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STABLE NUMERICAL SCHEMES FOR SOLVING HAMILTON–JACOBI–BELLMAN–ISAACS EQUATIONS[*]

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Abstract. Stable approximation schemes for solving Hamilton–Jacobi–Bellman–Isaacs equations in R^n are proposed. The efficiency of numerical procedures based on such approximations is demonstrated by solving several differential games in two, three, and four dimensions.

Key words. Hamilton–Jacobi and Bellman–Isaacs equations, differential game, finite difference scheme

AMS subject classifications. 49N90, 49L20, 49L25, 34K28

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1. Introduction.

Numerical methods for solving general Hamilton–Jacobi equations have been intensively developed within the last two or three decades (see, e.g., [9], [23], [17], [18], [1]). The present paper deals with Bellman–Isaacs equations arising in the theory of differential games. This is a special case of Hamilton–Jacobi equations where Hamiltonians are represented as a max-min of the scalar product of the impulse variable and the right-hand side of the conflict-controlled system. Such Hamiltonians are generally nonlinear in the state variable and neither convex nor concave in the impulse variable. There are different approaches to treat Bellman–Isaacs equations numerically. Papers [28], [29] propose a finite-difference method that requires the computation of generalized gradients of local convex hulls of approximate solutions. In [5], an approach based on the approximation of viability kernels [2] is presented. A discrete version of the dynamic programming method is developed in [3], [6]. The thesis [16] describes application of the level set method [22] to the computation of viscosity solutions of Bellman–Isaacs equations.

Our research relies on the works [23], [24], and [4], where finite-difference schemes based on monotonicity properties of the difference-scheme operator are proposed. It should be noted that results of [23], [24] cover differential games with fixed termination time. In [4], the approach of [23] and [24] is extended to differential games with nonfixed termination time. The monotonicity of the difference-scheme operator is achieved by a special change of variables of the controlled system, which transfers the Hamiltonian into a monotone increasing in impulse variables function. The disadvantage of such an approach is that the change of variables increases the Lipschitz constant of the Hamiltonian, which usually restricts the applicability of such a technique to the case of small time intervals.

In the paper [14], a sort of "upwind" finite-difference scheme for solving Bellman–Isaacs equations was proposed. Apparently, due to the absence of a rigorous proof of the convergence, this work remains unvalued. Nevertheless, the idea of the difference-scheme operator proposed there seems to be very fruitful. One of the objectives of our paper is to prove the convergence of the upwind difference scheme from [14].

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The idea of the proof is based on the results of [23] and [4]. Moreover, we consider another numerical method based on a max-min representation of solutions [24]. From the point of view of game theory such an operator implements the straightforward computation of the programming max-min [12], [13], [26].

Both the upwind difference scheme and the max-min grid operator are implemented as parallelized computer programs for multiprocessor computers. Several nontrivial test examples, including the two-dimensional homicidal chauffeur game, three-dimensional two cars and Dubins' car games, and four-dimensional isotropic rockets game are presented.

2. Differential games and Hamilton–Jacobi equations.

Consider the differential game

$$\dot{x} = f(t, x, u, v), \tag{2.1}$$

where $x \in R^n$ is the state vector, and u and v are control parameters of the first and second players restricted as

$$u \in P \subset R^p, \quad v \in Q \subset R^q. \tag{2.2}$$

Here, P and Q are given compacts. The game starts at $t_0 \in [0, t_f]$ and finishes at t_f. The payoff functional defined on the trajectories of system (2.1) is

$$J(x(\cdot)) = \max \ \text{(or)} \min_{t \in [t_0, t_f]} \sigma(t, x(t)), \tag{2.3}$$

where $\sigma : \ [0, t_f] \times R^n \to R$ is a given function. The objective of the control u (the first player) is to minimize functional (2.3); the objective of the control v (the second player) is the opposite. The players utilize feedback strategies that are arbitrary functions depending on time and state, which requires a special definition of trajectories of (2.1).

Assume that the following conditions are fulfilled:

(f1) The function f is uniformly continuous on $[0, t_f] \times R^n \times P \times Q$.

(f2) f is bounded, i.e.,

$$|f(t, x, u, v)| \leq M$$

for all $(t, x, u, v) \in [0, t_f] \times R^n \times P \times Q$.

(f3) f is Lipschitz-continuous in t, x, i.e.,

$$|f(t_1, x_1, u, v) - f(t_2, x_2, u, v)| \leq N(|t_1 - t_2| + |x_1 - x_2|)$$

for all $(t_i, x_i, u, v) \in [0, t_f] \times R^n \times P \times Q, \ i = 1, 2$.

