

Finite Volume Method Applied to Wave Propagation

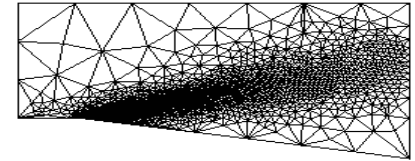
Talk at the
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by

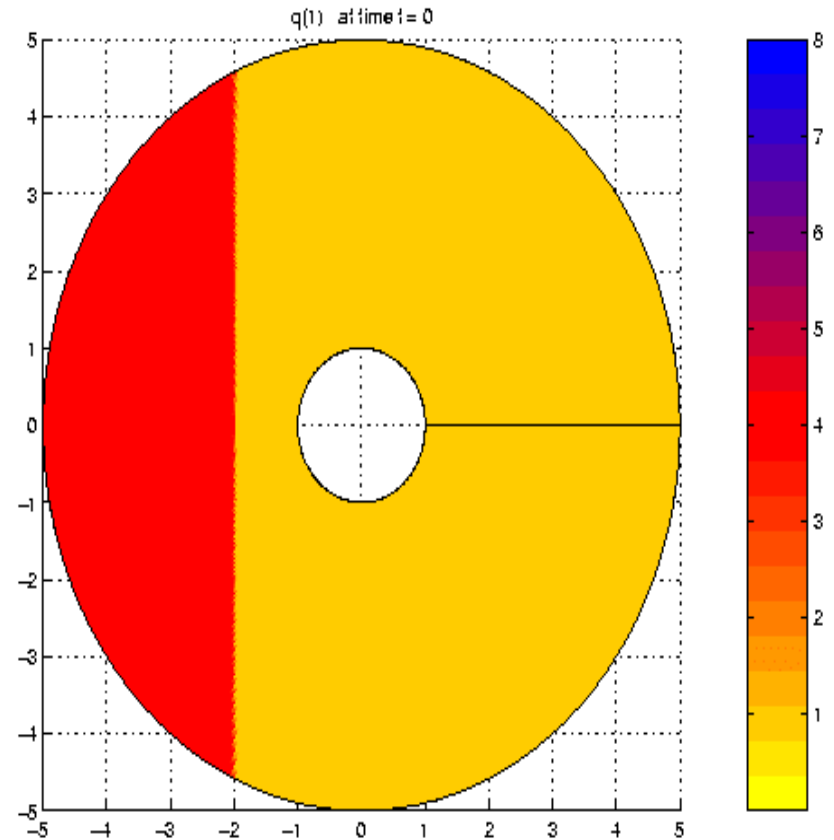
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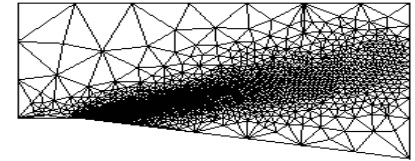
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1. Conservation Laws and Linear Hyperbolic PDEs
2. Riemann Problem in Heterogeneous Media
3. Finite Volume Methods
4. ADER
5. Outlook



Be aware: This talk contains no proofs!



Liquid flowing through a 1D-pipe

$q(x, t)$: density of chemical tracer (to be determined)

$u \equiv \text{const}$: velocity of fluid

$m_{1,2} = \int_{x_1}^{x_2} q(x, t) dx$: total mass of tracer in pipe section $x_1 \leq x \leq x_2$

$f(q, x, t) = u \cdot q(x, t)$: flux at point x and time t

We suppose mass conservation:

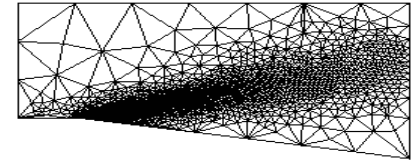
$$\frac{d}{dt} \int_{x_1}^{x_2} q(x, t) dx = - \left[f(q(x, t)) \right]_{x=x_1}^{x_2} = - \int_{x_1}^{x_2} \frac{\partial}{\partial x} f(q(x, t)) dx = 0$$

