SIAM J. Appl. Math.
Vol. 65, No. 3, pp. 983–1005

HOMOGENIZATION OF INTERFACES BETWEEN RAPIDLY OSCILLATING FINE ELASTIC STRUCTURES AND FLUIDS*

K.-H. HOFFMANN†, N. D. BOTKIN†, AND V. N. STAROVOITOV‡

Abstract. The paper studies the interaction of a periodic solid bristle structure with a fluid. Such problems arise, for example, when modelling biotechnological devices operating in liquids or when simulating epithelium surfaces of blood vessels. The fluid is described by the linearized Navier–Stokes equation whereas the solid part is governed by equations of linear elasticity. The interface conditions are accounted. A homogenized model of the structure is derived by employing the two-scale convergence technique. The model describes a new material which possesses some interesting properties.

Key words. homogenization, fluid-solid interface, biosensor, multi-layered structure

AMS subject classifications. 35B27, 74F10, 74Q10

DOI. 10.1137/S0036139903421572

1. Introduction. We study a mechanical system consisting of a fluid and a rapidly oscillating elastic fine structure interacting with the fluid. The goal is to obtain averaged equations which effectively describe the behavior of the system.

This investigation is motivated by modelling a surface acoustic wave sensor based on the generation and detection of horizontally polarized shear waves (see [3]). Acoustic shear waves are excited through an alternate voltage applied to electrodes deposited on a quartz crystal substrate. The waves are transmitted into a thin isotropic guiding layer covered by a thin gold film that contacts a liquid containing a protein to be detected. The protein adheres to a specific receptor (aptamer) placed on the surface of the gold film. The arising mass loading causes a phase shift in the electric signal to be measured by an electronic circuit.

One can impress the aptamer-protein layer as a periodic bristle or pin structure on the top of the gold film contacting with the liquid (see Figure 1). The thickness of the aptamer-protein layer is about 4 nm, and the number of bristles per surface unit is enormous large. Therefore, the direct numerical modelling of such a structure using fluid-solid interface conditions is impossible. Proper models can be derived using the homogenization technique from [12], [11], [1], [7], [8], and [5] along with the strict treatment of the solid-fluid interface (see, e.g., [6]).

Problems that are close to ours were studied in [13] and [2]. L. Baffico and C. Conca [2] considered the same geometry but the equations differ from ours. J. Sanchez-Hubert [13] investigated almost the same problem. She used techniques based on the Laplace transformation whereas we apply another approach which makes it possible to obtain an explicit representation of solutions to the cell equation, which allows us to investigate the limiting equations and to develop numerical algorithms.

2. Mathematical model. The coupled mechanical system under consideration is shown in Figure 1. The solid part consists of a substrate and pins located on its top.

*Received by the editors January 20, 2003; accepted for publication (in revised form) April 2, 2004; published electronically March 31, 2005.
http://www.siam.org/journals/siap/65-3/42157.html

†Center of Advanced European Studies and Research, Ludwig-Erhard-Allee 2, 53175 Bonn, Germany (phoffman@caesar.de, botkin@caesar.de).
‡Lavrentyev Institute of Hydrodynamics, 630090 Novosibirsk, Russia, Center of Advanced European Studies and Research, Ludwig-Erhard-Allee 2, 53175 Bonn, Germany (starovoitov@caesar.de).

The pin structure is assumed to be periodic in the plane (x_1, x_2) and independent of x_3. The domain of the coupled system is denoted by $\Omega \subset \mathbb{R}^3$. For simplicity, we suppose that Ω is the cube $\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, x_k \in (-1;+1),\ k = 1,2,3\}$. The domains occupied by the fluid and elastic continua are denoted by $\Omega_{_F}$ and $\Omega_{_S}$, respectively; the boundary separating the continua by Γ. Thus, $\Omega = \Omega_{_F} \cup \Gamma \cup \Omega_{_S}$. Let $(\partial\Omega)_{_F} = \partial\Omega \cap \overline{\Omega}_{_F}$ and $(\partial\Omega)_{_S} = \partial\Omega \cap \overline{\Omega}_{_S}$. Then the sets $\Gamma \cup (\partial\Omega)_{_F}$ and $\Gamma \cup (\partial\Omega)_{_S}$ are the boundaries of the domains $\Omega_{_F}$ and $\Omega_{_S}$, respectively.

2.1. Governing equations. We assume that the fluid is weakly compressible, which is physically correct because the operation frequency of the coupled structure lies in the acoustic range and the displacements of the fluid particles are small. This is a typical acoustic approximation which additionally utilizes linearized Navier–Stokes equations (see [9]).

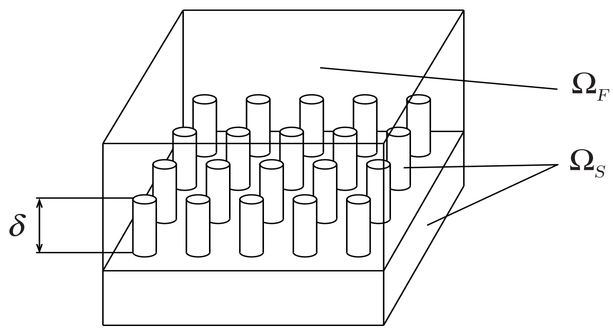

FIG. 1. *Coupled system:* $\Omega = \Omega_{_F} \cup \Gamma \cup \Omega_{_S}$

The solid part of the system will be described using the linear elasticity approach. This linear setting is supplemented by the assumption that the domains $\Omega_{_F}$ and $\Omega_{_S}$ remain unchangeable. Therefore, the coupled mechanical system is described by the following equations:

$$\rho_{_F} \boldsymbol{u}_t = -\nabla p + \operatorname{div} P\boldsymbol{u}_{\boldsymbol{x}} + \rho_{_F} \boldsymbol{f} \quad \text{in} \quad \Omega_{_F}, \tag{2.1}$$

$$\gamma p_t = -\operatorname{div} \boldsymbol{u} \quad \text{in} \quad \Omega_{_F}, \tag{2.2}$$

$$\rho_{_S} \boldsymbol{v}_{tt} = \operatorname{div} G\boldsymbol{v}_{\boldsymbol{x}} + \rho_{_S} \boldsymbol{f} \quad \text{in} \quad \Omega_{_S}. \tag{2.3}$$

Let $\boldsymbol{n}$ be the normal vector to the fluid-solid interface Γ. The no-slip and stress equilibrium conditions on Γ read

$$\boldsymbol{v}_t = \boldsymbol{u} \quad \text{on} \quad \Gamma, \tag{2.4}$$

$$G\boldsymbol{v}_{\boldsymbol{x}} \cdot \boldsymbol{n} = \big(-p\mathcal{I} + P\boldsymbol{u}_{\boldsymbol{x}} \big) \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma. \tag{2.5}$$

The boundary and initial conditions are prescribed:

$$\boldsymbol{u} = 0 \quad \text{on} \quad (\partial\Omega)_{_F}, \tag{2.6}$$

$$\boldsymbol{v} = 0 \quad \text{on} \quad (\partial\Omega)_{_S}, \tag{2.7}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^0,\ p|_{t=0} = p^0 \quad \text{in} \quad \Omega_{_F}, \tag{2.8}$$

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}^0,\ \boldsymbol{v}_t|_{t=0} = \boldsymbol{v}'^{\,0} \quad \text{in} \quad \Omega_{_S}. \tag{2.9}$$

Here, ρ_F and ρ_S are the constant densities of the fluid and of the solid parts, respectively; $\boldsymbol{u}$ is the velocity field of the fluid, p is the pressure in the fluid, $\boldsymbol{v}$ is the displacement field of the solid part, and $\boldsymbol{f}$ is an external force like the gravity. The coefficient γ characterizes the compressibility of the fluid. The fourth-rank tensor $P = \{P_{ijkl}\}$ is defined through the relation

$$P\boldsymbol{u}_{\boldsymbol{x}} = \lambda \mathcal{I} \operatorname{div} \boldsymbol{u} + 2\mu \mathcal{D}(\boldsymbol{u}). \tag{2.10}$$

The unit tensor $\mathcal{I}$ has the components $\mathcal{I}_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. The strain velocity tensor $\mathcal{D}(\boldsymbol{u})$ has, as is usual, the components $\mathcal{D}_{ij}(\boldsymbol{u}) = 1/2\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$. The symbols λ and μ denote positive balk and dynamic viscosity coefficients of the fluid, respectively. As is usual, the summation over repeating indices is assumed. The components G_{ijkl} of the elastic stiffness tensor G can be arbitrary up to base restrictions so that arbitrary *anisotropic solids* can be considered.

The model (2.1)–(2.9) was investigated in [10] where it was supposed to use the velocity instead of the displacement in (2.3). Following this approach, we introduce the integral operator

$$\mathcal{J}_t \boldsymbol{w} = \int_0^t \boldsymbol{w}(s) ds$$

that enables us to rewrite (2.3) in the form

$$\rho_S \boldsymbol{u}_t = \operatorname{div}\big(G\mathcal{J}_t \boldsymbol{u}_{\boldsymbol{x}}\big) + \operatorname{div} \mathcal{G}^0 + \rho_S \boldsymbol{f}, \tag{2.11}$$

where $\boldsymbol{u} = \boldsymbol{v}_t$, $\mathcal{G}^0 = G\boldsymbol{v}^0_{\boldsymbol{x}}$ in Ω_S. Similarly, the pressure p can be expressed from (2.3) as follows:

$$p = -\gamma^{-1} \operatorname{div} \mathcal{J}_t \boldsymbol{u} + p^0 \quad \text{in} \quad \Omega_F. \tag{2.12}$$

Let χ be the characteristic function of the domain Ω_F. Then (2.1), (2.2), and (2.3) can be written in the whole domain Ω as one equation with discontinuous coefficients

$$\rho\, \boldsymbol{u}_t = \operatorname{div}\big(\boldsymbol{M}^t \boldsymbol{u}_x\big) + \operatorname{div} \mathcal{N}^0 + \rho \boldsymbol{f}, \tag{2.13}$$

where

$$\boldsymbol{M}^t = \chi P + \big(\chi\gamma^{-1}\mathcal{I} \otimes \mathcal{I} + (1-\chi)G\big)\ \mathcal{J}_t,$$

$$\rho = \rho_F \chi + \rho_S (1-\chi), \quad \mathcal{N}^0 = -\chi p^0 \mathcal{I} + (1-\chi)\mathcal{G}^0.$$

The interface condition (2.4) is equivalent to the "continuity" of $\boldsymbol{u}$ on Γ but the condition (2.5) now assumes the form

$$\big(G\mathcal{J}_t \boldsymbol{u}_{\boldsymbol{x}} + \mathcal{G}^0\big) \cdot \boldsymbol{n} = \big(\gamma^{-1} \operatorname{div} \mathcal{J}_t\, \boldsymbol{u} I - p^0 I + P\boldsymbol{u}_{\boldsymbol{x}}\big) \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma \tag{2.14}$$

accounting (2.12). The boundary and initial data are

$$\boldsymbol{u} = 0 \qquad \text{on} \quad (\partial\Omega)_F, \tag{2.15}$$

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}^0 \qquad \text{in} \quad \Omega, \tag{2.16}$$

where the fluid initial condition $\boldsymbol{u}^0$ is extended to Ω_S by setting $\boldsymbol{u}^0(\boldsymbol{x}) = \boldsymbol{v}'^0(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega_S$.

Remark 2.1. One can forget the initial distribution $\boldsymbol{v}^0$ of the displacement when considering (2.13). It is sufficient to prescribe the initial velocity field $\boldsymbol{u}^0$ in Ω, the initial stress $\mathcal{G}^0$ in Ω_S (this replaces the information about $\boldsymbol{v}^0$), and initial pressure p^0 in Ω_F. The functions $\mathcal{G}^0$ and p^0 yield the function $\mathcal{N}^0$ involved in (2.13).

Remark 2.2. For mechanical reasons, the tensors P_{ijkl} and G_{ijkl} have the following properties:

$$Z_{ijkl} = Z_{ijlk} = Z_{klij} = Z_{jikl}, \quad Z_{ijkl}\mathcal{V}_{ij}\mathcal{V}_{kl} \geqslant 0,$$

$$Z_{ijkl}\mathcal{V}_{ij}\mathcal{V}_{kl} = 0 \quad \text{if and only if} \quad \mathcal{V}_{kl} + \mathcal{V}_{lk} = 0 \quad \text{for all} \quad k,l = 1,2,3.$$

Here, Z_{ijkl} stands for P_{ijkl} or G_{ijkl}.

2.2. Refinement of the structure. Let us define the structure of the regions Ω, Ω_F, and Ω_S more precisely. The pin structure (see Figure 1) is supposed to be (x_1, x_2)-periodic. Without loss of generality, we assume that the periodicity cell is a square with the side length equal to ε, where ε is a positive number. After scaling with the factor $1/\varepsilon$, the cell becomes the unit square $\Sigma = [0,1] \times [0,1]$. Let Σ_S be the $1/\varepsilon$-scaled projection of a solid pin to the (x_1, x_2)-plane. It is assumed to be a smooth, simply connected domain in Σ such that its boundary $\partial\Sigma_S$ does not meet $\partial\Sigma$. Denote by Σ_F the domain $\Sigma \setminus \overline{\Sigma}_S$ (see Figure 2).

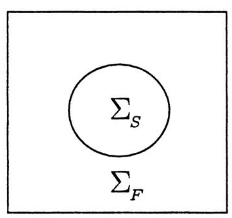

FIG. 2. *Structural cell* $\Sigma = [0,1] \times [0,1]$.