(f4) σ is bounded and Lipschitz-continuous in t, x, i.e.,

$$|\sigma(t, x)| \leq C_0$$

and

$$|\sigma(t_1, x_1) - \sigma(t_2, x_2)| \leq K(|t_1 - t_2| + |x_1 - x_2|)$$

for all (t, x), $(t_i, x_i) \in [0, t_f] \times R^n$, $i = 1, 2$.

(f5) f satisfies the saddle point condition, i.e.,

$$\min_{u \in P} \max_{v \in Q} \langle \ell, f(t, x, u, v) \rangle = \max_{v \in Q} \min_{u \in P} \langle \ell, f(t, x, u, v) \rangle$$

for any $\ell \in R^n$, $(t,x) \in [0,t_f] \times R^n$.

(f5a) $f(t,x,u,v) = f_1(t,x,u) + f_2(t,x,v)$, where f_1 (resp., f_2) is linear in u (resp., in v) for all fixed t, x. Obviously, (f5a) implies (f5).

Remark 2.1. Assumption (f2) is not limitative because the computations are restricted to a bounded domain.

The game (2.1)–(2.3) is formalized as in [12], [13], [26]. The players use feedback strategies which are arbitrary functions:

$$\mathcal{P} : [0,t_f] \times R^n \to P, \quad \mathcal{Q} : [0,t_f] \times R^n \to Q.$$

For any initial position $(t_0,x_0) \in [0,t_f] \times R^n$ and any strategies $\mathcal{P}$ and $\mathcal{Q}$, the two functional sets $X_1(t_0,x_0,\mathcal{P})$ and $X_2(t_0,x_0,\mathcal{Q})$ are defined. These sets consist of limits of step-by-step solutions of (2.1) generated by the strategies $\mathcal{P}$ and $\mathcal{Q}$, respectively (see [12], [13], [26]).

It is well known due to [12], [13], and [26] that, under assumptions (f1)–(f5), the differential game (2.1)–(2.3) has a value function $V : (t,x) \to V(t,x)$ defined by the relation

$$V(t,x) = \min_{\mathcal{P}} \max_{x(\cdot) \in X_1(t,x,\mathcal{P})} J(x(\cdot)) = \max_{\mathcal{Q}} \min_{x(\cdot) \in X_2(t,x,\mathcal{Q})} J(x(\cdot)).$$

The value function is bounded and Lipschitz-continuous in t, x (see [26]), i.e.,

$$|V(t,x)| \le C$$

and

$$|V(t_1,x_1) - V(t_2,x_2)| \le L(|t_1 - t_2| + |x_1 - x_2|)$$

for any $(t,x), (t_1,x_1), (t_2,x_2) \in [0,t_f] \times R^n$.

The value function is a viscosity solution of the following Isaacs–Bellman/Hamilton–Jacobi equation

$$V_t + H(t,x,V_x) = 0 \tag{2.4}$$

with the Hamiltonian

$$H(t,x,p) = \max_{v \in Q} \min_{u \in P} \langle p, f(t,x,u,v) \rangle$$

and the conditions

$$V(t,x) \le \sigma(t,x), \quad V(t_f,x) = \sigma(t_f,x).$$

Viscosity solutions in the case of functional (2.3) are defined (see [4]) through a pair of typical differential inequalities (cf. [8], [7]):

$$\frac{\partial \varphi}{\partial \tau}(s_0,y_0) + H(s_0,y_0,D\varphi(s_0,y_0)) \ge 0 \tag{2.5}$$

for any point $(s_0,y_0) \in (0,t_f) \times R^n$ and function $\varphi \in C^1((0,t_f) \times R^n)$ such that $V - \varphi$ attains a local maximum at (s_0,y_0); and

$$\frac{\partial \varphi}{\partial \tau}(s_0,y_0) + H(s_0,y_0,D\varphi(s_0,y_0)) \le 0 \tag{2.6}$$

for any point $(s_0, y_0) \in (0, t_f) \times R^n$, satisfying $V(s_0, y_0) < \sigma(s_0, y_0)$, and function $\varphi \in C^1((0, t_f) \times R^n)$ such that $V - \varphi$ attains a local minimum at (s_0, y_0). Inequalities (2.5) and (2.6) can be easily derived from the results obtained in [27] (see also [25]) by taking into account that the condition for the upper, respectively, lower directional derivative from [27] is locally equivalent to (2.5), respectively, (2.6).

In the next section, approximation schemes for solving Hamilton–Jacobi equation (2.4) will be discussed and corresponding convergence results will be given.

