\rho(x)$  and  $\rho_n(x)$  satisfy the first-order differential equations

$$(1.3.8) \quad (\sigma\rho)' = \tau\rho,$$

$$(1.3.9) \quad (\sigma\rho_n)' = \tau_n\rho_n.$$

So,

$$\frac{(\sigma\rho_n)'}{\rho_n} = \tau + n\sigma' = \frac{(\sigma\rho)'}{\rho} + n\sigma',$$

whence

$$\frac{\rho_n'}{\rho_n} = \frac{\rho'}{\rho} + n \frac{\sigma'}{\sigma}$$

and, consequently,

$$(1.3.10) \quad \rho_n(x) = \sigma^n(x)\rho(x).$$

Since  $\sigma\rho_n = \rho_{n+1}$  and  $v_n' = v_{n+1}$  we can rewrite (1.3.7) in the form

$$\rho_n v_n = -\frac{1}{\mu_n} (\rho_{n+1} v_{n+1})'.$$

Hence we obtain successively

$$\begin{aligned} \rho y = \rho_0 v_0 &= -\frac{1}{\mu_0} (\rho_1 v_1)' \\ &= \left(-\frac{1}{\mu_0}\right) \left(-\frac{1}{\mu_1}\right) (\rho_2 v_2)'' \\ &\quad \vdots \\ &= \frac{1}{A_n} (\rho_n v_n)^{(n)}, \end{aligned}$$

where

$$(1.3.11) \quad A_0 = 1, \quad A_n = (-1)^n \prod_{k=0}^{n-1} \mu_k.$$

If  $y = y_n(x)$  is a polynomial of degree  $n$ , then  $v_n = y_n^{(n)}(x) = \text{constant}$  and we arrive at the *Rodrigues formula* for polynomial solutions of (1.1.1),

$$(1.3.12) \quad y_n(x) = \frac{B_n}{\rho(x)} [\sigma^n(x)\rho(x)]^{(n)},$$

where  $B_n = A_n^{-1}y_n^{(n)}$  is a constant. These solutions correspond to the eigenvalues (1.3.5).

**Remark** We have also found the explicit series representations for polynomials (1.3.12) in Theorem 1.1.

### 1.3.3 Orthogonality

The polynomial solutions of (1.1.1) obey an orthogonality property. Let us write equations for polynomials  $y_n(x)$  and  $y_m(x)$  in the self-adjoint form,

$$\begin{aligned} (\sigma(x)\rho(x)y_n'(x))' + \lambda_n\rho(x)y_n(x) &= 0, \\ (\sigma(x)\rho(x)y_m'(x))' + \lambda_m\rho(x)y_m(x) &= 0, \end{aligned}$$

multiply the first equation by  $y_m(x)$  and the second by  $y_n(x)$ , subtract the second equality from the first one and then integrate the result over  $x$  on the interval  $(a, b)$ . Since

$$\begin{aligned} y_m(x) (\sigma(x)\rho(x)y_n'(x))' - y_n(x) (\sigma(x)\rho(x)y_m'(x))' \\ = \frac{d}{dx} (\sigma(x)\rho(x)W [y_m(x), y_n(x)]), \end{aligned}$$

where  $W(u, v) = uv' - vu'$  is the Wronskian, we get

$$(1.3.13) \quad \begin{aligned} (\lambda_m - \lambda_n) \int_a^b y_m(x)y_n(x)\rho(x) dx \\ = (\sigma(x)\rho(x)W [y_m(x), y_n(x)]) \Big|_a^b. \end{aligned}$$

If the conditions

$$(1.3.14) \quad \sigma(x)\rho(x)x^k \Big|_{x=a,b} = 0 \quad (k = 0, 1, 2, \dots)$$

are satisfied for some points  $a$  and  $b$ , then the right side of (1.3.13) vanishes because the Wronskian is a polynomial in  $x$ . Therefore, we arrive at the *orthogonality property*

$$(1.3.15) \quad \int_a^b y_m(x)y_n(x)\rho(x) dx = 0$$

provided that  $\lambda_n \neq \lambda_m$ . We can replace this condition by  $m \neq n$  due to the relation  $\lambda_n - \lambda_m = (m - n) \left( \tau' + \frac{n + m - 1}{2} \sigma'' \right)$ , if  $\tau' + (m + n - 1) \sigma'' / 2 \neq 0$ .

We shall refer to polynomial solutions of (1.1.1) obeying the orthogonality property (1.3.15) with respect to a positive weight function as *classical orthogonal polynomials*.

### 1.3.4 Classification

Equation (1.3.8) for the weight function  $\rho(x)$  is usually called the *Pearson equation*. By using the linear transformations of independent variable  $x$  we can reduce solutions of (1.3.8) to the following canonical forms

$$\rho(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta & \text{for } \sigma(x) = 1-x^2, \\ x^\alpha e^{-x} & \text{for } \sigma(x) = x, \\ e^{-x^2} & \text{for } \sigma(x) = 1. \end{cases}$$

The corresponding orthogonal polynomials are the *Jacobi polynomials*  $P_n^{(\alpha,\beta)}(x)$ , the *Laguerre polynomials*  $L_n^\alpha(x)$  and the *Hermite polynomials*  $H_n(x)$ .