Let $\hat{\boldsymbol{x}} = (x_1, x_2)$ and $\hat{\chi}(\hat{\boldsymbol{x}})$ be the Σ-periodic extension of the characteristic function of the domain Σ_F to all $\mathbb{R}^2$. Then the function χ introduced in the previous subsection can be represented as follows:

$$\chi(\boldsymbol{x}) = \chi(\hat{\boldsymbol{x}}, x_3) = \begin{cases} 1, & x_3 > \delta, \\ \hat{\chi}(\frac{\hat{\boldsymbol{x}}}{\varepsilon}), & 0 \leqslant x_3 \leqslant \delta, \\ 0, & x_3 < 0. \end{cases} \tag{2.17}$$

Remember that δ is the thickness of the pin layer. If $\varepsilon \to 0$, the pin structure becomes finer in the (x_1, x_2)-plane, whereas its height remains constant. Thus, the problem (2.13)–(2.16) depends in fact on ε. For this reason, we call it *Problem* S_ε.

DEFINITION 2.3. *A function* $\boldsymbol{u}$ *is called a weak solution to Problem* S_ε *if*

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega_F)), \quad \mathcal{J}_t\boldsymbol{u} \in L^\infty(0,T;H_0^1(\Omega)) \tag{2.18}$$

and the integral identity

$$\int_0^T \int_\Omega \Big(-\rho \boldsymbol{u} \cdot \boldsymbol{\varphi}_t + \boldsymbol{M}^t \boldsymbol{u}_x : \boldsymbol{\varphi}_x + \mathcal{N}^0 : \boldsymbol{\varphi}_x - \rho \boldsymbol{f} \cdot \boldsymbol{\varphi} \Big) d\boldsymbol{x} dt = \int_\Omega \rho \boldsymbol{u}^0 \cdot \boldsymbol{\varphi}^0 d\boldsymbol{x} \tag{2.19}$$

holds for every smooth function $\boldsymbol{\varphi}$ *such that* $\boldsymbol{\varphi}|_{t=T} = \boldsymbol{\varphi}|_{\partial\Omega} = 0$.

In this definition and further, T is an arbitrary positive number; the colon denotes the convolution of tensors so that $\mathcal{U} : \mathcal{V} = \mathcal{U}_{ij}\mathcal{V}_{ij}$ for all second-rank tensors $\mathcal{U}$ and $\mathcal{V}$; and the notation f^0 means $f|_{t=0}$. Remark that the second inclusion of (2.18) prevents jumps of $\boldsymbol{u}$ on Γ.

2.3. Solvability of Problem $\mathbf{S}_{\boldsymbol{\varepsilon}}$. It is not difficult to prove existence of a weak solution to Problem S_ε. This question was investigated in [10, section 9.1], and the following result was established.

THEOREM 2.4. *Let* $\boldsymbol{u}^0 \in L^2(\Omega)$, $\mathcal{N}^0 \in L^2(\Omega)$, *and* $\boldsymbol{f} \in L^2([0,T] \times \Omega)$. *Then there exists a unique weak solution to Problem* S_ε, *and the following energy estimate holds:*

$$\operatorname{ess\,sup}_{t\in(0,T)} \Big(\|\boldsymbol{u}(t)\|^2_{L^2(\Omega)} + \|\mathcal{D}\big(\mathcal{J}_t \boldsymbol{u}\big)\|^2_{L^2(\Omega_S)} \Big) + \int_0^T \|\mathcal{D}(\boldsymbol{u}(t))\|^2_{L^2(\Omega_F)}\, dt \leqslant C, \tag{2.20}$$

where C *is a constant which depends on* $\|\boldsymbol{u}^0\|_{L^2(\Omega)}$, $\|\mathcal{N}^0\|_{L^2(\Omega)}$, *and* $\|\boldsymbol{f}\|_{L^2([0,T]\times\Omega)}$ *but does not depend on* ε.

COROLLARY 2.5. *Under the conditions of Theorem* 2.4, *there exists an independent of* ε *constant* C *such that*

$$\operatorname{ess\,sup}_{t\in(0,T)} \|\mathcal{J}_t \boldsymbol{u}(t)\|_{H^1(\Omega)} \leqslant C. \tag{2.21}$$

Generally speaking, the estimates (2.20) and (2.21) are sufficient to fulfill the homogenization of Problem S_ε due to Proposition 3.9 which will be given below. However, some technical difficulties must be overcome in this case. To avoid that, a stronger estimate for $\boldsymbol{u}$ will be obtained under some compatibility conditions. The next theorem states such a result.

THEOREM 2.6. *Let* $\boldsymbol{u}^0 \in H^1(\Omega)$, $\mathcal{N}^0 \in L^2(\Omega)$, $\boldsymbol{f}, \boldsymbol{f}_t \in L^2([0,T] \times \Omega)$, *and*

$$\operatorname{div}\big(\chi P \boldsymbol{u}^0_{\boldsymbol{x}} + \mathcal{N}^0\big) \in L^2(\Omega). \tag{2.22}$$

Then the weak solution to Problem S_ε *satisfies the estimate*

$$\operatorname{ess\,sup}_{t\in(0,T)} \big(\|\boldsymbol{u}_t(t)\|_{L^2(\Omega)} + \|\boldsymbol{u}_x(t)\|_{L^2(\Omega)} \big) \leqslant C, \tag{2.23}$$

where C *is an independent of* ε *constant.*

Proof. Let us introduce a function $\boldsymbol{w}$ as a solution of the problem

$$\begin{aligned}
&\rho\, \boldsymbol{w}_t = \operatorname{div}\big(\boldsymbol{M}^t \boldsymbol{w}_x\big) + \operatorname{div}\big(\chi\gamma^{-1} I \operatorname{div} \boldsymbol{u}^0 + (1-\chi) G \boldsymbol{u}^0_{\boldsymbol{x}}\big) + \rho \boldsymbol{f}_t,\\
&\rho\, \boldsymbol{w}|_{t=0} = \rho\, \boldsymbol{w}_0 = \operatorname{div}\big(P u^0_{\boldsymbol{x}} + \mathcal{N}^0\big) + \rho \boldsymbol{f}^0,\\
&\boldsymbol{w}|_{\partial\Omega} = 0.
\end{aligned}$$

The energy estimate for this problem appears as follows:

$$\operatorname{ess\,sup}_{t\in(0,T)} \Big(\|\boldsymbol{w}(t)\|^2_{L^2(\Omega)} + \|\mathcal{D}\big(\mathcal{J}_t \boldsymbol{w}\big)\|^2_{L^2(\Omega_S)} \Big) + \int_0^T \|\mathcal{D}(\boldsymbol{w}(t))\|^2_{L^2(\Omega_F)}\, dt \leqslant C, \tag{2.24}$$

which yields

$$\operatorname{ess\,sup}_{t\in(0,T)}\|\mathcal{J}_t \boldsymbol{w}\|_{H^1(\Omega)} \leqslant C.$$

The assertion of the theorem is an immediate consequence of the last estimates because the function defined as

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_0^t \boldsymbol{w}(\boldsymbol{x},s)\,ds + \boldsymbol{u}^0(\boldsymbol{x}) = \mathcal{J}_t\boldsymbol{w}(\boldsymbol{x},t) + \boldsymbol{u}^0(\boldsymbol{x})$$

is the solution of Problem S_ε, and $\boldsymbol{u}_t = \boldsymbol{w}$. □

According to the definition of $\boldsymbol{u}^0$, the requirement $\boldsymbol{u}^0 \in H^1(\Omega)$ expresses the no-slip condition on Γ at the initial time instant $t = 0$. The requirement (2.22) expresses the stress equilibrium condition on Γ at $t = 0$. From the mechanical point of view, such conditions hold for any time instant including the initial one. Therefore, the requirements of the theorem are feasible.

3. Homogenization of the structure.

3.1. Two-scale convergence. Let us denote by $\boldsymbol{u}_\varepsilon$ the solution of Problem S_ε. In order to emphasize the dependence of χ on ε, we denote it by χ^ε. Our goal is to perform the passage to the limit in Problem S_ε as $\varepsilon \to 0$. To do this, we use the two-scale convergence method introduced by G. Nguetseng and developed by other mathematicians (see [12], [11], [1], [7]). Let us formulate the main results of this approach adapted to our situation.

THEOREM 3.7. *Let $\boldsymbol{w}_\varepsilon$ be a bounded sequence in $L^2\big([0,T]\times\Omega\big)$. There exists a subsequence, still denoted by $\boldsymbol{w}_\varepsilon$, and a function $\overline{\boldsymbol{w}}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}}) \in L^2\big([0,T]\times\Omega\times\Sigma\big)$ such that*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \boldsymbol{w}_\varepsilon(t,\boldsymbol{x})\boldsymbol{\phi}\left(t,\boldsymbol{x},\frac{\hat{\boldsymbol{x}}}{\varepsilon}\right)d\boldsymbol{x} = \int_0^T\int_\Omega\int_\Sigma \overline{\boldsymbol{w}}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})\boldsymbol{\phi}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})\,d\hat{\boldsymbol{\xi}}d\boldsymbol{x}dt$$

for every smooth function $\boldsymbol{\phi}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})$ which is Σ-periodic in $\hat{\boldsymbol{\xi}}$. Such a sequence $\boldsymbol{w}_\varepsilon$ is said to be two-scale convergent to $\overline{\boldsymbol{w}}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})$.

Recall the notation $\hat{\boldsymbol{x}} = (x_1,x_2)$ and $\hat{\boldsymbol{\xi}} = (\xi_1,\xi_2)$.

THEOREM 3.8. *Let a sequence $\boldsymbol{w}_\varepsilon$ converge weakly to $\boldsymbol{w}$ in $L^2\big(0,T;H^1(\Omega)\big)$. Then $\boldsymbol{w}_\varepsilon$ two-scale converges to $\boldsymbol{w}$ and there exists a function $\overline{\boldsymbol{w}}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})$ in $L^2\big([0,T]\times\Omega;H^1_\#(\Sigma)/\mathbb{R}\big)$ such that $\nabla\boldsymbol{w}_\varepsilon$ two-scale converges to $\nabla_x\boldsymbol{w}(t,\boldsymbol{x}) + \nabla_\xi\overline{\boldsymbol{w}}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})$ up to a subsequence.*

Here $H^1_\#(\Sigma)$ is the space of Σ–periodic functions which belong to the space $H^1(\Sigma)$. Since all functions under consideration do not depend on ξ_3, the notation $\nabla_\xi = (\partial_{\xi_1},\partial_{\xi_2},0)^\top$ is used below.

As a simple application of the theorems stated above, we formulate (without proof) the following result concerning the convergence of solutions of Problem S_ε.

PROPOSITION 3.9. *Let $\boldsymbol{u}_\varepsilon$ be the sequence of solutions to Problem S_ε. Then there exist a subsequence (still denoted by $\boldsymbol{u}_\varepsilon$) and a function $\boldsymbol{u}(t,\boldsymbol{x})$ such that*

1. *$\boldsymbol{u}_\varepsilon$ two-scale converges to $\boldsymbol{u}$, and $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ weakly in $L^2\big([0,T]\times\Omega\big)$;*
2. *$\mathcal{J}_t\boldsymbol{u}_\varepsilon$ two-scale converges to $\mathcal{J}_t\boldsymbol{u}$, and $\mathcal{J}_t\boldsymbol{u}_\varepsilon \to \mathcal{J}_t\boldsymbol{u}$ in $L^2\big([0,T]\times\Omega\big)$;*
3. *$\nabla\mathcal{J}_t\boldsymbol{u}_\varepsilon$ two-scale converges to $\nabla_x\mathcal{J}_t\boldsymbol{u} + \nabla_\xi\boldsymbol{\zeta}$, where $\boldsymbol{\zeta}(t,\boldsymbol{x},\hat{\boldsymbol{\xi}})$ is a function from $L^2\big([0,T]\times\Omega;H^1_\#(\Sigma)/\mathbb{R}\big)$.*

3.2. Passage to the limit in Problem S_ε. Let the initial data of Problem S_ε satisfy the conditions of Theorem 2.6. A solution $\boldsymbol{u}_\varepsilon$ of Problem S_ε satisfies the following integral identity,

