Remarks on the Beverton-Holt Equation

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11. April 2005

The Beverton-Holt equation (BHE):

$$x_{n+1} = \frac{\mu K x_n}{K + (\mu - 1)x_n}, \quad x_0 \geq 0$$

The BHE tries to model a salmon population, especially to outline the size of the salmon population at a specific moment and to show its dependence on the intrinsic growth rate μ and the carrying capacitiy K.

Problem. Prove that for $\mu > 1, K > 0$ all non-zero solutions converge to the equilibrium point $x^* = K$.

At first we consider the BHE as a function

$$f(x) = \frac{\mu K x}{K + (\mu - 1)x}.$$

Definition. A point x^* is said to be a fixed point of the map f or an equilibrium point of the equation $x_{n+1} = f(x_n)$ if $f(x^*) = x^*$.

Crucial Lemma. Let $x_0 \in X$ and $f : X \to X$ be a continuous function on a space X. If $lim_{n\to\infty} f(x_0)^n = a$, then a is a fixed point of f.

Proof.

Since f is continuous, we obtain

$$\lim_{n\to\infty} f^{n+1}(x_0) = \lim_{n\to\infty} f(f^n(x_0)) \stackrel{!}{=} f\left(\lim_{n\to\infty} f^n(x_0)\right).$$

If we assume that

$$\lim_{n\to\infty} f^n(x_0) = y,$$

we thus get $f(y) = y$. □

We will show that K is a fixed point. Let $\bar{x} \neq 0$ be a fixed point, with $f(\bar{x}) = \bar{x}$, then

$$\begin{aligned}
f(\bar{x}) = \bar{x} &= \frac{\mu K \bar{x}}{K + (\mu - 1)\bar{x}} \\
\bar{x}(K + (\mu - 1)\bar{x}) &= \mu K \bar{x} \\
K + (\mu - 1)\bar{x} &= \mu K \\
(\mu - 1)\bar{x} &= K(\mu - 1) \\
\bar{x} &= K
\end{aligned}$$

Therefore K is a fixed point of f. □

Now we search all x for which $0 < f'(x) < 1$. The relation $f'(x) < 1$ is equivalent to

$$\begin{aligned}
\mu K^2 &< (K + (\mu - 1)x)^2 \\
\sqrt{\mu} K &< K + (\mu - 1)x \\
\sqrt{\mu} K - K &< (\mu - 1)x \\
\frac{(\sqrt{\mu} - 1)K}{\mu - 1} &< x \\
\frac{\sqrt{\mu} - 1}{(\sqrt{\mu} - 1)(\sqrt{\mu} + 1)} K &< x.
\end{aligned}$$

Hence $f'(x) < 1$ is equivalent to

$$\frac{K}{\sqrt{\mu} + 1} < x.$$

Theorem 1. For each initial value

$$x_0 \in J := \left(\frac{K}{\sqrt{\mu} + 1}, \infty \right),$$

the sequence $x_n = f^n(x_0), n \geq 0$ converges to K.

Proof.

(a) We prove the theorem for

$$\frac{K}{\sqrt{\mu} + 1} < x_0 < x^*.$$

Using the MVT we obtain a ξ between x_0 and x^* with

$$0 < \frac{f(x^*) - f(x_0)}{x^* - x_0} = f'(\xi) < 1, \qquad \text{because } \forall x \in J, f'(x) < 1.$$

Thus we know that $f(x_0) < x^*$.
Because $f(x^*) = x^*$ it follows that $x^* - x_0 > x^* - f(x_0)$.
Repeating this process we obtain

$$\begin{aligned} 0 < \frac{x^* - f^2(x_0)}{x^* - f(x_0)} &< 1 \\ x^* - f^2(x_0) &< x^* - f(x_0) \\ &\dots \end{aligned}$$

Similarly,

$$x^* - x_0 > x^* - f(x_0) > x^* - f^2(x_0) > \ldots > x^* - f^n(x_0) > \ldots > 0 \tag{1}$$

The sequence $x^* - f^n(x_0)$ is decreasingly bounded by 0. Thus it converges to a $a \geq 0$.

$$\begin{aligned} x^* - f^n(x_0) &\quad \to a \geq 0 \\ -f^n(x_0) &\quad \to a - x^* \\ f^n(x_0) &\to x^* - a > 0 \end{aligned}$$

According to the Crucial Lemma $x^* - a$ is a fixed point. Since f has two fixed points we have to examine which point f converges to. From equation (1) we know that $f^n(x_0)$ is increasing and positive. Hence

$$x^* - a = \lim_{n \to \infty} f^n(x_0) > 0.$$

But the only positive fixed point is $K = x^*$. Therefore $x^* - a = x^* \;\Rightarrow\; a = 0$.

