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Structure of Viability Kernels for Some Linear Differential Games

N.D. Botkin · E.A. Ryazantseva

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Abstract A convenient form of necessary and sufficient conditions of viability for differential games with linear dynamics is proposed. These conditions are utilized to construct maximal viable subsets of state constraints, viability kernels, in two illustrative two-dimensional examples. These examples demonstrate the relative simplicity of the structure of the viability kernels. It was found that the boundaries of the viability kernels consist of segments of the boundary of the state constraint and of lines defined by the first integrals of the governing equations as the players use extremal constant controls. It is conjectured that such a structure holds in high dimensional cases too.

Keywords Linear differential games · Viability kernels · Exact solutions

1 Introduction

The following problem is typical in differential game theory. Given a controlled system subjected to unpredictable disturbances and given a target set and a set of admissible state positions, it is required to find a strategy which provides the achievement

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N.D. Botkin (✉)
Technische Universität München, Boltzmannstrasse 3, 85748 Garching b. Munich, Germany
e-mail: botkin@ma.tum.de

E.A. Ryazantseva
Institute of Mathematics and Mechanics, Ural Department of Russian Academy of Sciences,
S. Kovalevskaya str. 16, 620219 Ekaterinburg, Russia

of the target set without violation of the state constraints for any disturbances contained in a prescribed set. As stated in [1], such a strategy can be designed using the extremal aiming procedure applied to the maximal u-stable set. Difficulties arising in this regard are connected with the complexity of numerical procedures for finding maximal u-stable sets. Analytical methods are not easy to apply because of the complexity of the structure of maximal u-stable sets (see for example [2]). Nevertheless, in many technical control problems, there is no target set: it is only required to keep the controlled system within prescribed state constraints which are constant in time. The confinement should be maintained for a long (infinite) time. In such cases, it is relevant to consider viability kernels [3–6] as a tool for control design. Currently, the development of numerical procedures for finding viability kernels becomes very popular (see e.g. [7–13]). Therefore, reliable examples for verification of numerical results are required. This paper gives two examples where viability kernels are found analytically. These examples show that the structure of viability kernels may be much simpler than that of maximal u-stable sets. This gives hope that viability kernels can be found analytically for certain classes of problems.

2 Differential Games and Viability Kernels

Consider a differential game with the autonomous linear dynamics

$$\dot{x} = Ax + Bu + Cv, \quad x \in R^n,\ u \in P \subset R^p,\ v \in Q \subset R^q. \tag{1}$$

Here x is the state vector in R^n, u and v are control parameters of the first and the second players, respectively; P and Q are compact sets of the dimensions p and q, respectively; and A, B, and C are matrices of the dimensions $n \times n$, $n \times p$, and $n \times q$, respectively. Without loss of generality we can assume that P and Q be convex [1]. Measurable functions $u(\cdot)$, $v(\cdot)$ satisfying the conditions $u(t) \in P$, $v(t) \in Q$, $t \geq 0$ are called admissible open-loop controls (or, simply, controls) of the players.

Definition 2.1 (Viability property [4]) We say that $K \subset R^n$ has the *viability property* iff for any $x_* \in K$ and any constant control $v_*(\cdot) \equiv v_* \in Q$, there exists a control $u_*(\cdot)$ such that $x_*(t) \in K$, $t \geq 0$, where $x_*(\cdot)$ is the solution of (1) with the initial state $x_*(0) = x_*$ and with the controls $u_*(\cdot)$, $v_*(\cdot)$ substituted into (1).

Definition 2.2 (Viability kernels [4]) For a given compact $G \subset R^n$, we denote by $Viab(G)$ the largest subset of G with the viability property. This subset is called the *viability kernel* of G.

The following assertions are simple consequences of Definitions 2.1 and 2.2:

1) The closure of any set with the viability property is a set with the viability property.
2) The union of any family of sets with the viability property is a set with the viability property.
3) The convex hull of any set with the viability property is a set with the viability property (due to the linearity of (1) and convexity of P).

These assertions yield the existence of the convex closed viability kernel *Viab*(G) for any compact G (if there exists at last one subset of G with the viability property). So, henceforth, we shall consider closed convex sets with the viability property.

Remark 2.1 The definition of the viability property involves open loop controls of the first player and constant controls of the second player. This is nothing else as the slightly modified u-stability property utilized in differential game theory (comp. [1]). Therefore, the extremal aiming procedure (see [1]) applied to a set with the viability property yields a feedback strategy of the first player that can maintain the state vector within this set for any control law of the second player.

The following theorem gives a necessary and sufficient condition for viability.