(3.1)
$$\int_0^T \int_\Omega \Big(-\rho^\varepsilon \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\varphi}_t + \boldsymbol{M}^{\varepsilon t} \boldsymbol{u}_{\varepsilon x} : \boldsymbol{\varphi}_x + \mathcal{N}^{\varepsilon 0} : \boldsymbol{\varphi}_x - \rho^\varepsilon \boldsymbol{f} \cdot \boldsymbol{\varphi} \Big) d\boldsymbol{x} dt = \int_\Omega \rho^\varepsilon \boldsymbol{u}^0 \cdot \boldsymbol{\varphi}^0 d\boldsymbol{x},$$

where ρ^ε, $\boldsymbol{M}^{\varepsilon t}$, and $\mathcal{N}^{\varepsilon 0}$ are defined as in (2.13) but with χ replaced by χ^ε. Let us take

$$\boldsymbol{\varphi}(t, \boldsymbol{x}) = \boldsymbol{\phi}(t, \boldsymbol{x}) + \varepsilon \, \overline{\boldsymbol{\phi}}\Big(t, \boldsymbol{x}, \frac{\hat{\boldsymbol{x}}}{\varepsilon}\Big),$$

where $\boldsymbol{\phi}$ and $\overline{\boldsymbol{\phi}}$ are arbitrary functions that vanish for $\boldsymbol{x} \in \partial\Omega$ and at $t = T$. Theorem 3.8 enables the passage to the limit in (3.1) as $\varepsilon \to 0$. The limiting equations look as follows:

$$\begin{aligned}(3.2) \quad \int_0^T \int_\Omega \int_\Sigma \Big(-\rho \boldsymbol{u} \cdot \boldsymbol{\phi}_t + \boldsymbol{M}^t (\boldsymbol{u}_x + \overline{\boldsymbol{u}}_\xi) : \boldsymbol{\phi}_x + \mathcal{N}^0 : \boldsymbol{\phi}_x - \rho \boldsymbol{f} \cdot \boldsymbol{\phi} \Big) \, d\hat{\boldsymbol{\xi}} \, d\boldsymbol{x} dt \\ = \int_\Omega \int_\Sigma \rho \boldsymbol{u}^0 \cdot \boldsymbol{\phi}^0 \, d\hat{\boldsymbol{\xi}} \, d\boldsymbol{x},\end{aligned}$$

$$(3.3) \qquad \int_\Sigma \Big(\boldsymbol{M}^t (\boldsymbol{u}_x + \overline{\boldsymbol{u}}_\xi) : \overline{\boldsymbol{\phi}}_\xi + \mathcal{N}^0 : \overline{\boldsymbol{\phi}}_\xi \Big) \, d\hat{\boldsymbol{\xi}} = 0 \quad \text{in} \quad L^2\big([0, T] \times \Omega\big).$$

These equations hold for all functions $\boldsymbol{\phi} \in H^1([0, T] \times \Omega)$ and $\overline{\boldsymbol{\phi}} \in H^1_\#(\Sigma)$ such that $\boldsymbol{\phi}$ vanish on $\partial\Omega$ and at $t = T$. The coefficients ρ, $\boldsymbol{M}^t$, and $\mathcal{N}^0$ are defined as in (2.13) with $\chi(\boldsymbol{x})$ replaced by $\chi(\boldsymbol{x}, \hat{\boldsymbol{\xi}})$. The function $\chi(\boldsymbol{x}, \hat{\boldsymbol{\xi}})$ is defined as in subsection 2.2:

$$\chi(\boldsymbol{x}, \hat{\boldsymbol{\xi}}) = \begin{cases} 1, & x_3 > \delta, \\ \hat{\chi}(\hat{\boldsymbol{\xi}}), & 0 \leqslant x_3 \leqslant \delta, \\ 0, & x_3 < 0. \end{cases}$$

Equation (3.3) is called a *cell equation*.

Equations (3.2) and (3.3) are coupled through the auxiliary function $\overline{\boldsymbol{u}}$. The next step consists of finding $\overline{\boldsymbol{u}}$ from the cell equation (3.3) and substituting the obtained expression into (3.2).

4. Explicit solving of the cell equation.

4.1. Operator form of the cell equation in a Hilbert space. It is appropriate to rewrite (3.3) as an equation in the Hilbert space $H = H^1_\#(\Sigma)/\mathbb{R}$ with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Sigma \frac{\partial u_i}{\partial \xi_j} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}}.$$

The norm in H is denoted by $\| \cdot \|$. Let us define operators $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\langle \mathcal{A} \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Sigma \chi P_{ijkl} \frac{\partial u_k}{\partial \xi_l} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}}, \quad \langle \mathcal{B} \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Sigma \Big(\chi \gamma^{-1} \delta_{ij} \delta_{kl} + (1 - \chi) \, G_{ijkl} \Big) \frac{\partial u_k}{\partial \xi_l} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}}$$

for all functions $\boldsymbol{u}, \boldsymbol{v} \in H$. Due to the Riesz representation theorem, there exist $\boldsymbol{n}_0$, $\boldsymbol{a}_{kl}$, and $\boldsymbol{b}_{kl}$, $k, l = 1, 2, 3$, such that

$$\langle \boldsymbol{n}_0, \boldsymbol{v} \rangle = \int_\Sigma \mathcal{N}^0 : \boldsymbol{v}_\xi \, d\hat{\boldsymbol{\xi}}, \qquad \langle \boldsymbol{a}_{kl}, \boldsymbol{v} \rangle = \int_\Sigma \chi P_{ijkl} \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}},$$

$$\langle \boldsymbol{b}_{kl}, \boldsymbol{v} \rangle = \int_\Sigma \left(\chi \gamma^{-1} \delta_{ij} \delta_{kl} + (1 - \chi) \, G_{ijkl} \right) \frac{\partial v_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}}$$

for all $\boldsymbol{v} \in H$. Remark that $\mathcal{A}$, $\mathcal{B}$, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{n}_0$ do not depend on t and depend on the variable $\boldsymbol{x}$ just in the same way as the function $\chi(\boldsymbol{x}, \hat{\boldsymbol{\xi}})$. So we can consider $\boldsymbol{x}$ and t in (3.3) as parameters.

Now, the problem (3.3) transforms to the following equation in the space H:

$$\mathcal{A}\overline{\boldsymbol{u}} + \mathcal{B}\mathcal{J}_t \overline{\boldsymbol{u}} = \boldsymbol{g}, \tag{4.1}$$

where

$$\boldsymbol{g} = -\left(\boldsymbol{a}_{kl} + \boldsymbol{b}_{kl} \mathcal{J}_t \right) \frac{\partial u_k}{\partial x_l} - \boldsymbol{n}_0$$

and $\boldsymbol{u}(\boldsymbol{x}, t)$ is from (3.2) and (3.3).

Since the operators $\mathcal{A}$ and $\mathcal{B}$ are trivial whenever $x_3 \notin [0, \delta]$, we consider (4.1) for $x_3 \in [0, \delta]$, which corresponds to the treatment of the pin layer. In this case, the operators $\mathcal{A}$ and $\mathcal{B}$ are degenerated. Therefore, some difficulties appear when solving (4.1).

The next section is devoted to the study of the data of (4.1) to prepare tools for its explicit solving.

4.2. Properties of $\mathcal{A}$, $\mathcal{B}$, and g.

PROPOSITION 4.10. *The operator $\mathcal{A}$ has the following properties:*

1. *$\mathcal{A}$ is a bounded self-adjoint operator on H.*
2. *$\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0$ for all $\boldsymbol{u} \in H$.*
3. *The null-space $N(\mathcal{A}) = \{\boldsymbol{u} \in H : \ \boldsymbol{u}$ is constant in $\Sigma_F\}$, and $N(\mathcal{A})^\perp \subset \{\boldsymbol{u} \in H : \ \Delta \boldsymbol{u} = 0$ in $\Sigma_S\}$.*
4. *There exist positive constants c and C such that*

$$c \, \|\boldsymbol{u}\|^2 \leqslant \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u} \rangle \leqslant C \, \|\boldsymbol{u}\|^2 \tag{4.2}$$

for all $\boldsymbol{u} \in N(\mathcal{A})^\perp$.

5. *The range $R(\mathcal{A})$ is closed in H, $R(\mathcal{A}) = N(\mathcal{A})^\perp$, and $\mathcal{A}^{-1}$ is defined and bounded as an operator on $R(\mathcal{A})$.*

Proof. Assertions 1 and 2 are obvious (see Remark 2.2). The third assertion consists of two parts. In order to prove the first one we have only to establish that

$$N(\mathcal{A}) \subset \{\boldsymbol{u} \in H : \ \boldsymbol{u} \text{ is constant on } \Sigma_F\}$$

because the opposite inclusion is clearly true. Due to the positiveness of the operator $\mathcal{A}$, its null-space consists of functions $\boldsymbol{u}$ which satisfy the condition $\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u} \rangle = 0$. Thus, $\boldsymbol{u} \in N(\mathcal{A})$ implies

$$\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u} \rangle = \int_\Sigma \chi P_{ijkl} \frac{\partial u_k}{\partial \xi_l} \frac{\partial u_i}{\partial \xi_j} \, d\hat{\boldsymbol{\xi}} = 0.$$

Consequently, $\mathcal{D}(\boldsymbol{u}) = 0$ in Σ_F, and, hence, $\boldsymbol{u}$ is constant in Σ_F because of its periodicity.

Let $\boldsymbol{u} \in N(\mathcal{A})^\perp$. By definition, this means that

$$\int_\Sigma \frac{\partial u_k}{\partial \xi_l}\frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = \int_{\Sigma_S} \frac{\partial u_k}{\partial \xi_l}\frac{\partial v_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}} = 0$$

for any function $\boldsymbol{v} \in C^\infty(\Sigma)$ such that $\boldsymbol{v}$ is constant on $\overline{\Sigma}_F$. Consequently, $\boldsymbol{u}$ is harmonic in Σ_S, which proves the third assertion.

To validate assertion 3, we need only to prove the left inequality since the right one is obvious. Due to the Korn inequality (see, e.g., [14]), there exists a positive constant c_1 such that

$$\int_{\Sigma_F} |\boldsymbol{u}_\xi|^2\, d\hat{\boldsymbol{\xi}} \leqslant c_1 \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle$$

for every $\boldsymbol{u} \in H$. If $\boldsymbol{u} \in N(\mathcal{A})^\perp$, then $\boldsymbol{u}$ is harmonic in Σ_S and there exist positive constants c_2 and c_3 such that

$$c_2 \int_{\Sigma_S} |\boldsymbol{u}_\xi|^2\, d\hat{\boldsymbol{\xi}} \leqslant \|\boldsymbol{u}\|_{H^{1/2}(\partial\Sigma_S)} \leqslant c_3 \int_{\Sigma_F} |\boldsymbol{u}_\xi|^2\, d\hat{\boldsymbol{\xi}}.$$

That is, $\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle \geqslant c\,\|\boldsymbol{u}\|^2$ for some constant c.

When proving assertion 5, denote by $\mathcal{A}_R$ the restriction of $\mathcal{A}$ to $N(\mathcal{A})^\perp$. Due to the estimate (4.2), $R(\mathcal{A}_R)$ is closed in H. Since $R(\mathcal{A}) = R(\mathcal{A}_R)$, we conclude that $R(\mathcal{A})$ is also a closed subspace of H. This implies that $N(\mathcal{A})^\perp = \overline{R(\mathcal{A})} = R(\mathcal{A})$, and (4.2) is true for $\boldsymbol{u} \in R(\mathcal{A})$. Thus, $\mathcal{A}^{-1}$ exists and is bounded if $\mathcal{A}$ is considered being restricted to $R(\mathcal{A})$. The proposition is proved. ∎

PROPOSITION 4.11. *The operator $\mathcal{B}$ has the following properties:*

1. *$\mathcal{B}$ is a bounded self-adjoint operator on H.*
2. *$\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle \geqslant 0$ for all $\boldsymbol{u} \in H$.*
3. *The null-space $N(\mathcal{B}) = \{\boldsymbol{u} \in H :\ \mathcal{D}(\boldsymbol{u}) = 0$ in Σ_S and $\operatorname{div}\boldsymbol{u} = 0$ in $\Sigma_F\}$, and $N(\mathcal{B})^\perp \subset \{\boldsymbol{u} \in H :\ \Delta\boldsymbol{u} = \nabla q$ in Σ_F for some $q \in L^2(\Sigma)\}$.*
4. *There exist positive constants c and C such that*

$$c\,\|\boldsymbol{u}\|^2 \leqslant \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle \leqslant C\,\|\boldsymbol{u}\|^2 \tag{4.3}$$

for all $\boldsymbol{u} \in N(\mathcal{B})^\perp$.

5. *The range $R(\mathcal{B})$ is closed in H, $R(\mathcal{B}) = N(\mathcal{B})^\perp$, and $\mathcal{B}^{-1}$ is defined and bounded as an operator on $R(\mathcal{B})$.*

Proof. The first two assertions are obvious. To prove the third one, note that

$$\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle = \int_\Sigma \left(\chi\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\chi)G_{ijkl}\right)\frac{\partial u_i}{\partial \xi_j}\frac{\partial u_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}}$$
$$= \gamma^{-1}\int_{\Sigma_F} (\operatorname{div}\boldsymbol{u})^2\, d\hat{\boldsymbol{\xi}} + \int_{\Sigma_S} G_{ijkl}\frac{\partial u_i}{\partial \xi_j}\frac{\partial u_k}{\partial \xi_l}\, d\hat{\boldsymbol{\xi}}$$

for every $\boldsymbol{u} \in H$. Therefore, $\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u}\rangle = 0$ if and only if $\operatorname{div}\boldsymbol{u} = 0$ in Σ_F and $\mathcal{D}(\boldsymbol{u}) = 0$ in Σ_S.

