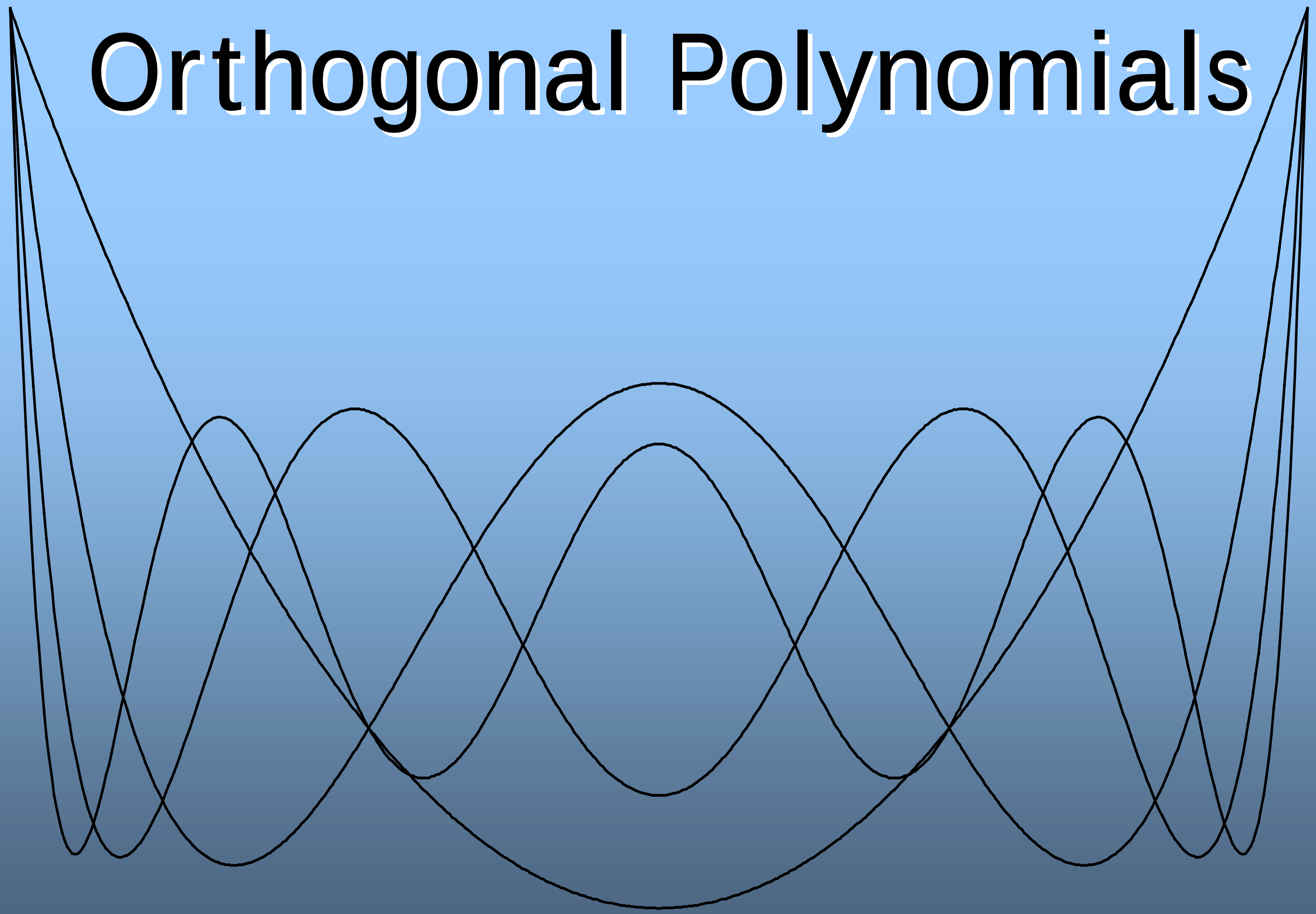
The Classical Orthogonal Polynomials



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Equation of hypergeometric type

$$u'' + \frac{\tilde{\tau}(z)}{\sigma(z)} u' + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} u = 0$$

$\sigma, \tilde{\sigma}$: polynomial of degree at most 2

$\tau, \tilde{\tau}$: polynomial of degree at most 1

λ : scalar

Substitution: $u = \phi(z)\, y$

$$\boxed{\sigma(z)\, y'' + \tau(z)\, y' + \lambda\, y = 0}$$

Important property:
All derivatives of functions of hypergeometric type are also of hypergeometric type