The basic information about classical orthogonal polynomials is presented in Table 1.1, which contains also the leading terms  $y_n(x) = a_n x^n + b_n x^{n-1} + \dots$  for these polynomials, squared norms,

$$(1.3.16) \quad d_n^2 = \int_a^b y_n^2(x)\rho(x) dx,$$

and the coefficients of the three-term recurrence relation

$$(1.3.17) \quad x y_n(x) = \alpha_n y_{n+1}(x) + \beta_n y_n(x) + \gamma_n y_{n-1}(x),$$

where

$$(1.3.18) \quad \alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \alpha_{n-1} \frac{d_n^2}{d_{n-1}^2}.$$

$y_n(x)$	$P_n^{(\alpha, \beta)}(x) (\alpha > -1, \beta > -1)$	$L_n^\alpha(x) (\alpha > -1)$	$H_n(x)$
$(a, b)$	$(-1, 1)$	$(0, \infty)$	$(-\infty, \infty)$
$\rho(x)$	$(1-x)^\alpha (1+x)^\beta$	$x^\alpha e^{-x}$	$e^{-x^2}$
$\sigma(x)$	$1-x^2$	$x$	$1$
$\tau(x)$	$\beta - \alpha - (\alpha + \beta + 2)x$	$1 + \alpha - x$	$-2x$
$\lambda_n$	$n(\alpha + \beta + n + 1)$	$n$	$2n$
$B_n$	$\frac{(-1)^n}{2^n n!}$	$\frac{1}{n!}$	$(-1)^n$
$a_n$	$\frac{\Gamma(\alpha + \beta + 2n + 1)}{2^n n! \Gamma(\alpha + \beta + n + 1)}$	$\frac{(-1)^n}{n!}$	$2^n$
$b_n$	$\frac{2^n (n-1)! \Gamma(\alpha + \beta + n + 1)}{(\alpha - \beta) \Gamma(\alpha + \beta + 2n)}$	$(-1)^{n-1} \frac{\alpha + n}{(n-1)!}$	$0$
$d^2$	$\frac{2^{\alpha+\beta+1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{n! (\alpha + \beta + 2n + 1) \Gamma(\alpha + \beta + n + 1)}$	$\frac{\Gamma(\alpha + n + 1)}{n!}$	$2^n n! \sqrt{\pi}$
$\alpha_n$	$\frac{2(n+1)(\alpha + \beta + n + 1)}{(\alpha + \beta + 2n + 1)(\alpha + \beta + 2n + 2)}$	$-(n+1)$	$\frac{1}{2}$
$\beta_n$	$\frac{\beta^2 - \alpha^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)}$	$\alpha + 2n + 1$	$0$
$\gamma_n$	$\frac{2(\alpha + n)(\beta + n)}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 1)}$	$-(\alpha + n)$	$n$

### 1.3.5 Functions of the second kind

Consider equation (1.1.1) in the complex  $z$ -plane for the eigenvalues (1.3.5). By using Theorem 1.2 we can choose a particular solution of the form

$$(1.3.19) \quad y = y_n(z) = \frac{B_n n!}{2\pi i \rho(z)} \int_C \frac{\rho_n(s) ds}{(s-z)^{n+1}},$$

where  $B_n$  is a constant,  $\rho_n(s) = \sigma^n(s)\rho(s)$  and  $C$  is a closed contour in the complex  $s$ -plane that encloses the point  $s = z$ . Here the conditions of the theorem hold.

The solution (1.3.19) defines *classical orthogonal polynomials*. In fact, in view of

$$\frac{d^n}{dz^n} \left( \frac{1}{s-z} \right) = \frac{n!}{(s-z)^{n+1}},$$

we arrive at the Rodrigues formula

$$\begin{aligned} y_n(z) &= \frac{B_n}{2\pi i \rho(z)} \int_C \rho_n(s) \frac{d^n}{dz^n} \left( \frac{1}{s-z} \right) ds \\ &= \frac{B_n}{2\pi i \rho(z)} \frac{d^n}{dz^n} \int_C \frac{\rho_n(s)}{s-z} ds = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\rho_n(z)] \end{aligned}$$

with the aid of the Cauchy integral formula.

As a second linearly independent solution of (1.1.1) for  $\lambda = \lambda_n$  we take a function of the form

$$(1.3.20) \quad y = Q_n(z) = \frac{B_n n!}{\rho(z)} \int_a^b \frac{\rho_n(s) ds}{(s-z)^{n+1}}, \quad z \neq s,$$

where  $B_n$  is the constant in the Rodrigues formula (1.3.12) and  $\rho_n(s) = \sigma^n(s)\rho(s)$ . The points  $a$  and  $b$  are chosen so that (1.3.14) holds, which is the case for the classical orthogonal polynomials. In this case the conditions of Theorem 1.2 hold.