Manipulating this equation leads to general form of a conservation law:

$$\frac{\partial}{\partial t} q(x, t) + \frac{\partial}{\partial x} f(q(x, t)) = 0$$



Linear Hyperbolic PDEs



$$\frac{\partial}{\partial t} \vec{q}(x, t) + \mathbf{A} \frac{\partial}{\partial x} \vec{q}(x, t) = \vec{0}$$

is said to be hyperbolic if $\mathbf{A} \in \text{Mat}(m \times m)$ is diagonalizable with real eigenvalues :

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$$

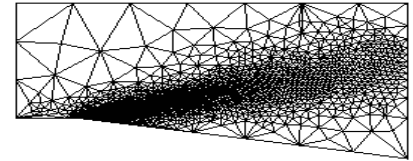
Complete set of eigenvectors: $\exists \vec{r}_i \in \mathbb{R}^m \left(\vec{r}_i \neq \vec{0}; i = 1, 2, \dots, m \right)$

Rewrite system of PDEs:

$$\mathbf{R} \equiv \left(\vec{r}_1 \mid \vec{r}_2 \mid \dots \mid \vec{r}_m \right) \Rightarrow \mathbf{R}^{-1} \mathbf{A} \mathbf{R} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$$

$$\vec{\omega}(x, t) \equiv \mathbf{R}^{-1} \vec{q}(x, t), \quad \mathbf{\Lambda} \equiv \mathbf{R}^{-1} \mathbf{A} \mathbf{R}$$

$$\Rightarrow \frac{\partial}{\partial t} \vec{\omega} + \mathbf{\Lambda} \frac{\partial}{\partial x} \vec{\omega} = \vec{0}$$



Λ diagonal, decoupling of the system of equations into m independent equations:

$$\forall_1 p \in \{1, 2, \dots, m\} : \frac{\partial}{\partial t} \omega_p + \lambda_p \frac{\partial}{\partial x} \omega_p = 0$$

$\lambda_p \in \mathbb{R} \Rightarrow$ physical sense: m waves travelling at "eigenspeeds" λ_p

Characteristic curves along which $\vec{q}(x, t)$ remains constant: $X(t) = x_0 + \lambda_p t$

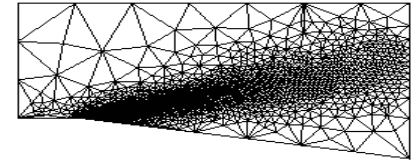
Application: elastic wave equation for pressure (i.e. longitudinal) waves in solids

$$\frac{\partial}{\partial t} \begin{pmatrix} \epsilon_{11} \\ u \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\frac{\lambda+2\mu}{\rho} & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \epsilon_{11} \\ u \end{pmatrix} = \vec{0}$$

eigenvalues of coefficient matrix: $\pm c_p = \pm \sqrt{(\lambda + 2\mu) / \rho}$



The Riemann Problem for Linear Hyperbolic PDEs



Riemann problem: hyperbolic equation together (with special initial data) which is piecewise constant with a single jump discontinuity

$$\underline{q}(x) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

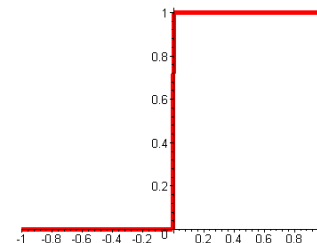
Here discussion of scalar case: $\partial_t q + u \partial_x q = 0$

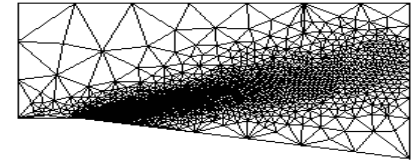
Solution: propagation of discontinuity $|q_l - q_r|$ along the single characteristic at speed u , the jump in $q(x, t)$ across p -th wave in the solution to Riemann problem equals $\alpha_p r_p$

$$\begin{aligned} \Rightarrow q(x, t) &= \underline{q}(x - ut) = \sum_{p: \lambda_p < x/t} w_p^r r_p + \sum_{p: \lambda_p > x/t} w_p^l r_p = \\ &= q_l + \sum_{p=1}^m H(x - \lambda_p t) \alpha_p r_p \quad \text{where } \forall p \in \{1, 2, \dots, m\} : \alpha_p \in \mathbb{R} \end{aligned}$$