Differentiate: $\sigma y'' + \tau y' + \lambda y = 0$

$$\sigma y''' + (\tau + \sigma')y'' + (\lambda + \tau')y' = 0 \qquad v_1 = y'$$

$$y = -\frac{1}{\lambda}\left[\sigma v_1' + \tau v_1\right]$$

$$\sigma(z)v_n'' + \tau_n(z)v_n' + \mu_n v_n = 0 \qquad \text{with} \qquad \tau_n(z) = \tau(z) + n\sigma'(z)$$

$$\mu_n = \lambda + n\tau' + \frac{n(n-1)}{2}\sigma''$$

If μ_n=0 we can construct solutions for y

Condition: $\lambda = -n\tau' - \frac{1}{2}n(n-1)\sigma''$

Rodrigues formula

Equation of hypergeometric type: $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$

$\sigma(z)$:	polynomial of degree at most 2
$\tau(z)$:	polynomial of degree at most 1
λ:	scalar

Self-adjoint form: $\left(\sigma\,\rho\,y'\right)' + \lambda\,\rho\,y = 0$

Pearson equation: $\left[\sigma(z)\ \rho(z)\right]' = \tau(z)\ \rho(z)$

Condition: $\lambda = -n\,\tau' - \frac{1}{2}n\,(n-1)\,\sigma''$

Rodrigues formula: $y_n(z) = \dfrac{B_n}{\rho(z)} \cdot \left[\sigma^n(z)\ \rho(z)\right]^{(n)}$

Classification

degree 2

$$\sigma(z) = (b - z)(z - a)$$

degree 1

$$\sigma(z) = z - a$$

degree 0

$$\sigma(z) = 1$$

What is $\rho(z)$ and $y_n(z)$?

σ=1

Pearson equation: $\left[\sigma(z)\ \rho(z)\right]' = \tau(z)\ \rho(z)$

Let $\tau = 2\alpha z + \beta \qquad \alpha, \beta \in \mathbb{C}$

$$\rho' = \left(2\alpha z + \beta\right)\rho \qquad \frac{d\rho}{\rho} = \left(2\alpha z + \beta\right)dz \qquad \ln\left(\frac{\rho}{C}\right) = \alpha z^2 + \beta z$$

$$\rho(z) = C \cdot e^{\alpha z^2 + \beta z}$$

Linear change of the variable z:

Canonical form: $\rho(z)=e^{-z^2}$

σ=z-a

Let $\tau = \alpha z + \beta + 1 \qquad \alpha, \beta \in \mathbb{C}$

$$((z-a)\rho)' = (\alpha z + \beta + 1)\rho \qquad \rho' = \frac{\alpha z + \beta}{z-a}\rho \qquad \ln\left(\frac{\rho}{C}\right) = \int \frac{\alpha z + \beta}{z-a} dz$$

$$\rho(z) = C\, e^{\alpha z}\, (z-a)^{a\alpha + \beta}$$

Canonical form: $\rho(z) = z^{\tilde{\alpha}}\, e^{-z}$

σ=(b-z)(z-a)

Let $\tau = (\alpha - 2)z + \beta \qquad \alpha, \beta \in \mathbb{C}$

$$((b-z)(z-a)\rho)' = ((\alpha - 2)z + \beta)\rho \qquad \rho' = \frac{\alpha z + \beta - a - b}{(b-z)(z-a)}\rho$$

$$\rho(z) = C\,(z-a)^{\frac{a+b-\beta-\alpha a}{a-b}}\,(z-b)^{\frac{\alpha b + b - a - b}{a-b}}$$

Canonical form: $\rho(z) = (1-z)^{\tilde{\alpha}}(1+z)^{\tilde{\beta}}$

Classification

normal forms		canonical forms	
$\sigma(z)$	$\rho(z)$	$\sigma(z)$	$\rho(z)$
$(b-z)\cdot(z-a)$	$(b-z)^{\alpha}\cdot(z-a)^{\beta}$	$1-z^2$	$(1-z)^{\alpha}\cdot(1+z)^{\beta}$
$z-a$	$(z-a)^{\alpha}\,e^{\beta z}$	z	$z^{\alpha}e^{-z}$
1	$e^{\alpha z^2+\beta z}$	1	e^{-z^2}

Polynomials

Rodrigues formula: $y_n(z) = \frac{B_n}{\rho(z)} \cdot \left[\sigma^n(z)\ \rho(z)\right]^{(n)}$

$\sigma(z) = 1 - z^2$

Jacobi polynomials: $P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}\left[(1-z)^{n+\alpha}(1+z)^{n+\beta}\right]$