The functions  $Q_n(z)$  defined by (1.3.20) are called *functions of the second kind*.

We can find a relation between the functions  $Q_n(z)$  and the polynomials  $y_n(z)$ . From definition

$$\begin{aligned} Q_n(z) &= -\frac{B_n(n-1)!}{\rho(z)} \int_a^b \rho_n(s) \frac{d}{ds} \left[ \frac{1}{(s-z)^n} \right] ds \\ &= \frac{B_n(n-1)!}{\rho(z)} \left( -\frac{\rho_n(s)}{(s-z)^n} \Big|_a^b + \int_a^b \frac{[\rho_n(s)]'}{(s-z)^n} ds \right). \end{aligned}$$

The first term vanishes by virtue of (1.3.14). Similarly,

$$Q_n(z) = \frac{B_n}{\rho(z)} \int_a^b \frac{[\rho_n(s)]^{(n)}}{s-z} ds,$$

and we find that

$$(1.3.21) \quad Q_n(z) = \frac{1}{\rho(z)} \int_a^b \frac{y_n(s)}{s-z} \rho(s) ds$$

due to the Rodrigues formula.

By setting  $y_n(s) = [y_n(s) - y_n(z)] + y_n(z)$  equation (1.3.21) can be written in the convenient form

$$(1.3.22) \quad Q_n(z) = \frac{1}{B_0} y_n(z) Q_0(z) + \frac{1}{\rho(z)} q_{n-1}(z),$$

where

$$(1.3.23) \quad Q_0(z) = \frac{B_0}{\rho(z)} \int_a^b \frac{\rho(s)}{s-z} ds$$

and

$$(1.3.24) \quad q_{n-1}(s) = \int_a^b \frac{y_n(s) - y_n(z)}{s-z} \rho(s) ds$$

is a polynomial of degree  $n-1$  in  $s$ , which is called a *polynomial of the second kind*. It follows from (1.3.22) that all the singularities of the second solution  $Q_n(z)$  in the complex  $z$ -plane are determined by the behavior of the functions  $Q_0(z)$  and  $1/\rho(z)$ .

It is possible to derive from (1.3.20)–(1.3.22) all the main properties of the functions of the second kind. Using the identity

$$\frac{1}{s-z} = -\frac{1}{z} \sum_{k=0}^p \left(\frac{s}{z}\right)^k + \frac{s^{p+1}}{(s-z)z^{p+1}}$$

and the orthogonality property

$$\int_a^b s^k y_n(s) \rho(s) ds = 0, \quad k < n$$

we obtain from (1.3.21) an expansion of the form

$$(1.3.25) \quad \rho(z) Q_n(z) = - \sum_{k=n}^p \frac{1}{z^{k+1}} \int_a^b s^k y_n(s) \rho(s) ds + \frac{r_p(z)}{z^{p+1}},$$

where

$$r_p(z) = \int_a^b s^{p+1} \rho(s) ds.$$

The equation (1.3.25) determines the *asymptotic behavior* of the functions  $Q_n(z)$  as  $|z| \rightarrow \infty$ . In particular, for  $p = n$ , (1.3.25) yields

$$(1.3.26) \quad Q_n(z) = \frac{(d_n^2/a_n)}{\rho(z)z^{n+1}} \left[ 1 + O\left(\frac{1}{z}\right) \right], \quad z \rightarrow \infty.$$

Hence, the functions of the second kind  $Q_n(z)$  and the classical orthogonal polynomials  $y_n(z)$  have different asymptotic behaviors at  $|z| \rightarrow \infty$ , so that they are two linearly-independent solutions of (1.1.1) for  $\lambda = \lambda_n$ .

According to (1.3.21), we have

$$z Q_n(z) = \frac{1}{\rho(z)} \int_a^b \frac{s y_n(s)}{s-z} \rho(s) ds - \frac{1}{\rho(z)} \int_a^b y_n(s) \rho(s) ds.$$

Therefore, the functions  $Q_n(z)$  satisfy a *three-term recurrence relation*

$$z Q_n(z) = \alpha_n Q_{n+1}(z) + \beta_n Q_n(z) + \gamma_n Q_{n-1}(z) \quad \text{if } n \geq 1$$

and

$$z Q_0(z) = \alpha_0 Q_1(z) + \beta_0 Q_0(z) - \frac{1}{\rho(z)} \frac{d_0^2}{B_0} \quad \text{if } n = 0,$$

where  $\alpha_n$ ,  $\beta_n$  and  $\gamma_n$  are the coefficients of the recurrence relation (1.3.17) for the classical orthogonal polynomials.