(b) We prove the theorem for

$$x^* < x_0 < \infty$$

Considering case 2 you can obviously draw the same conclusions which you obtained in case 1. □

After we have shown the convergence behaviour on the interval J, we will now extend that to the interval

$$\left(0, \frac{K}{\sqrt{\mu} + 1}\right).$$

Therefore it is to be shown that:
(1) $\forall x_0 \in \left(0, \frac{K}{\sqrt{\mu}+1}\right)$ the sequence $x_n = f^n(x_0)_{n \in \mathbb{N}_0}$ is increasing and bounded.

(2) The limit of that sequence is K.

Proof. **(1)** We show the boundedness of $(x_n)_{n\in\mathbb{N}_0}$, using mathematical induction:
We initialize the induction:

$$n = 0: \quad x_0 \leq \frac{K}{\sqrt{\mu}+1} < K$$

Induction step:

$$(n-1) \rightarrow n$$

$$\begin{aligned}
x_{n-1} &\leq K \\
Kx_{n-1} &\leq K^2 \\
0 &\leq K^2 - Kx_{n-1} \\
\mu K x_{n-1} &\leq K^2 + Kx_{n-1}\mu - Kx_{n-1} \\
\mu K x_{n-1} &\leq K^2 + Kx_{n-1}(\mu - 1) \\
\mu K x_{n-1} &\leq K^2 + Kx_{n-1}K(K(+(\mu-1)x_{n-1} \\
x_n = f(x_{n-1}) = \frac{\mu K x_{n-1}}{K + (\mu-1)x_{n-1}} &\leq K
\end{aligned}$$

Thus it follows that $x_n \leq K$ for all nonnegative integers n.

Afterwards we show the monotonicity $x_n \leq x_{n+1}$.
We initialize the induction:

$$\begin{aligned}
x_0 &\leq K \\
(\mu-1)x_0 + K(1-\mu) &\leq 0 \\
x_0((\mu-1)x_0 + K(1-\mu)) &\leq 0 \\
(\mu-1)x_0^2 + K(1-\mu) &\leq 0 \\
(\mu-1)x_0^2 + Kx_0 &\leq \mu K x_0 \\
x_0 &\leq f(x_0) = \frac{\mu K x_0}{K + (\mu-1)x_0}
\end{aligned}$$

For the induction step $n \to n+1$ we assume $x_n \leq K$. Therefore we obtain

$$\begin{aligned}
x_n(\mu-1) &\leq K(\mu-1) \\
(\mu-1)x_n + K(1-\mu) &\leq 0 \\
(\mu-1)x_n^2 + K(1-\mu)x_n &\leq 0 \\
(\mu-1)x_n^2 + Kx_n - \mu K x_n &\leq 0 \\
(\mu-1)x_n^2 + Kx_n &\leq \mu K x_n \\
x_n((\mu-1)x_n + K) &\leq \mu K x_n \\
x_n &\leq \frac{\mu K x_n}{(\mu-1)x_n + K} = f(x_n) = x_{n+1}
\end{aligned}$$

Thereby follows that the sequence $(x_n)_{n \in \mathbb{N}_0}$ converges. □

(2) We know that $x_1^* = 0$ and $x_2^* = K$.

The crucial lemma which we had proven above tells us that $f^n(x_0) \stackrel{n\to\infty}{\longrightarrow} x_1^*$ or x_2^*. By the monotonicity $x_0 \leq x_1 \leq \ldots$ and $x_0 > 0$ we get

$$x_n = f^n(x_0) \geq x_0 > 0 \quad \forall n \in \mathbb{N}_0$$

$$\implies \quad \lim_{n\to\infty} f^n(x_0) \geq x_0 > 0$$

From this, it follows that our sequence $(x_n)_{n \in \mathbb{N}_0}$ converges to the fixed point K.