Theorem 2.1 *A convex compact set $K \in R^n$ has the viability property iff*

$$\max_{x\in K^0(l)} \langle l, Ax\rangle + H(l) \le 0 \tag{2}$$

for any $l \in S$, where

$$S = \{l \in R^n : |l| = 1\},$$
$$K^0(l) = \{x \in K : \langle l, x\rangle = \max_{y\in K}\langle l, y\rangle\},$$
$$H(l) = \min_{u\in P}\langle l, Bu\rangle + \max_{v\in Q}\langle l, Cv\rangle.$$

Proof Let K be a compact convex set. Using the results of [3], we conclude that K has the viability property iff for any $v \in Q$ and any $x \in \partial K$, the following is true

$$T_K(x) \cap (Ax + BP + Cv) \ne \emptyset, \tag{3}$$

where $T_K(x)$ is the tangent cone to K at x, i.e.

$$T_K(x) = \overline{\text{cone}}(K - x).$$

Let $T_K^*(x)$ be the polar cone to $T_K(x)$ so that $N_K(x) = T_K^*(x) \cap S$ be the set of unit normals to K at x. Then we can rewrite (3) as follows

$$\min_{u\in P}\max_{l\in N_K(x)} [\langle l, Ax\rangle + \langle l, Bu\rangle + \langle l, Cv\rangle] \le 0, \quad \forall v \in Q,\ \forall x \in \partial K. \tag{4}$$

Replacing the “max” and “min” operations and taking into account that (4) must be valid for any $v \in Q$ and $x \in \partial K$, we obtain

$$\max_{x\in\partial K}\max_{l\in N_K(x)} [\langle l, Ax\rangle + \min_{u\in P}\langle l, Bu\rangle + \max_{v\in Q}\langle l, Cv\rangle] \le 0.$$

It is easily seen that

$$\{(l, x) : l \in N_K(x), x \in \partial K\} = \{(l, x) : x \in K^0(l), l \in S\}.$$

Indeed, if (l, x) belongs to the left-hand side, then $|l| = 1$ and $l \in T_K^*(x)$. Therefore, $\langle l, y - x\rangle \leq 0$ for all $y \in K$, which yields that $\max_{y\in K}\langle l, y\rangle = \langle l, x\rangle$, i.e. $x \in K^0(l)$, and, therefore, (l, x) belongs to the right-hand side. Visa verse, if (l, x) belongs to the right-hand side, then $|l| = 1$ and $x \in K^0(l)$. Therefore, $\max_{y\in K}\langle l, y\rangle = \langle l, x\rangle$, which yields that $\langle l, y - x\rangle \leq 0$ for all $y \in K$, i.e. $l \in T_K^*(x)$, and, therefore, (l, x) belongs to the left-hand side.

Thus,

$$\max_{x\in\partial K}\max_{l\in N_K(x)}[\ldots] = \max_{l\in S}\max_{x\in K^0(l)}[\ldots],$$

which completes the proof of the theorem. □

Note that condition (2) involves only boundary points of K. Let us assume that some subset $L \subset \partial K$ belongs to a level set of a *first integral* of (1) with some fixed constant controls u_0 and v_0. That is,

$$L \subset \{x \in R^n : F(x, u_0, v_0) = \text{const}\},$$

where $F(\cdot, u_0, v_0) : R^n \to R$ is some function satisfying the equation

$$\left\langle \frac{\partial F}{\partial x}, Ax + Bu_0 + Cv_0 \right\rangle = 0, \quad x \in R^n.$$

Consider an arbitrary normal l to the surface L. For any $x \in K^0(l)$ we have

$$\langle l, Ax + Bu_0 + Cv_0\rangle = \gamma \left\langle \frac{\partial F}{\partial x}, Ax + Bu_0 + Cv_0 \right\rangle = 0,$$

where γ is some scalar factor. Hence, for any $x \in K^0(l)$

$$\langle l, Ax\rangle = -\langle l, Bu_0\rangle - \langle l, Cv_0\rangle.$$

Therefore,

$$\max_{x\in K^0(l)}\langle l, Ax\rangle = -\langle l, Bu_0\rangle - \langle l, Cv_0\rangle.$$

And finally, we obtain the following corollary.

Corollary 2.1 *If a subset L of ∂K belongs to a level set of a* first integral *of* (1) *with some constant controls u_0, v_0, then* (2) *is equivalent to*

$$H(l) - \langle l, Bu_0\rangle - \langle l, Cv_0\rangle \leq 0 \tag{5}$$

for any normal l to L.

3 A Game of Tracking

Consider the following game. Let y be a one-dimensional mass point controlled by a force f. Assume there is another one-dimensional but noninertial point z which can instantly change its velocity g. Thus, the dynamic equations look as follows: $\ddot{y} = f$, $\dot{z} = g$. The point y has to follow the point z so that the restrictions $y \geq z$, $|y - z| \leq a$ must hold. With the variables $x_1 = y - z$, $x_2 = \dot{y}$, $u = f$, and $v = -g$, this game has the form

$$\begin{cases} \dot{x}_1 = x_2 + v \\ \dot{x}_2 = u. \end{cases} \tag{6}$$

The state constraints are

$$0 \leq x_1 \leq a.$$

We assume that the controls be bounded:

$$|u| \leq \mu, \qquad |v| \leq 1,$$

where μ is some positive constant.

It was pointed out in [14] that there exists a viable set with the boundary formed by two trajectories of (6) corresponding to the following combinations of controls

$$\text{I.} \quad u = -\mu, \qquad v = 1,$$
$$\text{II.} \quad u = \mu, \qquad v = -1.$$

Now this observation will be proved and a full description of the class of viable sets satisfying the state constraints will be given.