It follows from (1.3.20) and (1.2.8) that the derivative of  $Q_n(z)$  can be represented in the form

$$Q'_n(z) = \frac{\kappa_n B_n n!}{\sigma(z)\rho(z)} \int_a^b \frac{\rho_n(s)}{(s-z)^n} ds,$$

where  $\kappa_n = \tau' + (n-1)\sigma''/2$ . For  $n=0$  this leads to the simple differential equation for  $Q_0(z)$ ,

$$(1.3.27) \quad \sigma(z)\rho(z)Q'_0(z) = C_0,$$

where  $C_0 = \kappa_0 B_0 d_0^2$ . According to (1.3.27) we obtain

$$(1.3.28) \quad Q_0(z) = Q_0(z_0) + C_0 \int_{z_0}^z \frac{ds}{\sigma(s)\rho(s)},$$

where it is convenient to choose for  $z_0$  a value of  $z$  for which  $Q_0(z_0) = 0$ .

The functions  $Q_n(z)$  have not been defined when  $z \in [a, b]$ . On this interval it is convenient to set

$$(1.3.29) \quad \begin{aligned} \rho(x)Q_n(x) &= \frac{1}{2} [\rho(x-i0)Q_n(x-i0) + \rho(x+i0)Q_n(x+i0)] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2} [\rho(x-i\epsilon)Q_n(x-i\epsilon) + \rho(x+i\epsilon)Q_n(x+i\epsilon)]. \end{aligned}$$

We note also the relation

$$(1.3.30) \quad \rho(x-i0)Q_n(x-i0) - \rho(x+i0)Q_n(x+i0) = 2\pi i \rho(x)y_n(x),$$

which comes from (1.3.21) due to

$$\frac{1}{x-i0} - \frac{1}{x+i0} = 2\pi i \delta(x)$$

where  $\delta(x)$  is Dirac's delta function.

The explicit forms of the functions of the second kind can be easily found by comparing the definition (1.3.20) with the integral representations for the hypergeometric functions in (1.2.10), (1.2.12) and Ex. 1.10(ii). They are

$$(1.3.31) \quad Q_n^{(\alpha, \beta)}(z) = (-1)^n 2^{\alpha+\beta+n+1} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)} \\ \times (1-z)(1+z)^{-\beta} {}_2F_1 \left( \begin{matrix} n+1, \alpha+n+1 \\ \alpha+\beta+2n+2 \end{matrix}; 2(1-z)^{-1} \right), \quad z \notin [-1, 1];$$

$$(1.3.32) \quad Q_n^\alpha(z) = e^{-i\pi\alpha} \Gamma(\alpha+n+1) \\ \times e^z G(\alpha+n+1; \alpha+1; -z), \quad 0 < \arg z < 2\pi;$$

(see Ex. 1.10) and

$$(1.3.33) \quad Q_n(z) = 2^{n+1} n! \sqrt{\pi} e^{z^2 \mp i\frac{\pi}{2}(n-1)} H_{-n-1}(\mp iz) \quad (\text{Im } z > 0 \quad \text{or} \quad \text{Im } z < 0)$$

corresponding to the cases of the Jacobi, Laguerre and Hermite polynomials, respectively.

### 1.3.6 Complex orthogonality

Classical orthogonality polynomials obey an interesting orthogonality property with respect to a complex measure. To prove it, let us start with the identity

$$(1.3.34) \quad [\sigma(z)\rho(z)W(y_m, y_n)]' = (\lambda_m - \lambda_n) y_m(z)y_n(z)\rho(z),$$

derived in Section 1.3.3. Multiply both sides of (1.3.34) by the function  $Q_0(z)$  defined by (1.3.23) and integrate the result over a contour  $C$  in the complex  $z$ -plane. Upon integrating by parts we obtain with the aid of (1.3.27) that

$$(1.3.35) \quad (\lambda_m - \lambda_n) \int_C y_m(z)y_n(z)\rho(z)Q_0(z) dz \\ = (\sigma(z)\rho(z)Q_0(z)W[y_m(z), y_n(z)])|_{z_1}^{z_2} \\ - C_0 \int_C W[y_m(z), y_n(z)] dz.$$

Here,  $z_1$  and  $z_2$  are the end points of the contour  $C$ .

Suppose first that  $(a, b)$  is a finite interval on the real axis. Then for a closed contour  $C$ , which encloses interval  $(a, b)$  in the complex  $z$ -plane, the right side of (1.3.3) vanishes due to Cauchy's theorem. As a result, we arrive at the *complex orthogonality property*

$$(1.3.36) \quad \int_C y_m(z)y_n(z)\rho(z)Q_0(z) dz = 0, \quad m \neq n$$

provided that  $\tau' + (m + n - 1)\sigma''/2 \neq 0$ . This case corresponds to the Jacobi polynomials  $y_n = P_n^{(\alpha, \beta)}(z)$  if  $\alpha + \beta + 1 \neq -1, -2, \dots$  (Fig. 1.1). The complex weight function in (1.3.36) is a weight function in a wider range of parameters than the real weight function for the Jacobi polynomials. (See Ex. 1.25).