If $\boldsymbol{u} \in N(\mathcal{B})^\perp$, then the equalities

$$0 = \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_\Sigma \boldsymbol{u}_\xi \boldsymbol{v}_\xi \, d\hat{\boldsymbol{\xi}} = \int_\Sigma \mathcal{D}(\boldsymbol{u}) : \mathcal{D}(\boldsymbol{v}) \, d\hat{\boldsymbol{\xi}} = \int_{\Sigma_F} \mathcal{D}(\boldsymbol{u}) : \mathcal{D}(\boldsymbol{v}) \, d\hat{\boldsymbol{\xi}} \tag{4.4}$$

hold for every $\boldsymbol{v} \in N(\mathcal{B})$. Let $\boldsymbol{u}^k \in N(\mathcal{B})^\perp$ be a sequence of smooth functions that converges to $\boldsymbol{u}$ in H. Such a sequence exists because $C^\infty(\Sigma)$ is dense in $N(\mathcal{B})^\perp$. Relation (4.4) is also valid for all $\boldsymbol{u}^k$. If $\boldsymbol{v}$ is an arbitrary smooth function such that $\operatorname{div} \boldsymbol{v} = 0$ and $\operatorname{supp} \boldsymbol{v} \subset \Sigma_F$, then $\boldsymbol{v} \in N(\mathcal{B})$, and

$$0 = \int_{\Sigma_F} \mathcal{D}(\boldsymbol{u}^k) : \mathcal{D}(\boldsymbol{v}) \, d\hat{\boldsymbol{\xi}} = -\int_{\Sigma_F} \operatorname{div}\big(\mathcal{D}(\boldsymbol{u}^k)\big) \cdot \boldsymbol{v} \, d\hat{\boldsymbol{\xi}}.$$

Consequently, there exist functions $\tilde{q}^k \in L^2(\Sigma)$ such that $\operatorname{div}\mathcal{D}(\boldsymbol{u}^k) = \nabla \tilde{q}^k$ for all k. Passing to the limit yields $\operatorname{div}\mathcal{D}(\boldsymbol{u}) = \nabla \tilde{q}$. That is, $\Delta \boldsymbol{u} = \nabla q$, where $q = \tilde{q} - \operatorname{div}\boldsymbol{u}$. This proves the third assertion.

The right inequality of the fourth assertion is obvious. Let us prove the left one. According to the classical theory of the Stokes equations (see [4, Chap. 4]), the following estimate holds for all $\boldsymbol{u} \in N(\mathcal{B})^\perp$:

$$\int_{\Sigma_F} |\boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}} \leqslant c_1 \big(\|\operatorname{div} \boldsymbol{u}\|^2_{L_2(\Sigma_F)} + \|\boldsymbol{u}_\Gamma\|^2_{H^{1/2}(\partial\Sigma_S)/\mathbb{R}} \big),$$

where $\boldsymbol{u}_\Gamma$ is the trace of $\boldsymbol{u}$ on $\partial\Sigma_S$. On the other hand,

$$\|\boldsymbol{u}_\Gamma\|^2_{H^{1/2}(\partial\Sigma_S)/\mathbb{R}} \leqslant c_2 \int_{\Sigma_S} |\boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}}.$$

Thus, there exists a positive constant c_3 such that

$$\|\boldsymbol{u}\|^2 \leqslant c_3 \left(\int_{\Sigma_S} |\boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}} + \|\operatorname{div} \boldsymbol{u}\|^2_{L_2(\Sigma_F)} \right) \tag{4.5}$$

for every $\boldsymbol{u} \in N(\mathcal{B})^\perp$. In order to obtain (4.3), it is sufficient to prove that there exists a positive constant c_4 such that

$$\int_{\Sigma_S} |\boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}} \leqslant c_4 \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u} \rangle \tag{4.6}$$

for $\boldsymbol{u} \in N(\mathcal{B})^\perp$. This can be done using standard contradiction arguments. Assume the converse, i.e., there exists a sequence $\boldsymbol{u}^n \in N(\mathcal{B})^\perp$, $n \in \mathbb{N}$, such that $\int_{\Sigma_S} |\boldsymbol{u}^n_\xi|^2 \, d\hat{\boldsymbol{\xi}} = 1$ and $\langle \mathcal{B}\boldsymbol{u}^n, \boldsymbol{u}^n \rangle \to 0$ as $n \to 0$. The estimate (4.5) implies that the sequence $\{\boldsymbol{u}^n\}$ is bounded in H too. Thus, there exists its subsequence (still denoted by $\{\boldsymbol{u}^n\}$) that converges weakly in H and $H^1(\Sigma_S)/\mathbb{R}$ but strongly in $L^2(\Sigma)$ to a function $\boldsymbol{u}$. Note that $\boldsymbol{u} \in N(\mathcal{B})^\perp$ since $N(\mathcal{B})^\perp$ is weakly closed in H. Using the Korn inequality yields

$$\int_{\Sigma_S} |\boldsymbol{u}^n_\xi - \boldsymbol{u}_\xi|^2 \, d\hat{\boldsymbol{\xi}} \leqslant C \left(\langle \mathcal{B}(\boldsymbol{u}^n - \boldsymbol{u}), \boldsymbol{u}^n - \boldsymbol{u} \rangle + \|\boldsymbol{u}^n - \boldsymbol{u}\|^2_{L^2(\Sigma)} \right).$$

The passage to the limit in this inequality implies that $\langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and $\boldsymbol{u}^n \to \boldsymbol{u}$ in H. This means that $\boldsymbol{u} \in N(\mathcal{B})^\perp \cap N(\mathcal{B})$ and $\boldsymbol{u} = 0$ in H. On the other hand, $\int_{\Sigma_S} |\boldsymbol{u}_\xi|^2\, d\hat{\boldsymbol{\xi}} = \lim_{n\to\infty} \int_{\Sigma_S} |\boldsymbol{u}_\xi^n|^2\, d\hat{\boldsymbol{\xi}} = 1$. This contradiction proves (4.6) and, consequently, (4.3).

The proof of the fifth assertion is the same as for the operator $\mathcal{A}$ in Proposition 4.10. $\square$

PROPOSITION 4.12. *The following is true:*

$$\boldsymbol{a}_{kl}, \boldsymbol{b}_{kl}, \boldsymbol{n}_0 \in R(\mathcal{A}) \cap R(\mathcal{B}), \quad k,l = 1,2,3.$$

Consequently, $\boldsymbol{g} \in R(\mathcal{A}) \cap R(\mathcal{B})$ for almost all t and $\boldsymbol{x}$, where $\boldsymbol{g}$ is the right-hand side of the cell equation (4.1).

Proof. Due to Propositions 4.10 and 4.11, $\boldsymbol{w} \in R(\mathcal{A}) \cap R(\mathcal{B})$ if and only if $\langle \boldsymbol{w}, \boldsymbol{v}\rangle = 0$ for all $\boldsymbol{v} \in N(\mathcal{A}) \cup N(\mathcal{B})$. Let us verify this condition for $\boldsymbol{a}_{kl}$. The functions $\boldsymbol{b}_{kl}$ and $\boldsymbol{n}_0$ can be treated in the same way. Let $\boldsymbol{v}$ be an arbitrary function from $N(\mathcal{A})$. That is, $\boldsymbol{v}$ is a constant in Σ_F because of Proposition 4.10. Thus,

$$\langle \boldsymbol{a}_{kl}, \boldsymbol{v}\rangle = \int_{\Sigma_F} P_{ijkl} \frac{\partial v_i}{\partial \xi_j}\, d\hat{\boldsymbol{\xi}} = 0.$$

If $\boldsymbol{v} \in N(\mathcal{B})$ then $\mathcal{D}(\boldsymbol{v}) = 0$ in Σ_S according to Proposition 4.11, and

$$\begin{aligned}\langle \boldsymbol{a}_{kl}, \boldsymbol{v}\rangle &= \int_{\Sigma_F} P_{ijkl} \frac{\partial v_i}{\partial \xi_j}\, d\hat{\boldsymbol{\xi}} = \int_{\Sigma} P_{ijkl} \frac{\partial v_i}{\partial \xi_j}\, d\hat{\boldsymbol{\xi}} - \int_{\Sigma_S} P_{ijkl} \frac{\partial v_i}{\partial \xi_j}\, d\hat{\boldsymbol{\xi}} \\ &= \int_{\Sigma_S} P_{ijkl} \mathcal{D}_{ij}(\boldsymbol{v})\, d\hat{\boldsymbol{\xi}} = 0.\end{aligned}$$

Here, we used the periodicity of $\boldsymbol{v}$ in Σ and the symmetry of the tensor P (see Remark 2.2). This proves the proposition. $\square$

PROPOSITION 4.13.

$$N(\mathcal{A}) \cap N(\mathcal{B}) = \{0\}.$$

Proof. If $\boldsymbol{u} \in N(\mathcal{A}) \cap N(\mathcal{B})$, then $\mathcal{D}(\boldsymbol{u}) = 0$ in Σ due to Propositions 4.10 and 4.11. That is, $\boldsymbol{u}$ is constant in Σ because of its periodicity. This means that $\boldsymbol{u} = 0$ in H. $\square$

The result of Proposition 4.13 implies that the operator $\lambda\mathcal{A} + \mathcal{B}$ is invertible for every $\lambda > 0$. Besides that, it is not difficult to see that the operator $(\lambda\mathcal{A} + \mathcal{B})^{-1}$ is bounded in H: Let us introduce the following closed subspaces of H:

$$\begin{aligned}&E_A = (\lambda\mathcal{A} + \mathcal{B})^{-1} R(\mathcal{A}),\\ &E_B = (\lambda\mathcal{A} + \mathcal{B})^{-1} R(\mathcal{B}),\\ &E = E_A \cap E_B = (\lambda\mathcal{A} + \mathcal{B})^{-1}\big(R(\mathcal{A}) \cap R(\mathcal{B})\big).\end{aligned}$$

Note that the spaces E, E_A, and E_B do not depend on λ. More precisely, if $E_A^\lambda = (\lambda\mathcal{A} + \mathcal{B})^{-1} R(\mathcal{A})$ then $E_A^\lambda = E_A^\mu$ for all $\lambda > 0$ and $\mu > 0$. This follows from simple arguments like those. If $\boldsymbol{x} \in E_A^\lambda$, then $(\lambda\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{A})$ and $\mathcal{B}\boldsymbol{x} \in R(\mathcal{A})$. Consequently, $(\mu\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{A})$ and $\boldsymbol{x} \in E_A^\mu$. That is, $E_A^\lambda \subset E_A^\mu$. In the same way we can obtain that $E_A^\mu \subset E_A^\lambda$.

LEMMA 4.14. *The operator $\mathcal{A}$ maps the space E_B into $R(\mathcal{B})$, and the operator $\mathcal{B}$ maps the space E_A into $R(\mathcal{A})$.*

Proof. The first part is true due to the following implications:

$$\boldsymbol{x} \in E_{B} \Longrightarrow (\lambda\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{B}) \Longrightarrow \mathcal{A}\boldsymbol{x} \in R(\mathcal{B}).$$

The second part is being proved analogously. ☐

LEMMA 4.15. *If X is a closed subspace of H then $\mathcal{A}(X)$ and $\mathcal{B}(X)$ are closed in H.*

Proof. Let us verify this assertion for the operator $\mathcal{A}$ by taking an arbitrary sequence $\boldsymbol{u}_n \in \mathcal{A}(X)$ which converges to a function $\boldsymbol{u}$ in H. There exists a corresponding sequence $\boldsymbol{v}_n \in R(\mathcal{A}) \cap X$ such that $\boldsymbol{u}_n = \mathcal{A}(\boldsymbol{v}_n)$. Due to Proposition 4.10, the operator $\mathcal{A}^{-1}$ is bounded on $R(\mathcal{A})$. This implies that the sequence $\{\boldsymbol{v}_n\}$ converges in H to a function $\boldsymbol{v}$ which is in X because X is closed. In the limit, we have $\boldsymbol{u} = \mathcal{A}(\boldsymbol{v})$. That is, $\boldsymbol{u} \in \mathcal{A}(X)$, which proves the lemma. ☐

PROPOSITION 4.16.

$$\mathcal{B}E_{A} = \mathcal{A}E_{B} = R(\mathcal{A}) \cap R(\mathcal{B}).$$

That is, for every $\boldsymbol{\psi} \in R(\mathcal{A}) \cap R(\mathcal{B})$, there exist $\boldsymbol{\psi}_{B} \in E_{B}$ and $\boldsymbol{\psi}_{A} \in E_{A}$ such that $\boldsymbol{\psi} = \mathcal{A}\boldsymbol{\psi}_{B} = \mathcal{B}\boldsymbol{\psi}_{A}$.

Proof. Let us prove the first claim. Due to Lemma 4.14, $\mathcal{B}E_{A} \subset R(\mathcal{A}) \cap R(\mathcal{B})$. Besides that, Lemma 4.15 implies that $\mathcal{B}E_{A}$ is a closed subspace in H. Suppose that $\mathcal{B}E_{A} \neq R(\mathcal{A}) \cap R(\mathcal{B})$. Then there exists $\boldsymbol{x} \in R(\mathcal{A}) \cap R(\mathcal{B})$ such that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for every $\boldsymbol{y} \in \mathcal{B}E_{A}$. That is, $\langle \boldsymbol{x}, \mathcal{B}(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{z} \rangle = 0$ for all $\boldsymbol{z} \in H$, and

$$\langle \mathcal{A}(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}\boldsymbol{x}, \boldsymbol{z} \rangle = 0 \quad \text{for all} \quad \boldsymbol{z} \in H.$$

Consequently, $(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}\boldsymbol{x} \in N(\mathcal{A})$ and, hence, $\mathcal{B}\boldsymbol{x} \in (\lambda\mathcal{A} + \mathcal{B})N(\mathcal{A}) = \mathcal{B}N(\mathcal{A})$. That is, there exists $\boldsymbol{y} \in N(\mathcal{A})$ such that $\mathcal{B}\boldsymbol{x} = \mathcal{B}\boldsymbol{y}$ and, therefore, $\mathcal{B}(\boldsymbol{x} - \boldsymbol{y}) = 0$. This implies that $\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{y} \in N(\mathcal{B})$. Thus, $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{w}$, where $\boldsymbol{y} \in N(\mathcal{A})$, and $\boldsymbol{w} \in N(\mathcal{B})$. That is, $\boldsymbol{x} \in N(\mathcal{A}) \oplus N(\mathcal{B})$. Consequently, $\boldsymbol{x} = 0$ because $\big(N(\mathcal{A}) \oplus N(\mathcal{B})\big) \cap \big(R(\mathcal{A}) \cap R(\mathcal{B})\big) = \{0\}$. The proposition is proved. ☐

Let us introduce the restrictions $\mathcal{A}_{E}$ and $\mathcal{B}_{E}$ of the operators $\mathcal{A}$ and $\mathcal{B}$ to the space E.