The equation of the phase trajectories of (6) with constant controls u and v is

$$\frac{1}{2}(x_2 + v)^2 - ux_1 = C.$$

The stated controls I and II generate two families of parabolas

$$\text{I.} \quad \frac{1}{2}(x_2 + 1)^2 + \mu x_1 = C_1,$$
$$\text{II.} \quad \frac{1}{2}(x_2 - 1)^2 - \mu x_1 = C_2.$$

All curves of the families I and II are congruent parabolas with the apexes belonging to the lines $x_2 = -1$ and $x_2 = 1$ respectively (Fig. 1). For any values of C_1 and C_2, the intersection of the convex regions bounded by the parabolas I and II gives a convex compact set. We will consider the values C_1 and C_2 such that the resulting set is a subset of the state constraints. Note that in this example, the viability property does not depend on translations in the x_1-direction because the variable x_1 does not appear in the right-hand side of (6). So, we put $C_2 = 0$ which corresponds to the maximal admissible left location of the parabola II (Fig. 2). We will move the parabola I,

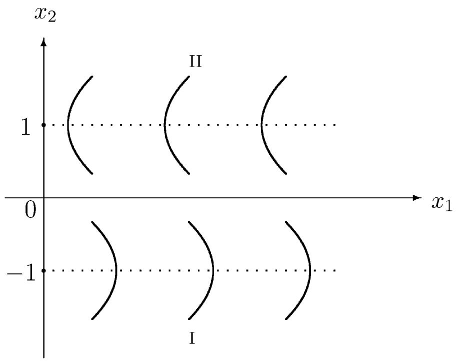


Fig. 1 Two families of parabolas

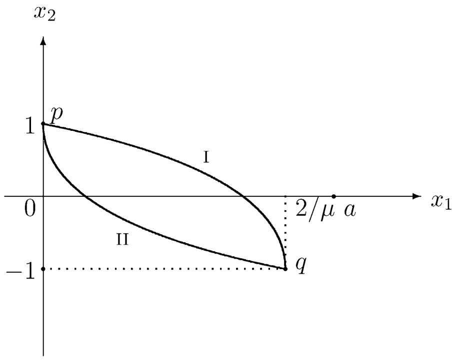


Fig. 2 The case of two parabolas

varying $C_1 \in [0, a\mu]$ (these bounds provide the embedding of the resulting set into the state constraints).

First, assume that $a\mu \geq 2$ and consider the case $C_1 = 2$. In this case, the corresponding parabola I passes through the apex of the previously chosen parabola II and vice versa (see Fig. 2). Let us check the viability property of the resulting set. For the arc pq of the parabola I, the left-hand side of (5) can be written as

$$\mu(x_2 + 1) - \mu - \mu|x_2 + 1| + \mu.$$

Taking into account $x_2 \in [-1, 1]$, we conclude that this expression is equal to zero. For the arc pq of the parabola II, the left-hand side of (5) has the form

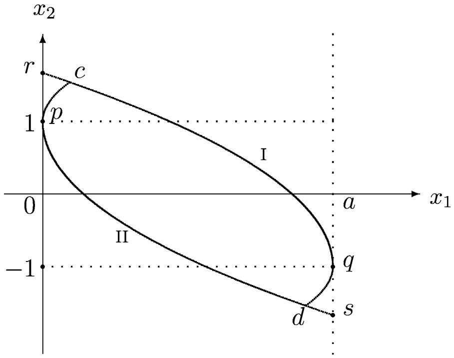
$$-\mu(x_2 - 1) - \mu - \mu|x_2 - 1| + \mu.$$

This is equal to zero, since $x_2 \in [-1, 1]$.

It remains to check the inequality (2) for normals at the points p and q. We do it for the point p; the point q can be handled similarly. Thus, we must check (2) for each normal $l \in \text{cone}\{l^I, l^{II}\}$, where $l^I = (\mu, 2)^T$ and $l^{II} = (-\mu, 0)^T$ (see Fig. 2). Any vector of this cone can be represented as follows

$$l = (1 - \lambda)l^I + \lambda l^{II} = \begin{pmatrix} \mu(1 - 2\lambda) \\ 2 - 2\lambda \end{pmatrix}, \quad \lambda \in [0, 1].$$

Fig. 3 The presence of two segments of the phase constraint



So, the left hand side of (2) becomes

$$\mu(1-2\lambda)+\mu|1-2\lambda|-2\mu|1-\lambda|,$$

and it is easy to see that this expression is equal to zero for $\lambda=0$ and $\lambda=1$. For the other values of λ, this expression is less then zero. This proves the viability property of the set shown in Fig. 2.

When we decrease C_1 from 2 to 0, the parabola I moves to the left and the resulting set is shrinking. The calculation shows that the resulting set is not viable whenever $C_1<2$ because the condition (2) is violated for normals at the points p and q. Therefore, if the condition $a\mu\geq 2$ is not fulfilled, then viable sets embedded in the state constraints can not be found in this way. Moreover, it can be easily shown that any trajectory of (6) will violate the state constraints, if the second player applies $v\equiv \operatorname{sign}(\mu x_1^0+x_2^0-a\mu/2)$, where (x_1^0,x_2^0) is the initial state. Hence, in this case, there exists no viable set. Thus, the set shown in Fig. 2 is the smallest viable set in this example.