Heaviside step function:

$$H(\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ 1 & \text{if } \xi > 0 \end{cases}$$





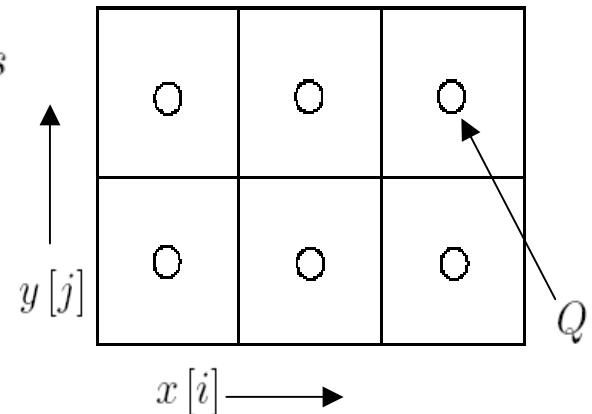
Assumptions:

- Quadrilateral (irregular) grids
- each discrete value Q : cell average over a grid cell
- Q modified due to fluxes through the edges of the cell (i. e. waves moving into cell from each edge)
- Apply this approach to any shape cell using integral form of conservation law – cf. Gauss' Theorem (here in 2D):

$$\frac{d}{dt} \iint_{\mathcal{C}} q(x, y, t) dx dy = - \int_{\partial\mathcal{C}} \vec{n}(s) \vec{f}(s, t) ds$$

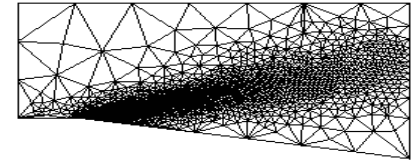
$$\text{where } \vec{f}(s, t) = \begin{pmatrix} f(q(x(s), y(s), t)) \\ g(q(x(s), y(s), t)) \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} : \text{boundary of } \mathcal{C} \text{ parametrised by arclength } s$$





Finite Volume Method Fluxes



Flux **per unit length per unit time** in direction $\vec{n}(s)$:

$$\check{F} = \vec{n}(s) \cdot \vec{f}(s, t) = n^x(s) f(q(x(s), y(s), t)) + n^y(s) \cdot g(q(x(s), y(s), t))$$

Integrate integral form conservation law from t_n to t_{n+1} and divide by $|\mathcal{C}|$ for cell average:

$$\frac{1}{|\mathcal{C}|} \iint_{\mathcal{C}} q(x, y, t_{n+1}) dx dy = \frac{1}{|\mathcal{C}|} \iint_{\mathcal{C}} q(x, y, t_n) dx dy - \frac{1}{|\mathcal{C}|} \int_{t_n}^{t_{n+1}} \int_{\partial\mathcal{C}} \vec{n}(s) \vec{f}(s, t) ds dt$$

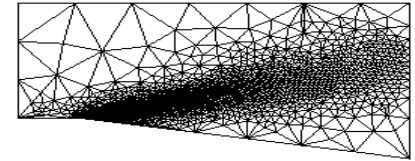
Let Q^n represent the cell average at time t_n :

$$Q^{n+1} = Q^n - \frac{\Delta t}{|\mathcal{C}|} \sum_{j=1}^N h_j \check{F}_j^n$$

N : number of cell sides
 h_j : length of j -th side of cell

Numerical approximation to average normal flux across j -th side of cell per unit length and unit time:

$$\check{F}_j^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left[\frac{1}{h_j} \int_{\text{side } j} \vec{n} \cdot \vec{f}(s, t) ds \right] dt$$



Consider uniform rectangular quadrilateral grid:

$$|\mathcal{C}| = \Delta x \Delta y; h_{i-1/2,j} = \Delta y; h_{i,j-1/2} = \Delta x$$