THEOREM 4.17.

1. *The operators $\mathcal{A}_{E}$ and $\mathcal{B}_{E}$ map E onto $R(\mathcal{A}) \cap R(\mathcal{B})$.*
2. *The operators $\mathcal{A}_{E}$, $\mathcal{B}_{E} : E \to R(\mathcal{A}) \cap R(\mathcal{B})$ are one to one.*
3. *There exist bounded operators $\mathcal{A}_{E}^{-1}$, $\mathcal{B}_{E}^{-1} : R(\mathcal{A}) \cap R(\mathcal{B}) \to E$.*

Proof. Let us prove these assertions for the operator $\mathcal{A}_{E}$ only. The operator $\mathcal{B}_{E}$ can be treated in the same way.

1. Since $E \subset E_{B}$, Proposition 4.16 and Lemma 4.15 imply that $\mathcal{A}E \subset R(\mathcal{A}) \cap R(\mathcal{B})$, and $\mathcal{A}E$ is a closed subspace in H. Suppose that $\mathcal{A}E \neq R(\mathcal{A}) \cap R(\mathcal{B})$. This means that there exists $\boldsymbol{x} \in R(\mathcal{A}) \cap R(\mathcal{B})$ such that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ for every $\boldsymbol{y} \in \mathcal{A}E$. That is,

$$\langle (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \mathcal{A}(\lambda\mathcal{A} + \mathcal{B})^{-1}\boldsymbol{z} \rangle = 0$$

for all $\boldsymbol{z} \in R(\mathcal{A}) \cap R(\mathcal{B})$. Thus, due to Proposition 4.16,

$$\langle (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x}, \mathcal{B}\boldsymbol{z} \rangle = 0 \quad \text{for all } \boldsymbol{z} \in E_{A}. \tag{4.7}$$

Since $(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x} \in E_{A}$, we can take $\boldsymbol{z} = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x}$. Then the relation (4.7) implies that $(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\boldsymbol{x} \in N(\mathcal{B})$, that is, $\mathcal{A}\boldsymbol{x} \in \mathcal{A}N(\mathcal{B})$. Consequently (see the end of the proof of Proposition 4.16), $\boldsymbol{x} = 0$, which proves the first assertion of the theorem.

2. We have to prove that $N(\mathcal{A}) \cap E = \{0\}$. Let $\boldsymbol{x} \in E$ and $\mathcal{A}\boldsymbol{x} = 0$. Then $\mathcal{B}\boldsymbol{x} = (\lambda\mathcal{A} + \mathcal{B})\boldsymbol{x} \in R(\mathcal{A}) \cap R(\mathcal{B})$, that is, $\mathcal{B}\boldsymbol{x} \in R(\mathcal{A})$. But $\boldsymbol{x} \in N(\mathcal{A}) = R(\mathcal{A})^{\perp}$ and, consequently, $\langle \mathcal{B}\boldsymbol{x}, \boldsymbol{x}\rangle = 0$. Since $\mathcal{B}$ is a positive operator, the last relation implies that $\boldsymbol{x} \in N(\mathcal{B})$. Thus, $\boldsymbol{x} \in N(\mathcal{A}) \cap N(\mathcal{B}) = \{0\}$, which proves the second assertion of the theorem.

3. This assertion is the consequence of parts 1 and 2. The theorem is proved. ∎

4.3. Solving the cell equation. Now we are in position to find an explicit representation of solutions to the cell equation (4.1). With a new unknown function $\overline{\boldsymbol{\zeta}} = \mathcal{J}_t\overline{\boldsymbol{u}}$, the problem (4.1) assumes the form

$$\mathcal{A}\overline{\boldsymbol{\zeta}}_t + \mathcal{B}\overline{\boldsymbol{\zeta}} = \boldsymbol{g}, \qquad \overline{\boldsymbol{\zeta}}(0) = 0. \tag{4.8}$$

As it follows from Theorem 4.17, the operator $\mathcal{A}_E$ ($\mathcal{A}$ restricted to E) is invertible, the operator $\mathcal{A}_E^{-1}\mathcal{B}_E$ bounded, and $\mathcal{A}_E^{-1}\boldsymbol{g} \in E$. Therefore, the problem

$$\overline{\boldsymbol{\zeta}}_t + \mathcal{A}_E^{-1}\mathcal{B}_E\overline{\boldsymbol{\zeta}} = \mathcal{A}_E^{-1}\boldsymbol{g}, \qquad \overline{\boldsymbol{\zeta}}(0) = 0 \tag{4.9}$$

is uniquely solvable on the subspace E, and the solution is of the form

$$\overline{\boldsymbol{\zeta}}(t) = \int_0^t e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\boldsymbol{g}(s)\,ds. \tag{4.10}$$

THEOREM 4.18. *Equations* (4.8) *and* (4.9) *are equivalent.*

Proof. Obviously, if $\overline{\boldsymbol{\zeta}}$ is a solution to (4.9), then $\overline{\boldsymbol{\zeta}}$ satisfies (4.8). If $\overline{\boldsymbol{\zeta}}$ is a solution to (4.8), then the function $\overline{\boldsymbol{\eta}} = e^{-\lambda t}\overline{\boldsymbol{\zeta}}$ solves the problem

$$\mathcal{A}\overline{\boldsymbol{\eta}}_t + (\lambda\mathcal{A} + \mathcal{B})\overline{\boldsymbol{\eta}} = e^{-\lambda t}\boldsymbol{g}, \qquad \overline{\boldsymbol{\eta}}(0) = 0. \tag{4.11}$$

Since the operator $\lambda\mathcal{A} + \mathcal{B}$ is nondegenerate for any $\lambda > 0$, we can rewrite (4.11) as follows:

$$(\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}\overline{\boldsymbol{\eta}}_t + \overline{\boldsymbol{\eta}} = e^{-\lambda t}(\lambda\mathcal{A} + \mathcal{B})^{-1}\boldsymbol{g}, \qquad \overline{\boldsymbol{\eta}}(0) = 0. \tag{4.12}$$

Due to Proposition 4.12, $\boldsymbol{g} \in R(\mathcal{A})$, and, hence $\overline{\boldsymbol{\eta}}(t)$ must belong to E_A for all t. Therefore, $\overline{\boldsymbol{\zeta}}(t) \in E_A$ for all t. On the other hand, (4.8) can be rewritten as follows:

$$(\lambda\mathcal{A} + \mathcal{B})\overline{\boldsymbol{\zeta}}_t - \mathcal{B}\overline{\boldsymbol{\zeta}}_t + \lambda\mathcal{B}\overline{\boldsymbol{\zeta}} = \lambda\boldsymbol{g}.$$

That is,

$$\overline{\boldsymbol{\zeta}}_t = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}(\overline{\boldsymbol{\zeta}}_t - \lambda\overline{\boldsymbol{\zeta}}) + \lambda(\lambda\mathcal{A} + \mathcal{B})^{-1}\boldsymbol{g}.$$

Taking into account that $\overline{\boldsymbol{\zeta}}(t)$ and $\overline{\boldsymbol{\zeta}}_t(t) \in E_A$ for all t, we establish, using Proposition 4.16, that $\overline{\boldsymbol{\zeta}}_t(t) \in E$ for all t. Since $\overline{\boldsymbol{\zeta}}(0) = 0$, we conclude that $\overline{\boldsymbol{\zeta}}(t) \in E$ for all t. Therefore, $\overline{\boldsymbol{\zeta}}$ is a solution of (4.9). The theorem is proved. ∎

Thus, the unique solution of the problem (4.8) is given by (4.10) and the unique solution $\overline{\boldsymbol{u}}$ of the problem (4.1) reads as

$$\overline{\boldsymbol{u}}(t) = \overline{\boldsymbol{\zeta}}_t(t) = \mathcal{A}_E^{-1}\boldsymbol{g}(t) - \mathcal{A}_E^{-1}\mathcal{B}_E\int_0^t e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\boldsymbol{g}(s)\,ds. \tag{4.13}$$

5. Homogenized structure.

5.1. Limiting equations. Substitution of the expression for $\boldsymbol{g}$ into (4.13) gives

(5.1)
$$\overline{\boldsymbol{u}}(t) = -e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\boldsymbol{n}_0 - \mathcal{A}_E^{-1}\boldsymbol{a}_{kl}\frac{\partial u_k(t)}{\partial x_l} - \int_0^t \boldsymbol{m}_{kl}(t-s)\frac{\partial u_k(s)}{\partial x_l}\,ds,$$

(5.2)
$$\mathcal{J}_t\overline{\boldsymbol{u}}(t) = \big(e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E} - \mathcal{I}\big)\mathcal{B}_E^{-1}\boldsymbol{n}_0 - \mathcal{B}_E^{-1}\boldsymbol{b}_{kl}\int_0^t \frac{\partial u_k(s)}{\partial x_l}\,ds - \int_0^t \widetilde{\boldsymbol{m}}_{kl}(t-s)\frac{\partial u_k(s)}{\partial x_l}\,ds,$$

where

$$\boldsymbol{m}_{kl}(t) = -\mathcal{A}_E^{-1}\mathcal{B}_E e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\big(\mathcal{A}_E^{-1}\boldsymbol{a}_{kl} - \mathcal{B}_E^{-1}\boldsymbol{b}_{kl}\big) \in E,$$

$$\widetilde{\boldsymbol{m}}_{kl}(t) = e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\big(\mathcal{A}_E^{-1}\boldsymbol{a}_{kl} - \mathcal{B}_E^{-1}\boldsymbol{b}_{kl}\big) \in E.$$

The integration by parts and the formula

$$\frac{d}{ds}e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E} = \mathcal{A}_E^{-1}\mathcal{B}_E e^{-(t-s)\mathcal{A}_E^{-1}\mathcal{B}_E}$$

are applied when deriving (5.1) and (5.2). Now we are in position to compute the principal term

$$\int_\Sigma M^t_{ijkl}\frac{\partial \overline{u}_k}{\partial \xi_l}\,d\hat{\boldsymbol{\xi}} = \langle \boldsymbol{a}_{ij}, \overline{\boldsymbol{u}}\rangle + \langle \boldsymbol{b}_{ij}, \mathcal{J}_t\overline{\boldsymbol{u}}\rangle$$

appearing in the limiting (homogenized) equation (3.2). Utilizing (5.1) and (5.2) and computing other terms in (3.2), we obtain the following limiting equation:

$$\begin{aligned}(5.3)\quad &\int_0^T\int_\Omega\Big(-\rho_\theta u_i\frac{\partial \phi_i}{\partial t} + \big(\theta P_{ijkl} - \alpha_{ijkl}\big)\frac{\partial u_k}{\partial x_l}\frac{\partial \phi_i}{\partial x_j}\\ &+ \int_0^t\Big(\theta\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\theta)G_{ijkl} - \beta_{ijkl} + \omega_{ijkl}(t-s)\Big)\frac{\partial u_k}{\partial x_l}\,ds\,\frac{\partial \phi_i}{\partial x_j}\Big)\,d\boldsymbol{x}dt\\ &= \int_0^T\int_\Omega\Big(\rho_\theta f_i\phi_i - \big(\nu_{ij} - \theta p^0\delta_{ij} + (1-\theta)\mathcal{G}^0_{ij}\big)\frac{\partial \phi_i}{\partial x_j}\Big)\,d\boldsymbol{x}dt + \int_\Omega \rho_\theta\boldsymbol{u}^0\cdot\boldsymbol{\phi}^0\,d\boldsymbol{x},\end{aligned}$$

where

$$\theta(\boldsymbol{x}) = \int_\Sigma \chi\,d\hat{\boldsymbol{\xi}}, \qquad \rho_\theta = \theta\rho_F + (1-\theta)\,\rho_S,$$

$$\begin{aligned}\nu_{ij} &= -\langle \boldsymbol{a}_{ij}, e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\boldsymbol{n}_0\rangle + \langle \boldsymbol{b}_{ij}, \big(e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E} - \mathcal{I}\big)\mathcal{B}_E^{-1}\boldsymbol{n}_0\rangle,\\ \alpha_{ijkl} &= \langle \boldsymbol{a}_{ij}, \mathcal{A}_E^{-1}\boldsymbol{a}_{kl}\rangle,\\ \beta_{ijkl} &= \langle \boldsymbol{b}_{ij}, \mathcal{B}_E^{-1}\boldsymbol{b}_{kl}\rangle,\\ \omega_{ijkl}(t) &= -\langle \boldsymbol{a}_{ij}, \boldsymbol{m}_{kl}\rangle - \langle \boldsymbol{b}_{ij}, \widetilde{\boldsymbol{m}}_{kl}\rangle.\end{aligned}$$

Let us denote by $\overline{P}$, $\overline{G}$, and $\mathcal{S}^0$ the tensors with components

$$\overline{P}_{ijkl} = \theta P_{ijkl} - \alpha_{ijkl}, \quad \overline{G}_{ijkl} = \theta\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\theta)G_{ijkl} - \beta_{ijkl},$$

$$\mathcal{S}^0_{ij} = \nu_{ij} - \theta p^0\delta_{ij} + (1-\theta)\mathcal{G}^0_{ij}.$$