Now we consider how one can find the maximal viable set (viability kernel). To do this, we will increase C_1 from 2 to $a\mu$ (Fig. 3). When doing this, the corresponding parabola moves to the right until it touches the right bound of the state constraints (this corresponds to $C_1=a\mu$). The resulting set bounded by the pair of parabolas is shown in Fig. 3. The calculations show that the resulting set has the viability property. Moreover, for the arcs pc and dq, the left hand side of (5) is less then zero. This means that the set $pcqd$ can be extended in the neighborhood of the arcs pc and qd while retaining the viability property. Let us continue the arcs qc and pd so that they reach the lines $x_1=0$ and $x_1=a$ respectively. Consider the set $prqs$. The left hand side of (2) takes the following values: it is zero for the arcs rq and sp, it is less then zero for the segments pr, qs, and for normals of the normal cones at the points r and s. Therefore the set $prqs$ has the viability property.

Let us prove the maximality of this set. It is sufficient to prove that, if the initial state does not belong to the set $prqs$, then there exists a constant control v of the second player such that the state vector violates the state constraints. Without loss of generality we may assume that the initial state (x_1^0,x_2^0) lies above the arc rq. The case where the initial state lies below the curve sp is symmetric. Let us put $v\equiv 1$ and suppose that there exists a control $u(\cdot)$ such that the corresponding trajectory

$(x_1(t), x_2(t))$ does not leave the state constraints. Let

$$f_I(x_1, x_2) \stackrel{def}{=} \frac{1}{2}(x_2 + 1)^2 + \mu x_1 - a\mu.$$

Note that the points which lie above the arc rq satisfy the inequality $f_I > 0$, and $f_I = 0$ for the points of rq. Calculation of the derivative of $f_I(x_1(t), x_2(t))$ gives

$$\frac{df_I}{dt} = (x_2(t) + 1)(\mu + u(t)).$$

Let $\theta = \sup\{t : x_2(\xi) \geq -1,\ \xi \in [0, t]\}$. Then we have $df_I/dt \geq 0$, $t \in [0, \theta]$. This gives

$$f_I(x_1(t), x_2(t)) \geq f_I(x_1^0, x_2^0) > 0, \quad t \in [0, \theta] \tag{7}$$

and hence $(x_1(t), x_2(t))$ lies above the arc rq for all $t \in [0, \theta]$. The last gives $x_2(\theta) > -1$, which is a contradiction with the definition of θ. Therefore, $\theta = \infty$. Moreover, the inequality (7) provides the inequality $x_2(t) \geq -1 + \varepsilon$ for some $\varepsilon > 0$ depending only on the initial state. The first equation of (6) gives $\dot{x}_1(t) \geq \varepsilon$ and therefore the state vector must leave the state constraints. This contradiction proves the assertion.

4 A Pendulum with a Movable Suspension Point

The second example deals with a pendulum with the movable suspension point. We assume that this point can move with the velocity u. The performance is disturbed by a force v caused by the wind, say. Let x_1 be the angle deflection from the vertical axis and x_2 be the velocity of the suspended load in a ground-fixed reference system. With these variables, the dynamic equations look as follows

$$\begin{cases} \dot{x}_1 = \beta(x_2 - u) \\ \dot{x}_2 = -\alpha x_1 + v. \end{cases} \tag{8}$$

Here, α and β are given positive coefficients depending on the values of the suspended mass and on the pendulum's length. The controls u and v are bounded

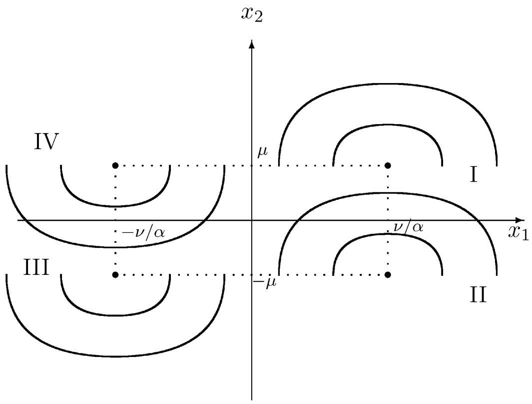
$$|u| \leq \mu, \qquad |v| \leq \nu.$$

The following state constraints are prescribed

$$|x_1| \leq a, \qquad |x_2| \leq b. \tag{9}$$

The first player dealing with the control parameter u seeks to maintain the validity of the state constraints. The second player uses the control parameter v to violate the state constraints. Without loss of generality we may let $\beta = 1$ because this can be achieved by an appropriate time scaling.