In the case  $b = +\infty$ , which corresponds to the Laguerre polynomials  $y_n = L_n^\alpha(z)$ , consider (1.3.35) for the contour  $C_\epsilon(R)$  on Fig. 1.2. Taking the limit  $\epsilon \rightarrow 0$  ( $\epsilon > 0$ ) due to (1.3.30) we obtain

$$(1.3.37) \quad \begin{aligned} Pv \int_{C(R)} y_m(z)y_n(z)\rho(z)Q_0(z) dz \\ = \frac{2\pi i}{\lambda_m - \lambda_n} \sigma(R)\rho(R) W[y_m(R), y_n(R)], \end{aligned}$$

where  $Pv$  denotes Cauchy's principal value integral and  $C(R) = \lim_{\epsilon \rightarrow 0} C_\epsilon(R)$  is the closed contour. In the limit  $R \rightarrow \infty$  we arrive at a complex orthogonality property, for the Laguerre polynomials  $L_n^\alpha(z)$  if  $\alpha > -1$ . (Ex. 1.26).

Finally, consider the case of the Hermite polynomials  $y_n = H_n(z)$  when  $a = -\infty$  and  $b = +\infty$ . In the limit  $\epsilon \rightarrow 0$  for the two contours  $C_\epsilon^\pm(R_1, R_2)$  on Fig. 1.3 we get

$$(1.3.38) \quad \begin{aligned} Pv \int_{C(R_1, R_2)} y_m(z)y_n(z)\rho(z)Q_0(z) dz \\ = \frac{2\pi i}{\lambda_m - \lambda_n} (\sigma(R_1)\rho(R_1) W[y_m(R_1), y_n(R_1)] \\ + \sigma(R_2)\rho(R_2) W[y_m(R_2), y_n(R_2)]), \end{aligned}$$

where  $C(R_1, R_2) = \lim_{\epsilon \rightarrow 0} C_\epsilon^+(R_1, R_2) \cup C_\epsilon^-(R_1, R_2)$  is the closed contour. Taking the limits  $R_{1,2} \rightarrow \mp\infty$  we obtain a complex orthogonality property of the Hermite polynomials (Ex. 1.27).

## Exercises 1

**1.1** (i) By using the Pearson equation  $(\sigma\rho)' = \tau\rho$  show that the function  $\rho_\nu(s) = \sigma^\nu(s)\rho(s)$  satisfies the equation

$$[\sigma(s)\rho_\nu(s)]' = \tau_\nu(s)\rho_\nu(s),$$

where  $\tau_\nu(s) = \tau(s) + \nu\sigma'(s)$ ;

(ii) using this equation and the expansions

$$\begin{aligned}\sigma(s) &= \sigma(z) + \sigma'(z)(s-z) + \frac{1}{2}\sigma''(s-z)^2, \\ \tau_\nu(s) &= \tau_\nu(z) + \tau'_\nu(s-z)\end{aligned}$$

verify the identity

$$\begin{aligned}\frac{d}{ds} [\sigma(s)\rho_\nu(s)(s-z)^\mu] &= \mu\sigma(z)(s-z)^{\mu-1}\rho_\nu(s) \\ &+ [\tau_\nu(z) + \mu\sigma'(z)](s-z)^\mu\rho_\nu(s) \\ &+ \left(\tau'_\nu + \frac{1}{2}\mu\sigma''\right)(s-z)^{\mu+1}\rho_\nu(s),\end{aligned}$$

where  $\mu$  and  $\nu$  are arbitrary complex numbers;

(iii) verify identity (1.1.6) for arbitrary complex values of  $n$ .

**1.2** Derive (1.1.10)–(1.1.13).

**1.3** (i) Show that the series (1.1.11) with arbitrary  $\alpha$  satisfies a non-homogeneous differential equation (1.1.1) with the right side

$$G_\alpha(x) = c_0\alpha\tau_{\alpha-1}(a)(x-a)^{\alpha-1}$$

if  $\sigma(a) = 0$ .

(ii) Show that the series of the form

$$\begin{aligned}u(x) &= x^\alpha \sum_{n=0}^{\infty} \frac{(\alpha+a)_n(\alpha+b)_n}{(\alpha+c)_n(\alpha+1)_n} x^n \\ &= x^\alpha {}_3F_2 \left( \begin{matrix} 1, \alpha+a, \alpha+b \\ \alpha+1, \alpha+c \end{matrix} ; x \right), \quad |x| < 1\end{aligned}$$

satisfies the non-homogeneous equation

$$x(1-x)u'' + [c - (a+b+1)x]u' - abu = \alpha(c + \alpha - 1)x^{\alpha-1}$$

(putting  $\alpha = 0$  or  $\alpha = 1 - c$ , we recover (1.1.5) and (1.1.17), respectively). Find the general solution of this non-homogeneous equation.

(iii) Show that the series

$$\begin{aligned} v(x) &= x^\alpha \sum_{n=0}^{\infty} \frac{(\alpha+a)_n}{(\alpha+c)_n(\alpha+1)_n} x^n \\ &= x^\alpha {}_2F_2 \left( \begin{matrix} 1, \alpha+a \\ \alpha+1, \alpha+c \end{matrix} ; x \right), \quad |x| < 1 \end{aligned}$$

satisfies the non-homogeneous equation

$$xv'' + (c-x)v' - av = \alpha(c + \alpha - 1)x^{\alpha-1}.$$

Find the general solution of this equation.