Let us divide the domain Ω into three parts:

$$\Omega^f = \{\boldsymbol{x} \in \Omega \,|\, x_3 > \delta\}, \quad \Omega^s = \{\boldsymbol{x} \in \Omega \,|\, x_3 < 0\}, \quad \Omega^h = \{\boldsymbol{x} \in \Omega \,|\, 0 < x_3 < \delta\}.$$

Let Γ^+_δ be the boundary between Ω^f and Ω^h, Γ^-_δ the boundary between Ω^s and Ω^h. That is, $\Omega = \Omega^f \cup \Gamma^+_\delta \cup \Omega^h \cup \Gamma^-_\delta \cup \Omega^s$. Note that $\theta(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \Omega^f$, $\theta(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \Omega^s$, and θ is a constant from the interval $(0,1)$ for $\boldsymbol{x} \in \Omega^h$. As for α_{ijkl}, β_{ijkl}, ν_{ij}, and ω_{ijkl}, they are constants for $\boldsymbol{x} \in \Omega^h$ and equal to zero if $\boldsymbol{x} \in \Omega^f \cup \Omega^s$, so that the integral identity (5.3) delivers the following equations which should be understood in the distributional sense:

(5.4)

$$\rho_F \boldsymbol{u}_t - \operatorname{div} P\boldsymbol{u}_x - \gamma^{-1}\nabla\operatorname{div}\mathcal{J}_t\boldsymbol{u} = -\nabla p^0 + \rho_F \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega^f,$$

(5.5)

$$\rho_S \boldsymbol{u}_t - \operatorname{div}\mathcal{J}_t G\boldsymbol{u}_x = \operatorname{div}\mathcal{G}^0 + \rho_S \boldsymbol{f}, \qquad \boldsymbol{x} \in \Omega^s,$$

(5.6)

$$\rho_\theta \boldsymbol{u}_t - \operatorname{div}\overline{P}\boldsymbol{u}_x - \operatorname{div}\mathcal{J}_t\overline{G}\boldsymbol{u}_x - \operatorname{div}\int_0^t \omega(t-s)\boldsymbol{u}_x(s)\,ds + \operatorname{div}\mathcal{S}^0 = \rho_\theta \boldsymbol{f}, \quad \boldsymbol{x} \in \Omega^h.$$

The natural interfacial boundary conditions at Γ^+_δ and Γ^-_δ can be derived from the integral identity (5.3). Equations (5.4) and (5.5) coincide with (2.1) and (2.11), respectively. That is, the governing equations for the pure fractions do not change after the homogenization, which have been expected. What is new is an integral-differential equation (5.6) which cannot be reduced to a pure differential equation by differentiating or by a substitution like $\boldsymbol{w} = \mathcal{J}_t\boldsymbol{u}$. The operators involved in the equation have to be investigated to confirm the parabolic type of its principal part.

5.2. Investigation of $\overline{P}$ and $\overline{G}$. It is not difficult to verify that the tensors $\overline{P}$ and $\overline{G}$ have the symmetry properties mentioned in Remark 2.2. Therefore, $\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ and $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ for every skew symmetric matrix $\mathcal{Z}$. The main objective of this subsection is to prove the strong positiveness of the tensor $\overline{P}$ and the nonnegativeness of $\overline{G}$ on the space of symmetric matrices. The null-space of $\overline{G}$ will be also described.

PROPOSITION 5.19. *For every second-rank tensor $\mathcal{Z}$, the following is valid:*

$$\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} \geqslant 0, \qquad \overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} \geqslant 0.$$

Proof. Let us prove the assertion for $\overline{P}$. Denote $\boldsymbol{z} = \boldsymbol{a}_{ij}\mathcal{Z}_{ij}$. Due to Proposition 4.12, $\boldsymbol{z} \in R(\mathcal{A}) \cap R(\mathcal{B})$ and, as it follows from Theorem 4.17, there exists a unique $\boldsymbol{y} \in E$ such that $\mathcal{A}_E\boldsymbol{y} = \boldsymbol{z}$. This means that

$$\int_\Sigma \chi P_{ijkl}\frac{\partial y_i}{\partial \xi_j}\frac{\partial v_k}{\partial \xi_l}\,d\hat{\boldsymbol{\xi}} = \langle \boldsymbol{z}, \boldsymbol{v}\rangle = \int_\Sigma \chi P_{ijkl}\mathcal{Z}_{ij}\frac{\partial v_k}{\partial \xi_l}\,d\hat{\boldsymbol{\xi}} \tag{5.7}$$

for all $\boldsymbol{v} \in H$. On the other hand, the definition yields

$$\alpha_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = \langle \boldsymbol{a}_{ij}\mathcal{Z}_{ij}, \mathcal{A}_E^{-1}\boldsymbol{a}_{kl}\mathcal{Z}_{kl}\rangle = \langle \boldsymbol{z}, \mathcal{A}_E^{-1}\boldsymbol{z}\rangle = \langle \mathcal{A}_E\boldsymbol{y}, \boldsymbol{y}\rangle.$$

From the last relation and (5.7) with $\boldsymbol{v} = \boldsymbol{y}$, we obtain

(5.8)
$$\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = \theta\, P_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \langle \mathcal{A}_{E}\boldsymbol{y}, \boldsymbol{y}\rangle = \int_\Sigma \Big(\chi P_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \chi P_{ijkl}\frac{\partial y_i}{\partial \xi_j}\frac{\partial y_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}}$$
$$= \int_\Sigma \chi P_{ijkl}\Big(\mathcal{Z}_{ij} - \frac{\partial y_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}_{kl} - \frac{\partial y_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}}.$$

The right-hand side of the last relation is clearly positive and the required assertion is proved for the tensor $\overline{P}$. Positiveness of the tensor $\overline{G}$ can be verified in the same way. □

The next theorem states the strong positiveness of the tensor $\overline{P}$.

THEOREM 5.20. *There exists a positive constant C such that*

$$\overline{P}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} \geqslant C\,|\mathcal{Z}|^2$$

for every symmetric second-rank tensor $\mathcal{Z}$. Here, $|\mathcal{Z}|^2 = \mathcal{Z}_{ij}\mathcal{Z}_{ij}$.

Proof. Assume that the assertion of the theorem is false. Then there exists a sequence $\{\mathcal{Z}^n\}$ such that $|\mathcal{Z}^n| = 1$ and $\overline{P}_{ijkl}\mathcal{Z}^n_{ij}\mathcal{Z}^n_{kl} \to 0$ as $n \to \infty$. The sequence $\{\mathcal{Z}^n\}$ is compact in $\mathbb{R}^3 \times \mathbb{R}^3$ and, therefore, it has a subsequence denoted again by $\{\mathcal{Z}^n\}$, which converges to a matrix $\mathcal{Z}^0$ such that $|\mathcal{Z}^0| = 1$. This means that the corresponding sequences $\boldsymbol{z}^n$ and $\boldsymbol{y}^n$, defined as $\boldsymbol{z}^n = \boldsymbol{a}_{ij}\mathcal{Z}^n_{ij}$ and $\boldsymbol{y}^n = \mathcal{A}_E^{-1}\boldsymbol{z}^n$, converge in H to $\boldsymbol{z}^0$ and $\boldsymbol{y}^0$, respectively. We use here the notations introduced in the proof of the previous proposition. Thus, the relation

$$\overline{P}_{ijkl}\mathcal{Z}^0_{ij}\mathcal{Z}^0_{kl} = \theta\, P_{ijkl}\mathcal{Z}^0_{ij}\mathcal{Z}^0_{kl} - \langle \mathcal{A}_{E}\boldsymbol{y}^0, \boldsymbol{y}^0\rangle = \int_\Sigma \chi P_{ijkl}\Big(\mathcal{Z}^0_{ij} - \frac{\partial y^0_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}^0_{kl} - \frac{\partial y^0_k}{\partial \xi_l}\Big)\, d\hat{\boldsymbol{\xi}} = 0.$$

holds due to (5.8). That is,

$$\chi P_{ijkl}\Big(\mathcal{Z}^0_{ij} - \frac{\partial y^0_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}^0_{kl} - \frac{\partial y^0_k}{\partial \xi_l}\Big) = 0 \qquad \text{in} \quad \Sigma,$$

and, consequently, $\mathcal{D}(\boldsymbol{y}^0) = \mathcal{Z}^0$ in Σ_F. This implies that $\mathcal{D}(\boldsymbol{y}^0 - \mathcal{Z}^0\boldsymbol{\xi}) = 0$ in Σ_F. Therefore, $\boldsymbol{y}^0(\boldsymbol{\xi})$ is a linear function of $\boldsymbol{\xi}$ for $\boldsymbol{\xi} \in \Sigma_F$. The only linear function satisfying the periodicity boundary conditions on $\partial\Sigma$ is a constant, which implies that $\mathcal{Z}^0 = 0$. This is impossible because $|\mathcal{Z}^0| = 1$. This contradiction proves the theorem. □

Remark that the arguments like those in the proof of Theorem 5.20 do not lead to a contradiction in the case of the tensor $\overline{G}$. The next theorem shows that the tensor $\overline{G}$ is degenerated and describes its null-space.

THEOREM 5.21. *The tensor $\overline{G}$ is degenerate, and $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ for a symmetric matrix $\mathcal{Z}$ if and only if $\mathcal{Z}_{11} + \mathcal{Z}_{22} = 0$ and $\mathcal{Z}_{33} = 0$.*

Proof. Let us denote $\boldsymbol{z} = \boldsymbol{b}_{ij}\mathcal{Z}_{ij}$. Due to Proposition 4.12 and Theorem 4.17, $\boldsymbol{z} \in R(\mathcal{A}) \cap R(\mathcal{B})$, and there exist unique elements $\boldsymbol{y}^E \in E$ and $\boldsymbol{y}^R \in R(\mathcal{B})$ such that

$$\mathcal{B}\boldsymbol{y}^E = \boldsymbol{z}, \qquad \mathcal{B}\boldsymbol{y}^R = \boldsymbol{z}. \tag{5.9}$$

It follows that $\boldsymbol{y}^N = \boldsymbol{y}^E - \boldsymbol{y}^R \in N(\mathcal{B})$. Besides that, $\mathcal{B}_E^{-1}\mathcal{B}\boldsymbol{y}^E = \boldsymbol{y}^E$. Therefore,

$$\langle \boldsymbol{z}, \mathcal{B}_E^{-1}\boldsymbol{z}\rangle = \langle \mathcal{B}\boldsymbol{y}^E, \boldsymbol{y}^E\rangle = \langle \mathcal{B}\boldsymbol{y}^R, \boldsymbol{y}^R + \boldsymbol{y}^N\rangle = \langle \mathcal{B}\boldsymbol{y}^R, \boldsymbol{y}^R\rangle.$$

The second equation in (5.9) implies that

$$\int_{\Sigma} K_{ijkl}(\chi)\frac{\partial y_i^R}{\partial \xi_j}\frac{\partial v_k}{\partial \xi_l}\,d\hat{\boldsymbol{\xi}} = \int_{\Sigma} K_{ijkl}(\chi)\mathcal{Z}_{ij}\frac{\partial v_k}{\partial \xi_l}\,d\hat{\boldsymbol{\xi}}, \tag{5.10}$$

for all $\boldsymbol{v} \in H$, where

$$K_{ijkl}(\chi) = \chi\gamma^{-1}\delta_{ij}\delta_{kl} + (1-\chi)G_{ijkl}.$$

As a consequence of this equation, we find

$$\begin{aligned}\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} &= K_{ijkl}(\theta)\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \langle \boldsymbol{z}, \mathcal{B}_E^{-1}\boldsymbol{z}\rangle = K_{ijkl}(\theta)\mathcal{Z}_{ij}\mathcal{Z}_{kl} - \langle \mathcal{B}\boldsymbol{y}^R, \boldsymbol{y}^R\rangle \\ &= \int_{\Sigma} K_{ijkl}(\chi)\Big(\mathcal{Z}_{ij} - \frac{\partial y_i^R}{\partial \xi_j}\Big)\Big(\mathcal{Z}_{kl} - \frac{\partial y_k^R}{\partial \xi_l}\Big)\,d\hat{\boldsymbol{\xi}} \\ &= \int_{\Sigma_F} \gamma^{-1}(\mathrm{tr}\mathcal{Z} - \mathrm{div}\,\boldsymbol{y}^R)^2\,d\hat{\boldsymbol{\xi}} + \int_{\Sigma_S} G_{ijkl}\big(\mathcal{Z}_{ij} - \mathcal{D}_{ij}(\boldsymbol{y}^R)\big)\big(\mathcal{Z}_{kl} - \mathcal{D}_{kl}(\boldsymbol{y}^R)\big)\,d\hat{\boldsymbol{\xi}}.\end{aligned}$$

Notice that (5.10) is the Euler–Lagrange equation for the functional

$$F_z(\boldsymbol{y}) = \int_{\Sigma} K_{ijkl}(\chi)\Big(\mathcal{Z}_{ij} - \frac{\partial y_i}{\partial \xi_j}\Big)\Big(\mathcal{Z}_{kl} - \frac{\partial y_k}{\partial \xi_l}\Big)\,d\hat{\boldsymbol{\xi}}.$$

Due to Proposition 4.11 (assertion 4), this functional is strictly convex on $R(\mathcal{B})$ and $\boldsymbol{y}^R$ is its unique minimizer there. That is,

$$\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = F_z(\boldsymbol{y}^R) = \min_{\boldsymbol{y}\in R(\mathcal{B})} F_z(\boldsymbol{y}).$$