Fig. 4 Four families of ellipses



We will prove the conjecture that the boundary of the viability kernel consists of parts of the boundary of the phase constraint and of segments of trajectories corresponding to some combinations of the extremal values of u and v. There are four combinations of such values

$$\begin{aligned} &\text{I.} \quad (\mu, \nu), \\ &\text{II.} \quad (-\mu, \nu), \\ &\text{III.} \quad (-\mu, -\nu), \\ &\text{IV.} \quad (\mu, -\nu). \end{aligned} \tag{10}$$

The equation of phase trajectories of (8) corresponding to constant controls u and v is

$$\frac{1}{\alpha}(\alpha x_1 - v)^2 + (x_2 - u)^2 = C, \tag{11}$$

and, therefore, each combination from (10) produces a family of ellipses (see Fig. 4). Calculation shows that (5) is satisfied for the upper parts of ellipses of families I and II. As for families III and IV, the condition (5) is valid for the lower parts of the ellipses (see Fig. 4).

Let us construct the viability kernel using these families. The construction depends on the relations between the values of a, b, μ, α. First of all, we assume that

$$\mu \geq \nu/\sqrt{\alpha}, \qquad a \geq \nu/\alpha. \tag{12}$$

Otherwise, viability sets do not exist at all because the control $v \equiv \nu$ provides the violation of the phase constraint. In the case of the failure of the first condition of (12), the assertion follows from the representation of solutions of (8) using the fundamental matrix. If the second condition of (12) fails, the violation of the constraint $x_1 \leq a$ immediately follows from the second equation of (8).

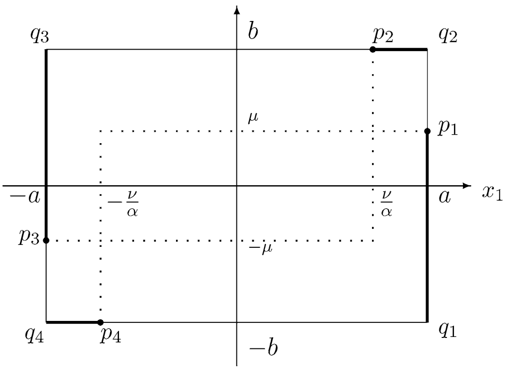


Fig. 5 Viable segments of the state constraint

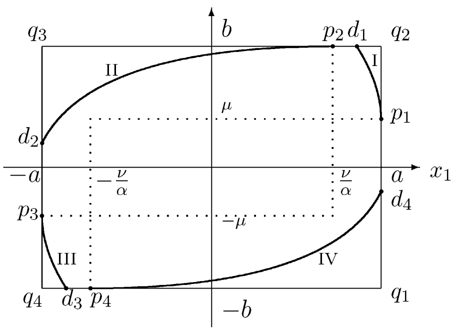


Fig. 6 Subcase 1

We begin with the case $b \geq \mu$. In this case, one can prove that the condition (2) holds for the following segments of the boundary of the state constraints

$$\begin{aligned} x_1 &= a, & -b &\leq x_2 \leq \mu, \\ x_1 &= -a, & -\mu &\leq x_2 \leq b, \\ x_2 &= b, & \nu/\alpha &\leq x_1 \leq a, \\ x_2 &= -b, & -a &\leq x_1 \leq -\nu/\alpha, \end{aligned} \tag{13}$$

(see Fig. 5). Therefore, these segments may be employed in the design of the viability kernel.

Consider the following subcases.

1. $|\sqrt{\alpha}a - b| \leq \mu - \nu/\sqrt{\alpha}$.

In this case, the boundary of the viability kernel includes parts of all segments (13). The construction is as follows (we will refer to Fig. 6). Choose the ellipse of family I passing through the point p_1 and draw it up to its intersection with the line $x_2 = b$.

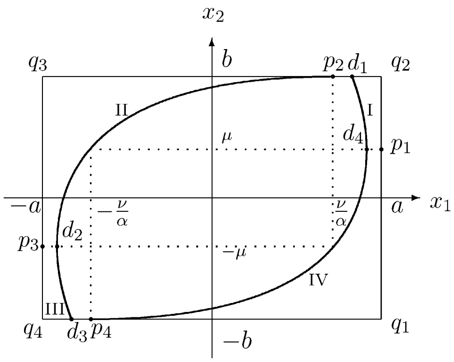


Fig. 7 Subcase 2

The first coordinate of the point d_1 of intersection is

$$\frac{\nu}{\alpha}+\sqrt{\left(a-\frac{\nu}{\alpha}\right)^2-\frac{1}{\alpha}(b-\mu)^2}.$$

Hence, d_1 lies on the admissible segment p_2, q_2. Now we choose the ellipse of family II passing through the point p_2 and draw it up to its intersection with the line $x_1 = -a$. The point d_2 of intersection lies on the admissible segment $p_3 q_3$.

We do the same with the symmetric points p_3 and p_4 using ellipses of families III and IV respectively. As a result, we obtain the symmetric curve $p_1 d_1 p_2 d_2 p_3 d_3 p_4 d_4$ which includes arcs of ellipses of all families and parts of all admissible segments (13). It follows from the choice of admissible arcs of the ellipses and admissible segments (13) that the condition (2) holds for all smooth pieces of the resulting curve (note that the conjunction at p_1, p_2, p_3 and p_4 is also smooth).