→

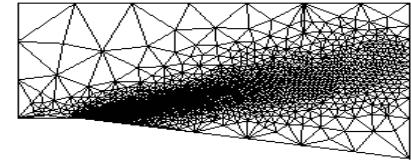
$$Q_{ij}^{n+1} = Q_{ij}^n - \frac{\Delta t}{\Delta x} (\check{F}_{i+1/2,j} - \check{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y} (\check{G}_{i,j+1/2} - \check{G}_{i,j-1/2})$$

Computational Grid	Physical Grid
$\Delta \eta, \Delta \xi$	$\vec{h}_{i-1/2,j}, \vec{n}_{i-1/2,j}$

$$\gamma_{i-1/2,j} \equiv \frac{h_{i-1/2,j}}{\Delta \eta}; \gamma_{i,j-1/2} \equiv \frac{h_{i,j-1/2}}{\Delta \xi}$$

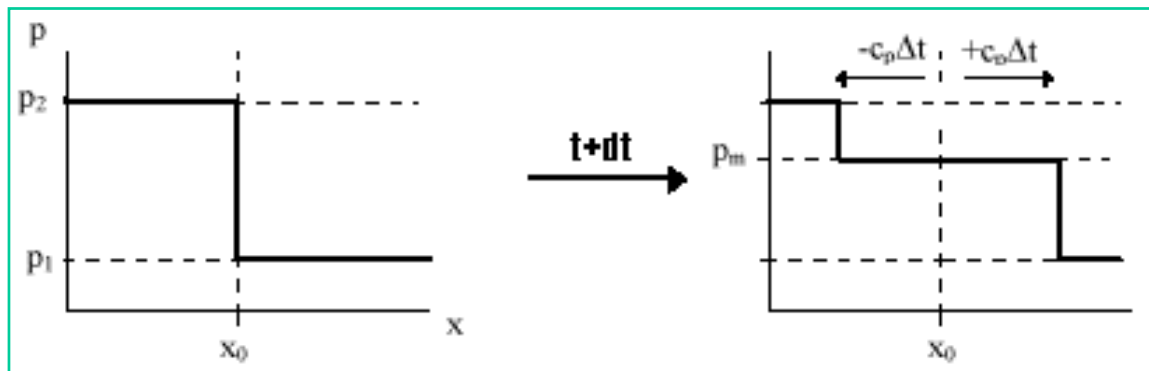
1D Riemann problem at the $(i - \frac{1}{2}, j)$ -side of a cell, flux variation only normal to edge \Rightarrow Godunov flux:

$$\check{F}_{i-1/2,j} = \vec{n}_{i-1/2,j} \cdot \vec{f}(Q_{i-1/2,j}^\downarrow)$$



Application: solve acoustic wave equation with a discontinuity \Rightarrow 2 acoustic waves which update the cell average on either side

$$\frac{\partial}{\partial t} \begin{pmatrix} p \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 & K_0 & 0 \\ 1/\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} p \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 & K_0 \\ 0 & 0 & 0 \\ 1/\rho & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} p \\ u \\ v \end{pmatrix} = \vec{0}$$



Obtain physical speeds by scaling speed of sound c by edge ratios γ , e.g.:

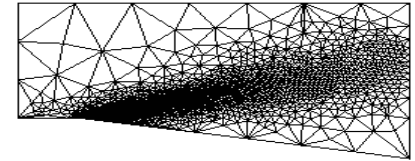
$$\gamma_{i,j-1/2} = \frac{h_{i,j-1/2}}{\Delta \xi} \Rightarrow c = \gamma \check{c}$$

Necessary knowledge for solution of this problem:

components of *velocity normal and tangential to edge in each of the two grid cells* on either side