(iv) Show that the function  $w(x) = u(x) - v(x)$  satisfies the homogeneous equation of the form

$$x^2w'' + (a+b)xw' + a(b-1)w = 0.$$

**1.4** (i) Show that the series (1.1.12) with arbitrary  $\alpha$  satisfies non-homogeneous differential equation (1.1.1) with the right side

$$G_\alpha(x) = c_0 \frac{\lambda - \lambda_{-\alpha}}{(x-a)^\alpha}$$

if  $\sigma(a) = 0$ .

(ii) Show that the series of the form

$$\begin{aligned} u(x) &= x^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha-c+1)_n}{(\alpha-a+1)_n(\alpha-b+1)_n} x^{-n} \\ &= x^{-\alpha} {}_3F_2 \left( \begin{matrix} 1, \alpha, \alpha-c+1 \\ \alpha-a+1, \alpha-b+1 \end{matrix} ; x^{-1} \right), \quad |x| > 1 \end{aligned}$$

satisfies the non-homogeneous equation

$$x(1-x)u'' + [c - (a+b+1)x]u' - abu = -(\alpha-a)(\alpha-b)x^{-\alpha}.$$

Find the general solution of this equation.

**1.5** Prove that

$$\begin{aligned} J_\nu(x) &= \frac{(x/2)^\nu}{\Gamma(\nu+1)} e^{-ix} {}_1F_1(\nu+1/2; 2\nu+1; 2ix) \\ &= \frac{(x/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(-; \nu+1; -x^2/4). \end{aligned}$$

**1.6** By using the special case of (1.1.6), prove that the elementary functions

$$\begin{aligned} u_0(x) &= e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ u_+(x) &= \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \\ u_-(x) &= \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

satisfy the equations  $u'' = u$  and  $u'' = -u$ , respectively.

**1.7** Classify all solutions of the hypergeometric-type equation (1.1.1) depending on degrees of  $\sigma(x)$  and  $\tau(x)$ .

**1.8** Prove that functions (1.2.3) satisfy (under proper boundary conditions) the differentiation formula

$$\sigma(z)y'_\nu(z) = \frac{\kappa_\nu}{\tau'_\nu} \left[ (\nu+1) \frac{C_\nu}{C_{\nu+1}} y_{\nu+1}(z) - \tau_\nu(z)y_\nu(z) \right]$$

and the recurrence relation

$$z y_\nu(z) = \alpha_\nu y_{\nu+1}(z) + \beta_\nu y_\nu(z) + \gamma_\nu y_{\nu-1}(z)$$

where

$$\begin{aligned} \alpha_\nu &= -\frac{(\nu+1)\kappa_\nu C_\nu}{\tau'_\nu \tau'_{\nu-1/2} C_{\nu+1}}, \\ \beta_\nu &= \nu \frac{\tau_{\nu-1}(0)}{\tau'_{\nu-1}} - (\nu+1) \frac{\tau_\nu(0)}{\tau'_\nu} \\ &= \frac{(\sigma'' - \tau') \tau(0) - \nu(2\tau' - (\nu-1)\sigma'') \sigma'(0)}{\tau'_{\nu-1} \tau'_\nu}, \end{aligned}$$

and

$$\gamma_\nu = -\frac{\tau'_{\nu-1}\sigma\left(-\tau_{\nu-1}(0)/\tau'_{\nu-1}\right)C_\nu}{\tau'_{\nu-1/2}C_{\nu-1}}.$$

Here  $\kappa_\nu = \tau' + (\nu - 1)\sigma''/2$ . (Nikiforov and Uvarov [1988], Suslov [1989])

**1.9** With the aid of the transformation  $u = \varphi(z)y$  reduce the equation

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)}u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u = 0,$$

where  $\sigma(z)$  and  $\tilde{\sigma}(z)$  are polynomials, at most of second degree, and  $\tilde{\tau}(z)$  is a polynomial, at most of first degree, to the hypergeometric form (1.1.1). (Nikiforov and Uvarov [1988])

**1.10** (i) Derive integral representations (1.2.10)–(1.2.12).

(ii) Prove that the function

$$G(\alpha; \gamma; z) = \frac{z^{-\alpha}}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} \left(1 + \frac{t}{z}\right)^{\gamma-\alpha-1} dt, \quad |\arg z| < \pi, \quad \operatorname{Re} \alpha > 0$$

satisfies the confluent hypergeometric equation.

(iii) Prove that

$$\begin{aligned} G(\alpha; \gamma; z) &= \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} {}_1F_1(\alpha; \gamma; z) \\ &+ \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} {}_1F_1(1+\alpha-\gamma; 2-\gamma; z). \end{aligned}$$

**1.11** (i) Transform the Bessel equation (1.2.13) to the form (1.2.14) and derive integral representations (1.2.15)–(1.2.16).