A direct calculation shows that the left-hand-side of (2) is nonpositive for all normals of the normal cones at d_1, d_2, d_3, d_4 where the resulting curve is nonsmooth. Therefore, the set obtained has the viability property.

Note that, if $\mu = \nu/\sqrt{\alpha}$ and hence $a\sqrt{\alpha} = b$, then $d_1 = p_2$, $d_2 = p_3$, $d_3 = p_4$ and $d_4 = p_1$.

2. $\sqrt{\alpha}a - b > \mu - \nu/\sqrt{\alpha}$.

In this case, the viability kernel has no points in common with the lines $x_1 = -a$ and $x_1 = a$ (see Fig. 7). The construction of the viability kernel goes as follows. We choose the ellipse of family II passing through the point p_2. In this case, it does not meet the line $x_1 = -a$, and we draw it up to the intersection with the line $x_2 = -\mu$, obtaining the intersection point d_2. Now we choose the ellipse of family III passing through the point d_2 and draw it up to the intersection with the line $x_2 = -b$. Using the relations between the parameters in this case, one can easily check that the point d_3 belongs to the admissible segment $q_4 p_4$. The symmetric curve $p_4 d_4 d_1$ is formed in the same way. Note that the conjunctions at p_2, d_2, p_4, and d_4 are smooth.

A direct calculation shows that the left-hand side of (2) is nonpositive for all normals of the normal cones at d_1 and d_3 where the conjunctions are nonsmooth. Therefore, the region bounded by the curve $p_2 d_2 d_3 p_4 d_4 d_1 p_2$ has the viability property.

Note, if $\mu = \nu/\sqrt{\alpha}$ then $d_1 = p_2$ and $d_3 = p_4$.

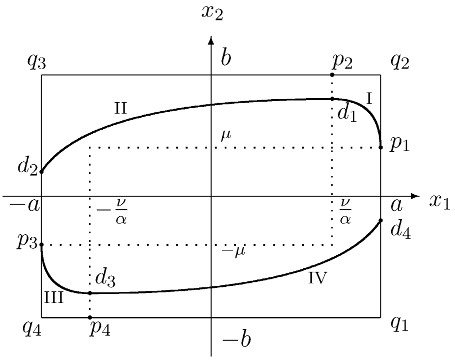


Fig. 8 Subcase 3

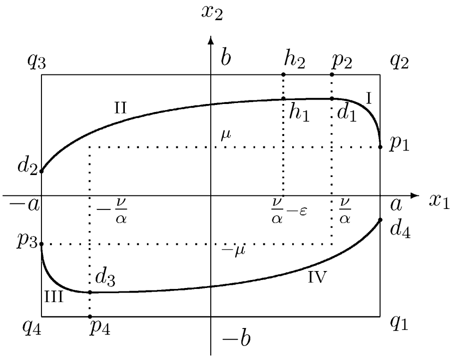


Fig. 9 An aid to the proof of subcase 3

3. $\sqrt{\alpha}a - b < \nu/\sqrt{\alpha} - \mu$.

In this case, the viability kernel has no points in common with the lines $x_2 = -b$ and $x_2 = b$ (Fig. 8). The construction goes as follows. We draw the ellipse of family I passing through the point p_1 up to the intersection with the line $x_1 = \nu/\alpha$ at d_1. Then we draw the ellipse of family II passing through d_1 up to the intersection with the line $x_1 = -a$ at d_2. One can check that d_2 belongs to the admissible segment p_3q_3. So, we obtain the curve $p_1d_1d_2$. The curve $p_3d_3d_4$ is formed using the symmetry. Note that the conjunctions at p_1, d_1, p_3, and d_3 are smooth. If $\mu = \nu/\sqrt{\alpha}$ then $d_2 = p_3$ and $d_4 = p_1$.

Now assume that $b < \mu$. In this case, the condition (2) is fulfilled for the following segments of the boundary of the state constraints

$$
\begin{aligned}
&x_1 = a, && -b \le x_2 \le b,\\
&x_1 = -a, && -b \le x_2 \le b,\\
&x_2 = b, && \nu/\alpha \le x_1 \le a,\\
&x_2 = -b, && -a \le x_1 \le -\nu/\alpha.
\end{aligned}
\tag{14}
$$

The viability kernel is formed using ellipses of families II and IV. Figures 10 and 11 show the viability kernels in the cases $\sqrt{\alpha}a - b \le \mu - \nu/\sqrt{\alpha}$ and $\sqrt{\alpha}a - b > \mu - \nu/\sqrt{\alpha}$ respectively.