Thus, $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ if and only if there exists $\boldsymbol{y}^R \in R(\mathcal{B})$ such that $F_z(\boldsymbol{y}^R) = 0$. It is not difficult to see that $F_z(\boldsymbol{y}) = F_z(\boldsymbol{y}+\boldsymbol{w})$ for every $\boldsymbol{w} \in N(\mathcal{B})$. Since $R(\mathcal{B}) \oplus N(\mathcal{B}) = H$, the existence of $\boldsymbol{y}^R \in R(\mathcal{B})$ with $F_z(\boldsymbol{y}^R) = 0$ is equivalent to the existence of a function $\boldsymbol{y} \in H$ which satisfies the condition $F_z(\boldsymbol{y}) = 0$. Due to the positiveness of the functional F_z, we can conclude that $\overline{G}_{ijkl}\mathcal{Z}_{ij}\mathcal{Z}_{kl} = 0$ if and only if there exists $\boldsymbol{y} \in H$ such that

$$\mathrm{div}\,\boldsymbol{y} = \mathrm{tr}\,\mathcal{Z} \quad \text{as} \quad \hat{\boldsymbol{\xi}} \in \Sigma_F, \tag{5.11}$$

$$\mathcal{D}(\boldsymbol{y}) = \mathcal{Z} \qquad \text{as} \quad \hat{\boldsymbol{\xi}} \in \Sigma_S. \tag{5.12}$$

Suppose that both of the last conditions are satisfied. Since functions from H do not depend on ξ_3, (5.12) implies that $\mathcal{Z}_{33} = 0$. Moreover, due to (5.12), $\mathrm{div}\,\boldsymbol{y} = \mathrm{tr}\,\mathcal{Z}$ in Σ_S. That is, $\mathrm{div}\,\boldsymbol{y} = \mathrm{tr}\,\mathcal{Z}$ in Σ. Integrating this equality over Σ we find that $\mathrm{tr}\,\mathcal{Z} = 0$ because $\boldsymbol{y}$ is periodic. Thus, we have proved the assertion of the theorem in one direction (the necessity).

Let us suppose that $\mathcal{Z}_{11} + \mathcal{Z}_{22} = 0$ and $\mathcal{Z}_{33} = 0$. In order to complete the proof of the theorem, we have to prove that there exists a function $\boldsymbol{y} \in H$ satisfying (5.11) and (5.12). Equation (5.12) is easy to solve. Namely, its solution appears as follows:

$$\boldsymbol{y}(\boldsymbol{\xi}) = \mathcal{Z}\boldsymbol{\xi} + \mathcal{Q}\boldsymbol{\xi} + \boldsymbol{y}_0, \qquad \hat{\boldsymbol{\xi}} \in \Sigma_S,$$

where $\mathcal{Q}$ is a skew-symmetric matrix and $\boldsymbol{y}_0$ is a constant which can be dropped because functions from the space H are defined up to a constant. Let us denote $\mathcal{T} =$

$\mathcal{Z} + \mathcal{Q}$. Since functions from H do not depend on ξ_3, we find that $\mathcal{T}_{i3} = 0\,(i = 1, 2, 3)$ and $y_3 = \mathcal{T}_{31}\xi_1 + \mathcal{T}_{32}\xi_2$ for $\hat{\boldsymbol{\xi}} \in \Sigma_S$. We extend y_3 to the whole domain Σ in such a way that it would be a periodic function (assuming equal values on the opposite edges of Σ).

In order to determine y_1 and y_2 in Σ_F, we have to solve the problem

$$\begin{aligned}
&\frac{\partial y_1}{\partial \xi_1} + \frac{\partial y_2}{\partial \xi_2} = 0, && \hat{\boldsymbol{\xi}} \in \Sigma_F,\\
&\boldsymbol{y}(\hat{\boldsymbol{\xi}}) = \mathcal{T}\boldsymbol{\xi}, && \hat{\boldsymbol{\xi}} \in \partial\Sigma_S,\\
&y_1 \quad \text{and} \quad y_2 \quad \text{are} \quad \text{periodic} \quad \text{in} \quad \Sigma.
\end{aligned}$$

This problem is clearly solvable, and the theorem is completely proved. ∎

As one can see from (5.6), the tensor $\overline{G}$ describes elastic stresses in the homogenized continuum. Theorem 5.21 says that the homogenized material has rather strange properties. Namely, it does not resist to the deformation, if the first invariant and the component (3,3) of the corresponding strain tensor are equal to zero. In other words, such deformations do not produce any stresses. The described class of deformations is sufficiently large. It contains all deformations which do not change volume. The following assertion is a simple consequence of Theorem 5.21.

COROLLARY 5.22. *If $i \neq j$ and $k \neq l$, then $\overline{G}_{ijkl} = 0$.*

This property of the tensor $\overline{G}$ yields an interesting conclusion about the passage to the limit as $\theta \to 0$. If we set $\theta = 0$ formally, the elastic structure will occupy the whole layer Ω^h. Therefore, it can seem that the limiting material must be the same as the original elastic one so that $\lim_{\theta\to 0} \overline{G} = G$. Nevertheless, it is wrong in general because the properties of the tensor $\overline{G}$ stated in Theorem 5.21 and in Corollary 5.22 do not depend on θ. Thus, if, for instance, the tensor G is not degenerate or $G_{1212} \neq 0$, then $\lim_{\theta\to 0} \overline{G} \neq G$. The physical reason is that the elastic structure consists of separate bristles for each $\theta > 0$, which differs from the bulk material corresponding to $\theta = 0$.

6. Numerical procedures. The formulas for the coefficients $\overline{P}$, $\overline{G}$, and ω contain the functions $\boldsymbol{a}_{kl}$, $\boldsymbol{b}_{kl}$, $\boldsymbol{n}_0$, the operators $\mathcal{A}_E$, $\mathcal{B}_E$, and their inverse defined in $H = H^1_\#(\Sigma)/\mathbb{R}$. From the mathematical point of view, all these functions and operators are well defined and completely described. However, numerical implementation of these formulas requires some effort. The computation of the functions $\boldsymbol{a}_{kl}$, $\boldsymbol{b}_{kl}$, and $\boldsymbol{n}_0$ is not difficult if one uses the finite element method. The situation with the operators $\mathcal{A}$, $\mathcal{B}$ is not so trivial, because they must be restricted to the subspace E, which creates additional problems when using finite elements. Below, we propose numerical procedures that can be implemented using conventional finite element software.

6.1. Calculation of $\boldsymbol{a_{kl}}$, $\boldsymbol{b_{kl}}$, and $\boldsymbol{n_0}$. Let us introduce functions $\sigma_k \in H$, $k = 1, 2, 3$, as solutions of the following problems:

$$\int_\Sigma \frac{\partial \sigma_k}{\partial \xi_i}\frac{\partial v}{\partial \xi_i}\, d\hat{\boldsymbol{\xi}} = \int_\Sigma \chi\, \frac{\partial v}{\partial \xi_k}\, d\hat{\boldsymbol{\xi}} \qquad \text{for all} \quad v \in H.$$

These problems can be easily solved applying the finite element method. Note that $\sigma_3 = 0$ because functions from the space H do not depend on ξ_3. It is not difficult to see that

$$\begin{aligned}
a^i_{kl} &= P_{kli1}\sigma_1 + P_{kli2}\sigma_2,\\
b^i_{kl} &= \gamma^{-1}\delta_{kl}\sigma_i - G_{kli1}\sigma_1 - G_{kli2}\sigma_2,\\
n^i_0 &= -p^0\sigma_i - \mathcal{G}^0_{i1}\sigma_1 - \mathcal{G}^0_{i2}\sigma_2
\end{aligned}$$

for $i, k, l \in \{1, 2, 3\}$. Here, the superscript i denotes the components of the vectors $\boldsymbol{a}_{kl}$, $\boldsymbol{b}_{kl}$, and $\boldsymbol{n}_0$.

6.2. Calculation of $\mathcal{A}_E^{-1}$ and $\mathcal{B}_E^{-1}$. The problem can be formulated as follows: *for every $\boldsymbol{w} \in R(\mathcal{A}) \cap R(\mathcal{B})$, find $\boldsymbol{u}, \boldsymbol{v} \in E$ such that $\mathcal{A}\boldsymbol{u} = \boldsymbol{w}$ and $\mathcal{B}\boldsymbol{v} = \boldsymbol{w}$.* It is enough to solve this problem for the operator $\mathcal{A}$. The operator $\mathcal{B}$ can be treated similarly. Let us consider the equation

$$(\mathcal{A} + \varepsilon\mathcal{B})\,\boldsymbol{u}_\varepsilon = \boldsymbol{w}. \tag{6.1}$$

As follows from Proposition 4.13, the operator $\mathcal{A} + \varepsilon\mathcal{B}$ is invertible in H for every $\varepsilon > 0$. Thus, there exists a unique $\boldsymbol{u}_\varepsilon \in H$ that satisfies (6.1). Moreover, $\boldsymbol{u}_\varepsilon \in E$ for every $\varepsilon > 0$ by definition of the subspace E. Equation (6.1) can be easily solved numerically with finite elements. Let us show that $\boldsymbol{u}_\varepsilon$ is an approximation of a function $\boldsymbol{u} \in E$ that satisfies the equation $\mathcal{A}\boldsymbol{u} = \boldsymbol{w}$. Since $\boldsymbol{u}_\varepsilon \in E$, we can rewrite (6.1) as $(\mathcal{A}_E + \varepsilon\mathcal{B}_E)\,\boldsymbol{u}_\varepsilon = \boldsymbol{w}$. Consequently,

$$\boldsymbol{u}_\varepsilon = \mathcal{A}_E^{-1}(\boldsymbol{w} - \varepsilon\mathcal{B}_E\boldsymbol{u}_\varepsilon). \tag{6.2}$$

Due to Theorem 4.17, the operator $\mathcal{A}_E^{-1} : R(\mathcal{A}) \cap R(\mathcal{B}) \to E$ is bounded. Therefore, there exists an independent of ε constant C such that

$$\|\boldsymbol{u}_\varepsilon\| \leqslant C(1 + \varepsilon\|\boldsymbol{u}_\varepsilon\|).$$

This means that the sequence $\{\boldsymbol{u}_\varepsilon\}_\varepsilon$ is weakly compact in H. That is, there exist $\boldsymbol{u} \in H$ and a subsequence $\{\boldsymbol{u}_\varepsilon\}_\varepsilon$ such that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$ weakly in H as $\varepsilon \to 0$. Since E is weakly closed, $\boldsymbol{u} \in E$. The passage to the limit in (6.1) yields the desired relation $\mathcal{A}\boldsymbol{u} = \boldsymbol{w}$. Moreover, the whole sequence $\{\boldsymbol{u}_\varepsilon\}_\varepsilon$ converges to $\boldsymbol{u}$ in H. In reality, $\boldsymbol{u}_\varepsilon - \boldsymbol{u} = -\varepsilon\mathcal{A}_E^{-1}\mathcal{B}_E\boldsymbol{u}_\varepsilon$, which implies

$$\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\| \leqslant C\varepsilon. \tag{6.3}$$

Thus, the order of the approximation is obtained.

6.3. Calculation of $e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\mathcal{B}_E$. Problem: *For every $\boldsymbol{w} \in E$ and all $t > 0$, find $\boldsymbol{u}(t) = e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\mathcal{B}_E\boldsymbol{w}$.* Let us consider the following equation:

$$\boldsymbol{u}_t + \mathcal{A}_E^{-1}\mathcal{B}_E\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0) = \mathcal{A}_E^{-1}\mathcal{B}_E\boldsymbol{w}. \tag{6.4}$$

It is obvious that $\boldsymbol{u}(t) = e^{-t\mathcal{A}_E^{-1}\mathcal{B}_E}\mathcal{A}_E^{-1}\mathcal{B}_E\boldsymbol{w} \in E$ is a unique solution of this equation (see section 4.3). Let us construct an approximate solution to problem (6.4) using the semidiscretization method. Fix t and introduce $\tau = t/N$, $N \in \mathbb{N}$. Define functions $\boldsymbol{u}_n$, $n = 1, 2, \ldots, N$, as solutions of the following problem:

$$(\mathcal{A} + \tau\,\mathcal{B})\boldsymbol{u}_n = \mathcal{A}\boldsymbol{u}_{n-1}, \qquad \boldsymbol{u}_0 = \mathcal{A}_E^{-1}\mathcal{B}_E\boldsymbol{w}. \tag{6.5}$$

That is,

$$\boldsymbol{u}_n = (\mathcal{A} + \tau\,\mathcal{B})^{-1}\mathcal{A}\boldsymbol{u}_{n-1}.$$

Note that $\boldsymbol{u}_n \in E$ for all $n \geq 0$ because $\boldsymbol{u}_0 \in E$ and $\boldsymbol{u}_{n-1} \in E$ implies $\boldsymbol{u}_n \in E$ for any $n > 0$. Therefore, the operators $\mathcal{A}$ and $\mathcal{B}$ can be replaced by $\mathcal{A}_E$ and $\mathcal{B}_E$ in (6.5).