$\sigma(z) = z$

Laguerre polynomials: $L_n^{\alpha}(z) = \frac{1}{n!}e^z z^{-\alpha}\frac{d^n}{dz^n}\left[z^{\alpha+n}e^{-z}\right]$

$\sigma(z) = 1$

Hermite polynomials: $H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\left[e^{-z^2}\right]$

Integral Representation

Rodrigues formula: $y_n(z) = \frac{B_n}{\rho(z)} \cdot \left[\sigma^n(z)\ \rho(z) \right]^{(n)}$

With Cauchy‘s integral formula:

$$y_n(z) = \frac{C_n}{\rho(z)} \int_C \frac{\sigma^n(s)\,\rho(s)}{(s-z)^{n+1}} ds \qquad C_n = \frac{B_n\, n!}{2\pi i}$$

Idea: Can this be used to get solutions for arbitrary complex n respective λ ?

And the answer is: YES! ;)

Condition for the (open or closed) contour C: $\left. \frac{\sigma^{\nu+1}(s)\ \rho(s)}{(s-z)^{\nu+2}} \right|_{s_1}^{s_2} = 0$

Orthogonality

Scalar product for functions:

$$< y_n(z), y_m(z) >= \int_a^b y_n(z) \cdot y_m(z)\ \rho(z)\ dz$$

Functions are orthogonal when: $< y_n(z), y_m(z) >= 0$

For n≠m:

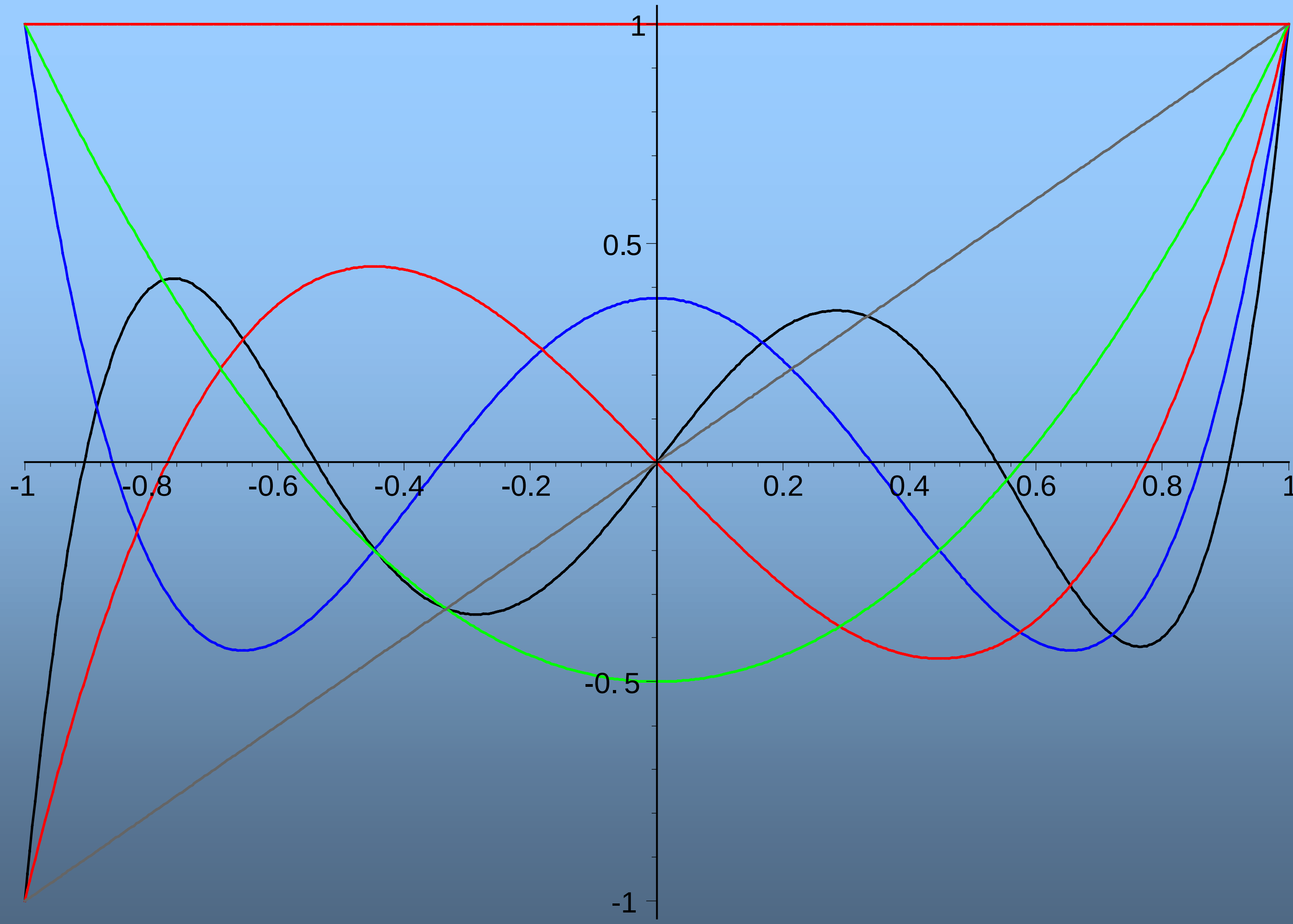
$$< J_n^{(\alpha,\beta)}(z), J_m^{(\alpha,\beta)}(z) >= 0$$