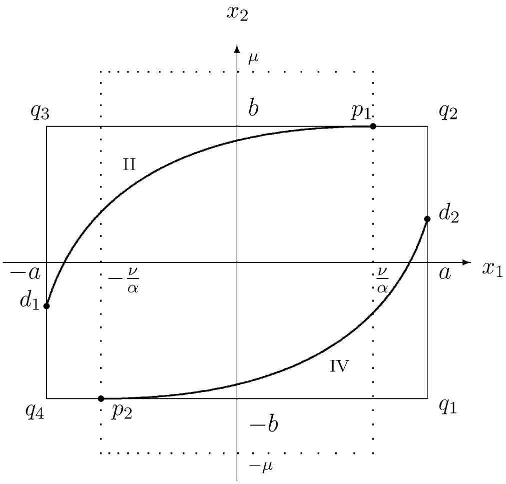


Fig. 10 The case $b < \mu$ (four segments of the phase constraint)

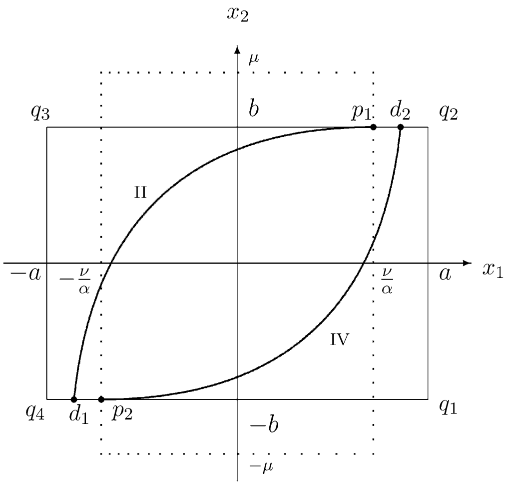


Fig. 11 The case $b < \mu$ (two segments of the phase constraint)

Let us prove the maximality of the sets we have found. We consider only the case $b \geq \mu$ because the arguments in the opposite cases are similar.

First, consider subcase 1 (Fig. 6). Assume the initial state (x_1^0, x_2^0) belongs to the region $p_1 d_1 q_2$ and lies above the arc $p_1 d_1$. Let us put $v \equiv \nu$ and suppose that there exists a control $u(\cdot)$ such that the corresponding trajectory $(x_1(t), x_2(t))$ does not leave the region specified by the state constraints. Let

$$f_I(x_1, x_2) \stackrel{def}{=} \frac{1}{\alpha}(\alpha x_1 - \nu)^2 + (x_2 - \mu)^2 - \frac{1}{\alpha}(\alpha a - \nu)^2.$$

The points of the region $p_1 d_1 q_2$ which lie above the arc $p_1 d_1$ satisfy the inequality $f_I > 0$, and $f_I = 0$ for the points of $p_1 d_1$. Calculating the derivative of

$f_I(x_1(t), x_2(t))$, we obtain

$$\frac{df_I}{dt} = 2(\alpha x_1(t) - \nu)(\mu - u(t)).$$

Let $\theta = \sup\{t : x_1(\xi) \geq \nu/\alpha,\ \xi \in [0, t]\}$. Then the function $f_I(x_1(t), x_2(t))$ is not decreasing on $[0, \theta]$. Arguing as in the first example, we can prove that $\theta = \infty$. Hence,

$$f_I(x_1(t), x_2(t)) \geq f_I(x_1^0, x_2^0) > 0 \tag{15}$$

for all $t \geq 0$. Therefore, the trajectory cannot meet the arc p_1d_1. Moreover, (15) provides the inequality $x_2(t) \geq \mu + \varepsilon$ with some $\varepsilon > 0$ depending only on the initial state. The first equation of (8) gives $\dot{x}_1(t) \geq \varepsilon$, and hence the state vector must leave the state constraints. This contradiction proves the assertion in the case where (x_1^0, x_2^0) belongs to the region $p_1d_1q_2$.

Now assume the initial state (x_1^0, x_2^0) belongs to the region $p_2d_2q_3$. The arguments are similar to the above described ones. We put $v \equiv \nu$ and assume there exists a trajectory $(x_1(t), x_2(t))$ which does not leave the state constraints. Consider the function

$$f_{II}(x_1, x_2) \stackrel{def}{=} \frac{1}{\alpha}(\alpha x_1 - \nu)^2 + (x_2 + \mu)^2 - (b + \mu)^2.$$

This function is greater than zero in the region $p_2d_2q_3$ excluding the arc p_2d_2 and equal to zero on this arc. Taking the derivative of the function $f_{II}(x_1(t), x_2(t))$, gives

$$\frac{df_{II}}{dt} = -2(\alpha x_1(t) - \nu)(\mu + u(t)) \geq 0$$

for any $t \in [0, \theta]$, where $\theta = \sup\{t : x_1(\xi) \leq \nu/\alpha,\ \xi \in [0, t]\}$. Arguing as before, we obtain $\theta = \infty$ and hence

$$f_{II}(x_1(t), x_2(t)) \geq f_{II}(x_1^0, x_2^0) > 0$$

for all $t \geq 0$. Therefore, the state vector cannot meet the arc p_2d_2. Moreover, this inequality provides the inequality $x_1(t) \leq \nu/\alpha - \varepsilon$ with some $\varepsilon > 0$ depending only on the initial state. The second equation of (8) gives $\dot{x}_2(t) \geq \alpha\varepsilon$, and therefore the state vector must leave the state constraints.

The cases where the initial state lies in the region $p_3p_4q_4$ and in the region $p_4d_4q_1$ can be considered analogously because of the symmetry.

Now we must consider cases 2, 3 (Figs. 7 and 8). The arguments in both cases are similar, and for brevity we consider only case 3 (Fig. 8).