To estimate $\boldsymbol{u}_N - \boldsymbol{u}(t)$ we prove first that there exists a constant C such that $\|\boldsymbol{u}_n\| \leqslant C$ for all n. Indeed,

$$\boldsymbol{u}_n = \boldsymbol{u}_0 - \tau \mathcal{A}_{_E}^{-1}\mathcal{B}_{_E} \sum_{k=1}^{n} \boldsymbol{u}_k,$$

which implies due to the boundness of $\mathcal{A}_{_E}^{-1}$ and $\mathcal{B}_{_E}^{-1}$ (see Theorem 4.17), the existence of a constant C such that

$$\|\boldsymbol{u}_n\| \leqslant \|\boldsymbol{u}_0\| + C\,\tau \sum_{k=1}^{n} \|\boldsymbol{u}_k\|.$$

The Gronwall inequality now implies the required estimate. The boundness of $\|\boldsymbol{u}_n\|$ and (6.5) provide the following estimate:

$$\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\| \leqslant C\tau\, \|\mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}\| \|\boldsymbol{u}_n\| \leqslant C\tau. \tag{6.6}$$

Let us introduce two time interpolations of $\{u_n\}$,

$$\hat{\boldsymbol{u}}^\tau(s) = \boldsymbol{u}_n\Big(1 - n + \frac{s}{\tau}\Big) + \boldsymbol{u}_{n-1}\Big(n - \frac{s}{\tau}\Big) \qquad \text{as} \quad s \in [(n-1)\tau, n\tau],$$
$$\bar{\boldsymbol{u}}^\tau(s) = \boldsymbol{u}_n \qquad \text{as} \quad s \in ((n-1)\tau, n\tau].$$

Due to (6.6),

$$\int_0^t \|\hat{\boldsymbol{u}}^\tau(s) - \bar{\boldsymbol{u}}^\tau(s)\|\, ds = \sum_{n=1}^{N} \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\| \int_{(n-1)\tau}^{n\tau} \Big(n - \frac{s}{\tau}\Big)\, ds$$
$$= \frac{\tau}{2} \sum_{n=1}^{N} \|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\| \leqslant C\tau.$$

Moreover, we have

$$\frac{\partial \hat{\boldsymbol{u}}^\tau}{\partial t} + \mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}\bar{\boldsymbol{u}}^\tau = 0.$$

Therefore,

$$\|\boldsymbol{u}_N - \boldsymbol{u}(t)\| = \|\hat{\boldsymbol{u}}^\tau(t) - \boldsymbol{u}(t)\| \leqslant \|\mathcal{A}_{_E}^{-1}\mathcal{B}_{_E}\| \int_0^t \|\bar{\boldsymbol{u}}^\tau(s) - \boldsymbol{u}(s)\|\, ds$$
$$\leqslant C \int_0^t \big(\|\hat{\boldsymbol{u}}^\tau(s) - \boldsymbol{u}(s)\| + \|\hat{\boldsymbol{u}}^\tau(s) - \bar{\boldsymbol{u}}^\tau(s)\|\big)\, ds \leqslant C\Big(\tau + \int_0^t \|\hat{\boldsymbol{u}}^\tau(s) - \boldsymbol{u}(s)\|\, ds\Big).$$

Finally, the Gronwall inequality yields

$$\|\boldsymbol{u}_N - \boldsymbol{u}(t)\| \leqslant C\tau.$$

6.4. Numerics. In this section we give some examples which demonstrate properties of the homogenized continuum for various values of θ. We consider the system consisting of the water and an isotropic elastic material (polymer) with the following properties:

$$P\boldsymbol{u}_{\boldsymbol{x}} = \lambda_f\, \mathcal{I} \operatorname{div} \boldsymbol{u} + 2\,\mu_f\, \mathcal{D}(\boldsymbol{u}), \qquad G\boldsymbol{u}_{\boldsymbol{x}} = \lambda_s\, \mathcal{I} \operatorname{div} \boldsymbol{u} + 2\,\mu_s\, \mathcal{D}(\boldsymbol{u}),$$
$$\lambda_f = 1.\mathsf{e}{-3}, \quad \mu_f = 1.\mathsf{e}{-3},$$
$$\lambda_s = 2.777778\mathsf{e}+9, \quad \mu_s = 4.166667\mathsf{e}+9.$$

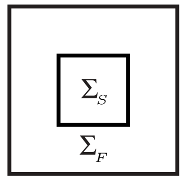

FIG. 3. *Structural cell* $\Sigma = [0,1] \times [0,1]$.

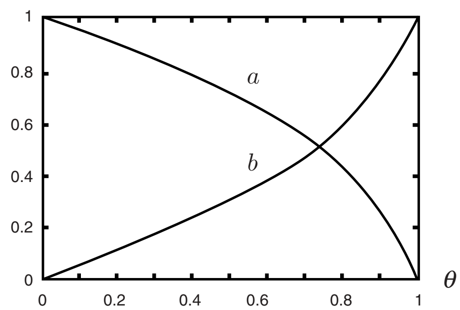

FIG. 4. *Dependence of* $|P - \overline{P}|/|P|$ *(curve a) and* $|\overline{P}|/|P|$ *(curve b) on* θ.

The constant which characterizes compressibility of the water is $\gamma = 4.597696\mathsf{e} - 10$. We take the structural cell of the form shown in Figure 3.

First, we investigate properties of the tensor $\overline{P}$. The graphics in Figure 4 present the dependence of $|P - \overline{P}|/|P|$ and $|\overline{P}|/|P|$ on θ, where $|P| = \big(\sum_{ijkl} P_{ijkl}P_{ijkl}\big)^{1/2}$. As one can see, $\lim_{\theta\to 1} \overline{P} = P$ and $\lim_{\theta\to 0} \overline{P} = 0$.

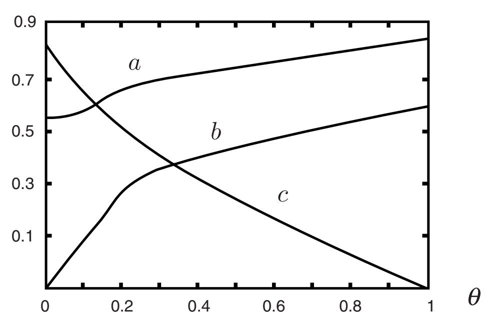

FIG. 5. *Dependence of* $|\overline{G} - G|/|G|$ *(curve a),* $|\overline{G} - Q|/|G|$ *(curve b), and* $|\overline{G} - \theta R|/|G|$ *(curve c) on* θ.

The dependence of the tensor $\overline{G}$ on θ is more complex. Let us introduce two tensors R and Q having the following components:

$$R_{ijkl} = \gamma^{-1}\delta_{ij}\delta_{kl}, \qquad Q_{ijkl} = \begin{cases} G_{ijkl} & \text{if} \quad i = j \quad \text{and} \quad k = l, \\ 0 & \text{otherwise.} \end{cases}$$

The curves in Figure 5 show the dependence of $|\overline{G} - G|/|G|$, $|\overline{G} - \theta R|/|G|$ and $|\overline{G} - Q|/|G|$ on θ. One can see that $\overline{G}$ does not tend to G as $\theta \to 0$ (see curve a). This fact was already noted in section 5.2 after Corollary 5.22. It is not surprising that

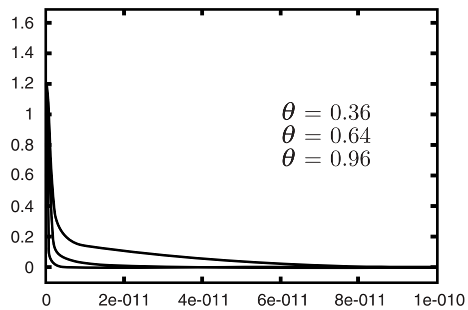

FIG. 6. *Graphics of the function* $|\omega(t)|/|\overline{G}|$ *for various values of* θ. *The time unit is* 1 *second.*

$\lim_{\theta\to 1}\overline{G} \neq 0$. It can be explained by the presence of the tensor R in the definition of $\overline{G}$. Really, curve c in Figure 5 shows that $\lim_{\theta\to 1}(\overline{G} - R) = 0$. Curve b shows that $\lim_{\theta\to 0}\overline{G} = Q$. This means that, in the limit case as $\theta \to 0$ (the elastic continuum occupies the whole domain Ω^h), the tensor $\overline{G}$ can be obtained from the tensor G by vanishing all of the "nondiagonal" components.

Thus, the limits of (5.6) as $\theta \to 0$ and $\theta \to 1$ look as follows:

$$\rho_S \boldsymbol{u}_t - \operatorname{div} \mathcal{J}_t Q \boldsymbol{u}_x - \operatorname{div} \int_0^t \omega(t-s)\boldsymbol{u}_x(s)\,ds = \rho_S \boldsymbol{f}, \qquad (\theta \to 0), \tag{6.7}$$

$$\rho_F \boldsymbol{u}_t - \operatorname{div} P \boldsymbol{u}_x - \gamma^{-1}\nabla \operatorname{div} \boldsymbol{u} - \operatorname{div} \int_0^t \omega(t-s)\boldsymbol{u}_x(s)\,ds = \rho_F \boldsymbol{f}, \quad (\theta \to 1). \tag{6.8}$$

We take here the initial data being equal to zero.

In Figure 6, the graphs of the function $|\omega(t)|/|\overline{G}|$ are presented for several values of θ. The function $|\omega(t)|$ decreases very rapidly. In fact, $|\omega(t)|/|\overline{G}|$ vanishes practically at the time $t \sim 10^{-10}\,s$. Thus, the memory term in (5.6) is very small and can be dropped in applications, if high frequency oscillations are not present.

7. Conclusions. Homogenization of a fine elastic structure immersed in a viscous weakly compressible fluid yields a continuum that possesses very interesting and rather unexpected properties. Equation (5.6) describing the behavior of the resulting continuum includes three basic terms. Two of them containing the tensors $\overline{P}$ and $\overline{G}$ are related to stresses. The third integral-term represents a memory effect that is responsible for viscoelastic properties of the resulting material. The presence of such a memory is not surprising because similar results were already obtained by other authors (see, for instance, [13]). More interesting from the mathematical and mechanical viewpoints is the investigation of the above mentioned stress terms. The term containing the tensor $\overline{P}$ describes a viscous damping and originates from the fluid part of the structure. Theorem 5.20 states the strict positiveness of $\overline{P}$, which implies the ellipticity of the corresponding differential operator. The term containing the tensor $\overline{G}$ represents elastic stresses. The tensor $\overline{G}$ is degenerate, its kernel is described in Theorem 5.21. The theorem implies that volume conserving deformations (shear deformations in particular) do not produce elastic stresses.

All of the coefficients involved in the homogenized equation are found in an explicit form. Although expressions representing them are rather complex, the coefficients can be computed numerically using the algorithms given in section 6. The numerical treatment delivers another interesting properties of the homogenized model. It is stated numerically that the memory effect is very weak. The system "forgets" the current history in a very short time. Therefore, the memory term can be dropped in most of the applications. Another interesting question is the dependence of properties of the homogenized continuum on the parameter θ which represents the volume fracture of the fluid so that the pure fluid corresponds to $\theta = 1$. As was expected, the homogenized equations coincide in the limit ($\theta \to 1$) with the ones being used for the description of the original fluid. In the opposite limiting case ($\theta \to 0$) the homogenized equations differ from the model of the original elastic continuum. In particular, the limiting elastic continuum can be nonisotropic even though the original material is isotropic. Thus, the limiting continuum inherits certain geometric properties of the fine elastic structure even if the fluid vanishes and the solid occupies the whole volume.

REFERENCES

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., 23 (1992), pp. 1482–1518.

[2] L. Baffico, C. Conca, *A mixing procedure of two viscous fluids using some homogenization tools*, Comput. Methods Appl. Mech. Engrg. 190 (2001), pp. 4245–4257.

[3] N. Botkin, M. Schlensog, M. Tewes, and V. Turova, *A mathematical model of a biosensor*, in Technical Proceedings of the Fourth International Conference on Modeling and Simulation of Microsystems, Hilton Head Island, SC, 2001, 231–234.

[4] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes equations*, Vol. 1. *Linearized Steady Problems*, Springer Tracts in Natural Philosophy 38, Springer-Verlag, New York, 1994.

[5] K.-H. Hoffmann and N. D. Botkin, *Homogenization of von Kármán plates excited by piezoelectric patches*, ZAMM Z. Angew. Math. Mech., 80 (2000), pp. 579–590.

[6] K.-H. Hoffmann and V. N. Starovoitov, *On a motion of a solid body in a viscous fluid. Two-dimensional case*, Adv. Math. Sci. Appl., 9 (1999), pp. 633–648.

[7] U. Hornung, *Homogenization and Porous Media*, Springer-Verlag, New York, 1997.

[8] W. Jäger and A. Mikelić, *On the roughness-induced effective boundary conditions for an incompressible viscous flow*, J. Differential Equations, 170 (2001), pp. 96–122.

[9] L. Landau and E. Lifshits., *Lehrbuch der theoretischen Physik*, Band VI: *Hydrodynamik*, 5th ed., Akademie-Verlag, Berlin, 1991.

[10] J.-L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Etudes mathematiques, Gauthier-Villars, Paris, 1969.

[11] D. Lukkassen, G. Nguetseng, and P. Wall, *Two–scale convergence*, Int. J. Pure Appl. Math., 2 (2002), pp. 35–86.

[12] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., 20 (1989), pp. 608–623.

[13] J. Sanchez-Hubert, *Asymptotic study of the macroscopic behaviour of a solid-fluid mixture*, Math. Methods Appl. Sci., 2 (1980), pp. 1–11.

[14] R. Temam, *Problèmes mathématiques en plasticité*, Méthodes Mathématiques de l'Informatique 12, Gauthier-Villars, Montrouge, 1983.