$$< L_n^{\alpha}(z), L_m^{\alpha}(z) >= 0$$

$$< H_n(z), H_m(z) >= 0$$

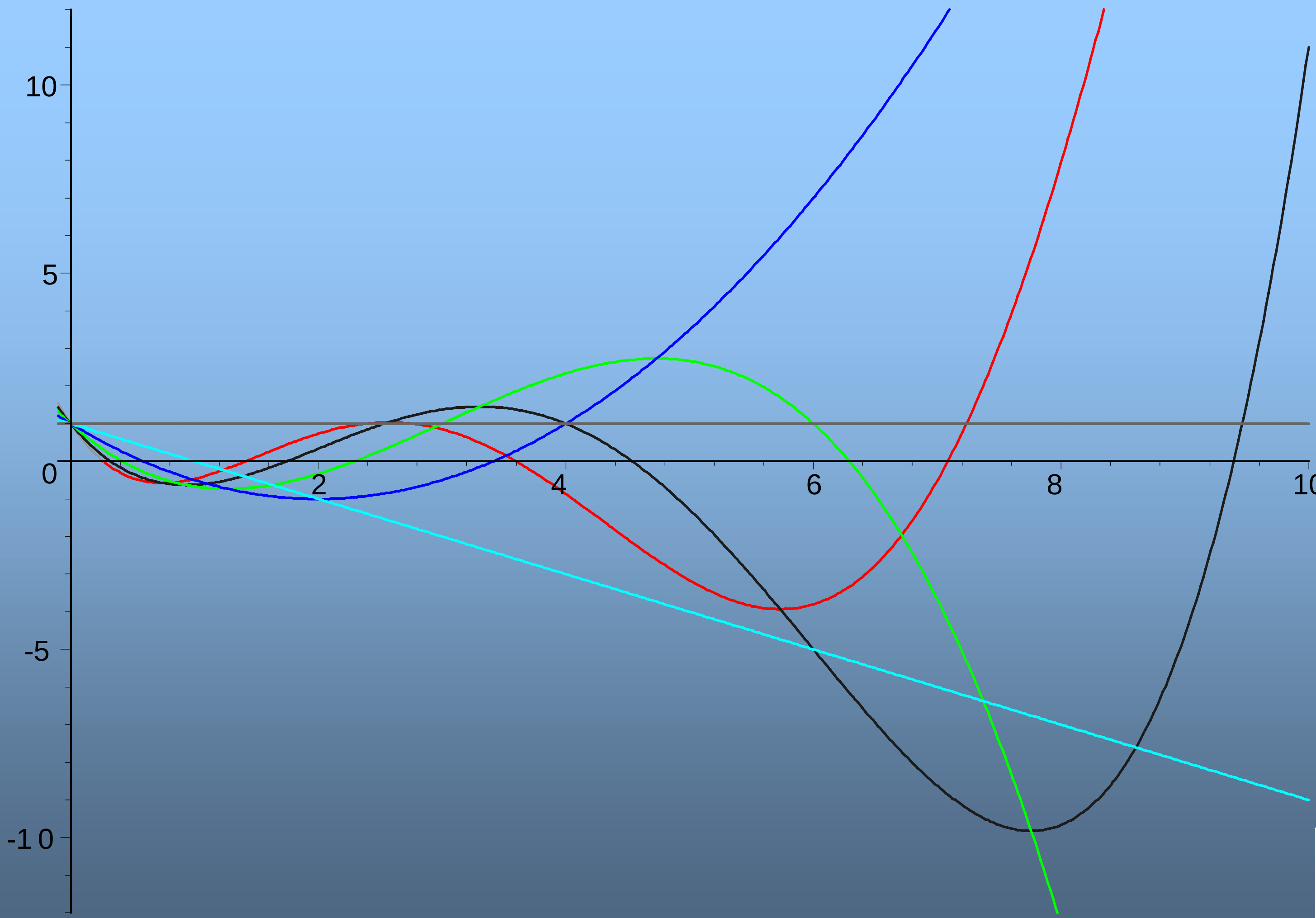
Jacobi polynomials $P_n^{(0,0)}(z)$

n=0..5