(ii) Prove that

$$\begin{aligned} \frac{d}{dz} \left( z^{\pm\nu} J_\nu(z) \right) &= \pm z^{\pm\nu} J_{\nu\mp 1}(z) \\ J_{\nu-1}(z) + J_{\nu+1}(z) &= \frac{2\nu}{z} J_\nu(z), \\ J_{\nu-1}(z) - J_{\nu+1}(z) &= 2J'_\nu(z). \end{aligned}$$

**1.12** Prove that

$$y_n^{(m)}(x) = \frac{A_{mn}B_n}{\sigma^m(x)\rho(x)} \frac{d^{n-m}}{dx^{n-m}} [\sigma^n(x)\rho(x)],$$

where

$$A_{mn} = (-1)^m \prod_{k=0}^{m-1} (\lambda_n - \lambda_k) = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left( \tau' + \frac{n+k-1}{2} \sigma'' \right), \quad A_{0n} = 1.$$

**1.13** Prove that the differentiation formula

$$\sigma(x)y'_n(x) = \frac{\lambda_n}{n\tau'_n} \left[ \tau_n(x)y_n(x) - \frac{B_n}{B_{n+1}} y_{n+1}(x) \right]$$

is valid for classical orthogonal polynomials and the functions of the second kind.

**1.14** Prove (1.3.17)–(1.3.18) for general orthogonal polynomials.

**1.15** Find the expressions

$$\begin{aligned} a_n &= \frac{A_{nn}B_n}{n!} = B_n \prod_{k=0}^{n-1} \left( \tau' + \frac{n+k-1}{2} \sigma'' \right), \quad a_0 = B_0; \\ \frac{b_n}{a_n} &= n \frac{\tau_{n-1}(0)}{\tau'_{n-1}} \end{aligned}$$

for the leading terms  $y_n(x) = a_n x^n + b_n x^{n-1} + \dots$  of the classical orthogonal polynomials.

**1.16** Show that the following relation

$$d_n^2 = (-1)^n A_{nn} B_n^2 \int_a^b \sigma^n(x) \rho(x) dx$$

is valid for the squared norms (1.3.16) of the classical orthogonal polynomials.

**1.17** Prove that

$$\begin{aligned} \frac{d}{dx} P_n^{(\alpha, \beta)}(x) &= \frac{1}{2} (\alpha + \beta + n + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x), \\ \frac{d}{dx} L_n^\alpha(x) &= -L_{n-1}^{\alpha+1}(x), \\ \frac{d}{dx} H_n(x) &= 2n H_{n-1}(x). \end{aligned}$$

**1.18** Prove the symmetry relations

$$\begin{aligned} P_n^{(\alpha, \beta)}(-x) &= (-1)^n P_n^{(\beta, \alpha)}(x), \\ H_n(-x) &= (-1)^n H_n(x). \end{aligned}$$

**1.19** Prove that

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(-n, \alpha + \beta + n + 1; \alpha + 1; \frac{1-x}{2}\right) \\ &= (-1)^n \frac{(\beta + 1)_n}{n!} {}_2F_1\left(-n, \alpha + \beta + n + 1; \beta + 1; \frac{1+x}{2}\right); \\ L_n^\alpha(x) &= \frac{(\alpha + 1)_n}{n!} {}_1F_1(-n, \alpha + 1; x) \\ &= \frac{(-1)^n}{n!} x^n {}_2F_0\left(-n, -n - \alpha; -; -\frac{1}{x}\right); \end{aligned}$$

and

$$\begin{aligned} H_{2n}(x) &= (-1)^n 2^{2n} \left(\frac{1}{2}\right)_n {}_1F_1\left(-n; \frac{1}{2}; x^2\right), \\ H_{2n+1}(x) &= (-1)^n 2^{2n+1} \left(\frac{3}{2}\right)_n x {}_1F_1\left(-n; \frac{3}{2}; x^2\right), \\ H_n(x) &= (2x)^n {}_2F_0\left(-n/2, (1-n)/2; -; -x^{-2}\right). \end{aligned}$$

**1.20** Define moments for classical orthogonal polynomials by

$$C_m = \int_a^b (s - \xi)^m \rho(s) ds.$$

By using the identity from Ex. 1.1(i), prove that

$$\frac{C_{m+1}}{C_m} = -\frac{\tau(a) + m\sigma'(a)}{\tau' + m\sigma''/2},$$

if  $\sigma(\xi) = 0$  and  $(\sigma(s)\rho(s)(s - \xi)^n)|_a^b = 0$ ,  $n = 0, 1, 2, \dots$  (Atakishiyev and Suslov [1988])

**1.21** Find relations between moments for the Jacobi, Laguerre and Hermite polynomials.

**1.22** Let  $\{\varphi_k\}_{k=0}^\infty$  and  $\{\psi_k\}_{k=0}^\infty$  be sequences of polynomials  $\varphi_k(x)$  and  $\psi_k(x)$  of exact degree  $k$ . Prove that the orthogonal polynomials  $p_n(x)$  for a given distribution  $d\mu$  can be expressed as Gram determinants

$$p_n(x) = \begin{vmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,n} \\ C_{1,0} & \dots & & C_{1,n} \\ \vdots & & & \\ C_{n-1,0} & \dots & & C_{n-1,n} \\ \varphi_0(x) & \varphi_1(x) & \dots & \varphi_n(x) \end{vmatrix}$$

with  $C_{i,k} = \int \psi_i \varphi_k d\mu$ .