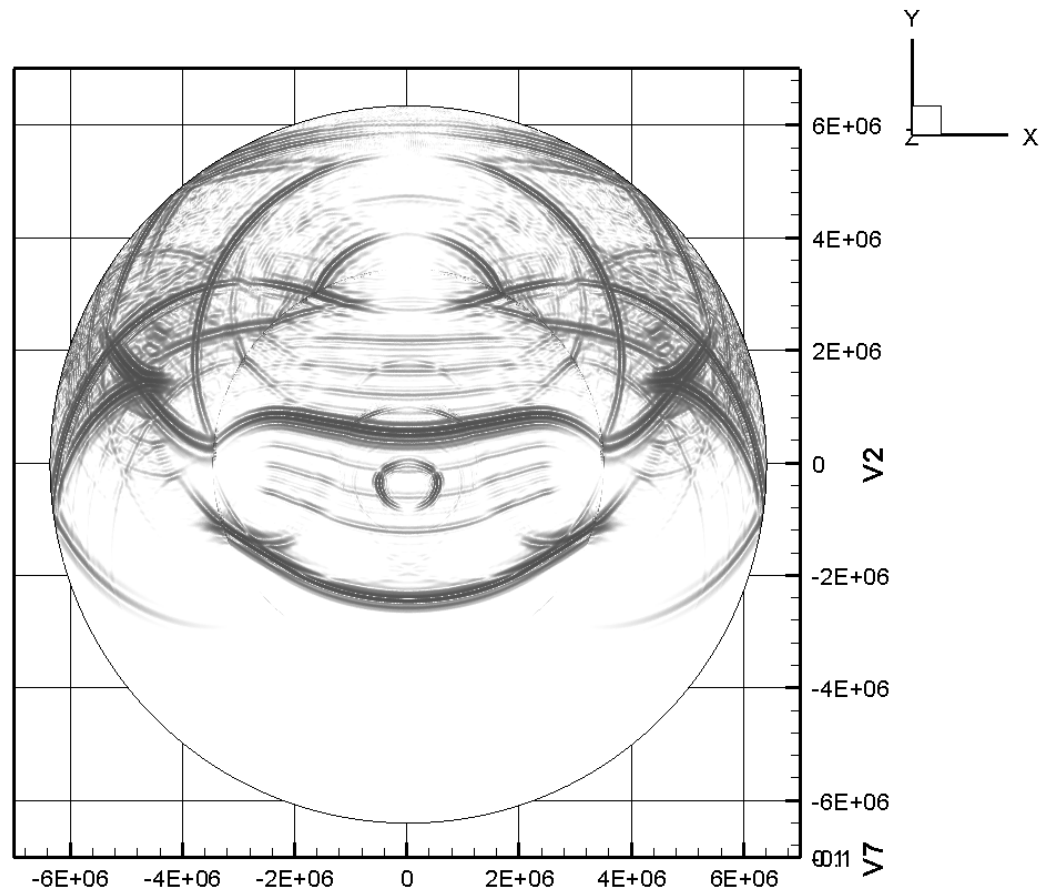


ADER Irregular meshes

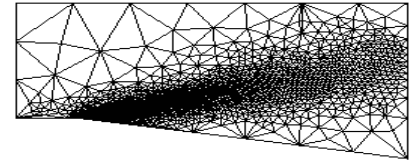


ADER: Arbitrary high order schemes using DERivatives

(Acronyms are cool... 😊)



pers. comm.
M. Käser



- [1] **LeVeque**, R. J.: Finite Volume Methods for Hyperbolic Problems, Cambridge University Press 2003
- [2] **Schwartzkopff**, T., Dumbser, M., Munz, C.-D.: Fast high order ADER schemes for linear hyperbolic equations and their numerical dissipation and dispersion, report by research group Sonderforschungsbereich 404 (Mehrfeldprobleme in der Kontinuumsmechanik) at Stuttgart University, 2003
- [3] **Schwartzkopff**, T., Dumbser, M., Munz, C.-D.: Fast high order ADER schemes for linear hyperbolic equations, J. Comp. Phys. 197 (2004) 532 – 539

Further reading:

- [4] **Käser**, M., Iske, A.: ADER Schemes on Adaptive Triangular Meshes for Scalar Conservation Laws, to appear in J. Comp. Phys.