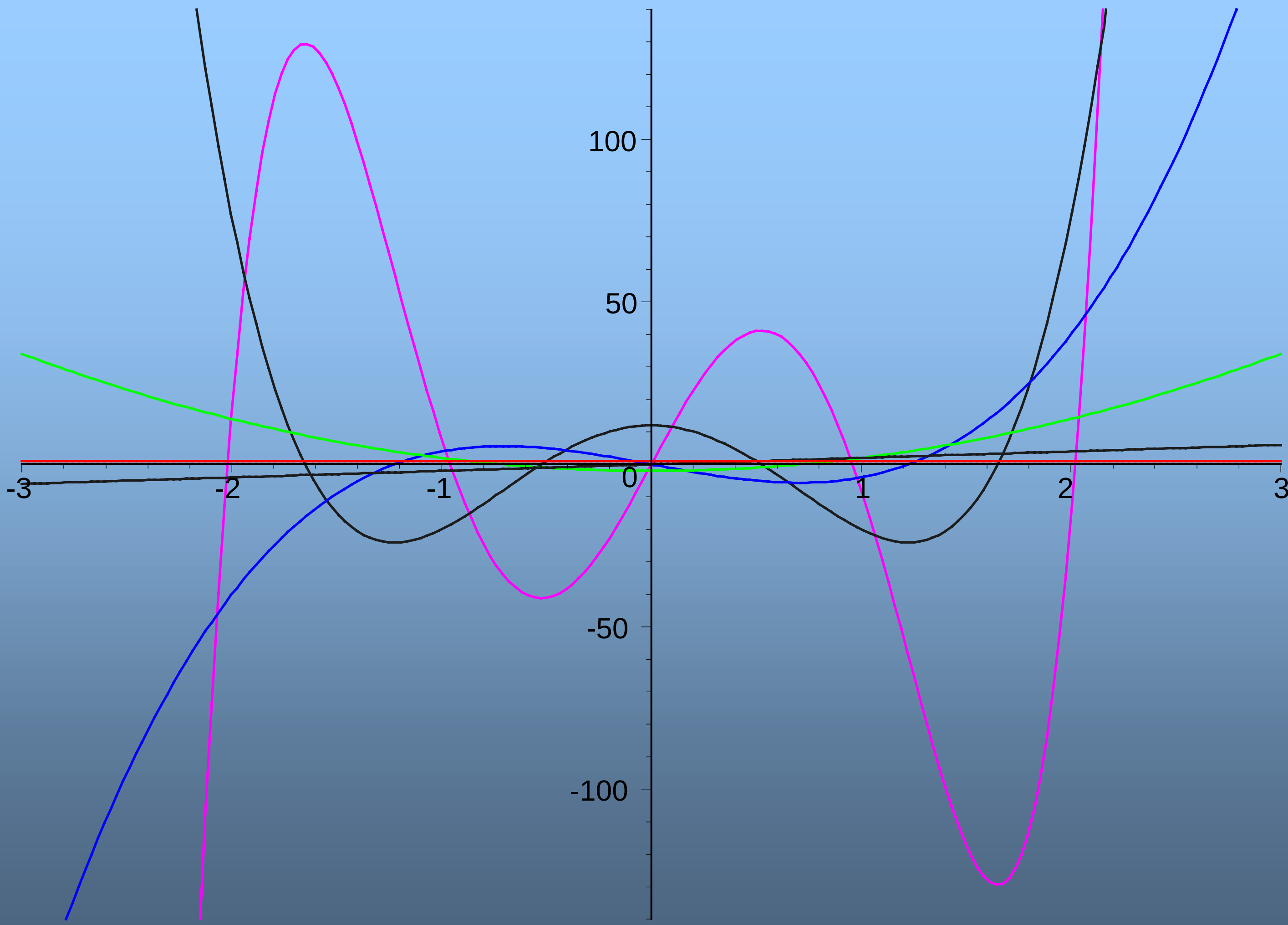
Laguerre polynomials $L_n^0(z)$

n=0..5



Hermite polynomials $H_n(z)$

n=0..5



Thank you
for your attention!