Derive the explicit series representation for the Jacobi polynomials  $P_n^{(\alpha,\beta)}(x)$  from the Gram determinant. (Wilson [1978, 1991])

**1.23** Prove (1.3.31)–(1.3.33).

**1.24** Let  $a, b < \infty$ . Prove that

$$\begin{aligned} & \frac{1}{2\pi i} \int_C y_m(z) y_n(z) \rho(z) Q_0(z) dz \\ &= -B_0 \int_a^b y_m(s) y_n(s) \rho(s) ds, \end{aligned}$$

where  $Q_0(z)$  is defined by (1.3.23) and  $C$  is a closed counter-clockwise contour which encloses the interval  $[a, b]$  in the complex  $z$ -plane. (Geronimus [1977], Ismail, Masson and Rahman [1991])

**1.25** Prove the complex orthogonality relation for the Jacobi polynomials,

$$\begin{aligned} & \frac{1}{2\pi i} P_V \int_C P_n^{(\alpha,\beta)}(z) P_m^{(\alpha,\beta)}(z) \chi^{(\alpha,\beta)}(z) dz \\ &= 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)} \delta_{mn}, \end{aligned}$$

provided that  $\alpha + \beta + 1 \neq -1, -2, \dots$ . Here

$$\begin{aligned} \chi^{(\alpha,\beta)}(z) &= -(1-z)^\alpha (1+z)^\beta Q_0^{(\alpha,\beta)}(z) \\ &= \begin{cases} -2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)(1-z)} {}_2F_1 \left( \begin{matrix} 1, \alpha+1 \\ \alpha+\beta+2 \end{matrix} ; \frac{2}{(1-z)} \right) & \text{if } |z-1| > 2, \\ 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)(1-z)} {}_2F_1 \left( \begin{matrix} 1, \beta+1 \\ \alpha+\beta+2 \end{matrix} ; \frac{2}{(1+z)} \right) & \text{if } |z+1| > 2 \end{cases} \end{aligned}$$

and  $C$  is a counter-clockwise closed contour which encloses the interval  $[-1, 1]$  in the complex  $z$ -plane (Fig. 1.1). (Rusev [1984], Ismail, Masson and Rahman [1991])

**1.26** (i) Let  $\alpha \neq -k$  and  $k = 1, 2, \dots$ . Prove the complex orthogonality relation for the Laguerre polynomials,

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{C(R)} L_n^\alpha(z) L_m^\alpha(z) \chi^\alpha(z) dz = \frac{(\alpha + 1)_n}{n!} \delta_{mn},$$

where

$$\chi^\alpha(z) = -\frac{z^\alpha e^{-z}}{\Gamma(\alpha + 1)} Q_0^\alpha(z) = G(1; 1 - \alpha; -z)$$

and  $C(R) = \lim_{\epsilon \rightarrow 0} C_\epsilon(R)$  is a closed contour (for the definition of the contour  $C_\epsilon(R)$  see Fig. 1.2).

(ii) Let  $\alpha = -k$  and  $k = 1, 2, \dots$ . Prove that a finite number of the Laguerre polynomials  $\{L_n^{-k}(x)\}_{n=0}^{k-1}$  satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} L_n^{-k}(x) L_m^{-k}(x) d\mu(x) = \frac{(1 - k)_n}{n!} \delta_{mn}$$

where the measure may be expressed as a real distribution with support at  $x = 0$ :

$$d\mu(x) = \left(1 + \frac{d}{dx}\right)^{k-1} \delta(x) dx = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \delta^{(\ell)}(x) dx.$$

(Ismail, Masson and Rahman [1991])

**1.27** Prove the complex orthogonality relation for the Hermite polynomials,

$$\lim_{R_{1,2} \rightarrow \mp\infty} \frac{1}{\pi i} P_V \int_{C(R_1, R_2)} H_n(z) H_m(z) \chi(z) dz = 2^n n! \delta_{mn},$$

where

$$\chi(z) = -\frac{e^{-z^2}}{2\sqrt{\pi}} Q_0(z) = \mp i H_{-1}(\mp iz) \quad (\text{Im } z > 0 \quad \text{or} \quad \text{Im } z < 0)$$

and  $C(R_1, R_2) = \lim_{\epsilon \rightarrow 0} C_\epsilon^+(R_1, R_2) \cup C_\epsilon^-(R_1, R_2)$  is the counter-clockwise contour (for the definition of  $C_\epsilon^\pm(R_1, R_2)$  see Fig. 1.3).



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