Assume the initial state lies in the region $p_1d_1d_2q_3p_2q_2$ and above the curve $p_1d_1d_2$ (the case where the initial state belongs to the region $p_3d_3d_4q_1p_4q_4$ is symmetric).

Let

$$f_I(x_1, x_2) \stackrel{def}{=} \frac{1}{\alpha}(\alpha x_1 - \nu)^2 + (x_2 - \mu)^2 - \frac{1}{\alpha}(\alpha a - \nu)^2,$$

$$f_{II}(x_1, x_2) \stackrel{def}{=} \frac{1}{\alpha}(\alpha x_1 - \nu)^2 + (x_2 + \mu)^2 - \left(2\mu + \frac{1}{\sqrt{\alpha}}(\alpha a - \nu)\right)^2.$$

Note that, for any point of the region $p_1d_1p_2q_2$ excluding the arc p_1d_1, we have $f_I(x_1,x_2)>0$. For any point of the arc p_1d_1, we have $f_I(x_1,x_2)=0$. The same is valid for $f_{II}(\cdot)$ with respect to the region $d_1d_2q_3p_2$ and the arc d_1d_2. Let us put $v\equiv\nu$ and let $u(\cdot)$ be a control of the first player. Denote by $(x_1(t),x_2(t))$ the corresponding trajectory of (8). First, assume that (x_1^0,x_2^0) belongs to the region $p_1d_1p_2q_2$ and lies above the arc p_1d_1. Suppose that the trajectory does not leave the state constraints. Denote

$$\theta=\sup\{t: x_1(\xi)\ge\nu/\alpha \ \&\ x_2(\xi)\ge\mu,\ \xi\in[0,t]\}.$$

Taking the derivative of the function $f_I(x_1(t),x_2(t))$, we obtain

$$\frac{df_I}{dt}=2(\alpha x_1(t)-\nu)(\mu-u(t))\ge 0,\quad t\in[0,\theta]. \tag{16}$$

From the first equation of (8), we have

$$\dot{x}_1(t)=x_2(t)-u(t)\ge 0,\quad t\in[0,\theta]. \tag{17}$$

Since $x_2(0)=x_2^0>\mu$, there exists $\delta>0$ such that

$$\dot{x}_1(t)>0,\quad t\in[0,\delta]. \tag{18}$$

From (16), (17) and (18), we obtain the inequalities

$$\begin{gathered} f_I(x_1(t),x_2(t))\ge f_I(x_1^0,x_2^0)>0,\\ x_1(t)>\nu/\alpha \end{gathered} \tag{19}$$

for all $t\in[0,\theta]$. By the definition of θ, this means that θ cannot be finite. So, $\theta=\infty$. Hence, the state vector never comes over the segment d_1p_2 and never meets the arc p_1d_1. Moreover, (19) implies the existence of $\varepsilon>0$ depending only on the initial state such that $x_2(t)\ge\mu+\varepsilon$ for all $t\ge 0$. Then from (17), we have $\dot{x}_1(t)\ge\varepsilon$ for all $t\ge 0$. Therefore, the trajectory must violate the state constraints.

Now assume the initial state belongs to the region $d_1d_2q_3p_2$, and the trajectory $(x_1(t),x_2(t))$ does not leave the state constraints. Since d_1 lies above the line $x_2=\mu$, we can choose a positive ε such that the segment h_1h_2 (Fig. 9) lies above the line $x_2=\mu$. Let $\theta=\sup\{t: x_1(\xi)\le\nu/\alpha-\varepsilon,\ \xi\in[0,t]\}$. From the second equation of (8), we have $\dot{x}_2(t)\ge\alpha\varepsilon$ for all $t\in[0,\theta]$. If $\theta=\infty$, then the last inequality implies a violation of the state constraints. If $\theta<\infty$, we consider the function $f_{II}(x_1(t), x_2(t))$. We have

$$\frac{df_{II}}{dt}=-2(\alpha x_1(t)-\nu)(\mu+u(t))\ge 0$$

for any $t\in[0,\theta]$, from which

$$f_{II}(x_1(\theta),x_2(\theta))\ge f_{II}(x_1^0,x_2^0)>0.$$

Accounting for the definition of θ and the assumption that the state vector does not leave the state constraints, we conclude that $(x_1(\theta),x_2(\theta))$ belongs to the segment h_1h_2. Now, arguing in the same way as in the case where the initial state lies in the region $p_1d_1p_2q_2$, we find that the trajectory comes over the line $x_1=a$.

5 Concluding Remarks

The main feature of the examples considered is that the viability kernels possess a relatively simple structure. Their boundary consists of phase trajectories of the controlled system with constant controls and of segments of the state constraints. The authors conjecture that the structure of viability kernels for linear differential games with polyhedral sets P and Q (see (1)) is generally of this form. That is, the boundary consists of some pieces of the boundary of state constraints and of some pieces of level sets of *first integrals* of (1) corresponding to some combinations of vertices u_i and v_j of P and Q. Numerical experiments with programs for solving differential games are in agreement with this conjecture. This provides hope that the approach outlined in the paper can be extended to multidimensional case.

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