3. Approximation schemes and convergence results. In the paper [4], a finite-difference scheme for finding viscosity solutions to (2.4) with the payoff functional of the form (2.3) was proposed. The scheme is based on a solution operator which can be considered as a modification of the abstract solution operator introduced in [23]. Similarly to [23], the convergence result is based on the monotonicity property of the solution operator. This monotonicity property can be achieved if the Hamiltonian is monotonic in the impulse variable. The required monotonicity of the Hamiltonian can be obtained by a certain transformation of the variables t and x. However, the change of variables disturbs the Lipschitz constant of the Hamiltonian in the impulse variables. The latter forces another change of variables, etc., which restricts the applicability of such a technique very strongly.

In this paper, two finite-difference schemes which are free from the above drawback are proposed. The first scheme is based on a difference-scheme operator introduced in [14]. The main idea here is to use either the right or the left divided differences for the approximation of the spatial derivatives V_{x_i}, $i = 1, \ldots, n$, depending on the sign of f_i, where f_i is the ith component of the right-hand side of the controlled system. The second scheme, which can be referred as the direct one, is related to the computation of the alternate programming max min (see [12]) and uses linear or multilinear interpolations of grid functions.

Let $\tau, \Delta_{x_1}, \ldots, \Delta_{x_n}$ be time and space discretization step sizes. Denote $t_m = m\tau$ and introduce the following notation:

$$V^m(x_{i_1}, \ldots, x_{i_n}) = V(t_m, i_1\Delta_{x_1}, \ldots, i_n\Delta_{x_n}), \quad m = 0, \ldots, \mathcal{M}, \text{ where } \mathcal{M}\tau = t_f.$$

Consider the difference scheme

$$V^{m-1}(x_{i_1}, \ldots, x_{i_n}) = V^m(x_{i_1}, \ldots, x_{i_n}) + \tau H(t_m, x_{i_1}, x_{i_2}, \ldots, x_{i_n}, V^m_{x_1}, \ldots, V^m_{x_n}),$$

$$V^{\mathcal{M}}(x_{i_1}, \ldots, x_{i_n}) = \sigma(t_f, x_{i_1}, \ldots, x_{i_n}).$$

Here, the symbols $V^m_{x_1}, \ldots, V^m_{x_n}$ denote finite-difference approximations (left, right, central, etc.) of the corresponding partial derivatives.

The scheme can be considered as the successive application of an operator Π to the grid functions:

$$V^{m-1} = \Pi(V^m; t_m, \tau, \Delta_{x_1}, \ldots, \Delta_{x_n}), \quad m = \mathcal{M}, \mathcal{M} - 1, \ldots, 1.$$

Note that such an operator can be naturally extended to continuum functions.

Definition 3.1. *The operator Π is monotone if the following implication holds:*

$$V \le W \Rightarrow \Pi(V; t, \tau, \Delta_{x_1}, \ldots, \Delta_{x_n}) \le \Pi(W; t, \tau, \Delta_{x_1}, \ldots, \Delta_{x_n}),$$

where the pointwise order is assumed.

Definition 3.2. *The operator* Π *has the generator property if the following estimate holds:*

$$\left|\frac{\Pi(\phi; t, \tau, a_1\tau, \dots, a_n\tau)(x) - \phi(x)}{\tau} - H(t, x, D\phi(x))\right| \le C\bigl(1 + \|D\phi\| + \|D^2\phi\|\bigr)\tau$$

for every $\phi \in C_b^2(R^n)$*,* $x \in R^n$*, and fixed* $a_1, \dots, a_n > 0$*.*

Here $C_b^2(R^n)$ is the space of twice continuously differentiable functions defined on R^n and bounded together with their two derivatives; $\|\cdot\|$ denotes the pointwise maximum norm, and $D\phi$ and $D^2\phi$ denote the gradient and the Hessian matrix of ϕ.

Theorem 3.3 (convergence [4]). *Assume that the operator* $\Pi(\cdot; t, \tau, a_1\tau, \dots, a_n\tau)$ *is monotone for any* $\tau > 0$ *and satisfies the generator property; then the grid function obtained by the procedure*

$$V^{m-1} = \max \quad \textit{(or)} \quad \min\{\Pi(V^m; t_m, \tau, a_1\tau, \dots, a_n\tau), \quad \sigma\}, \qquad V^{\mathcal{M}} = \sigma,$$

converges pointwise to the value function of the differential game (2.1)–(2.3) *as* $\tau \to 0$*, and the convergence rate is* $\sqrt{\tau}$*.*

Remark 3.4. Theorem 3.3 refers only to the monotonicity and generator properties of the operator Π. Really, some secondary properties have to hold to claim the convergence (see [23] and [4]). We omit here the discussion of them because they obviously hold for the operators Π considered below.

The next lemma gives conditions which provide the monotonicity of Π. Denote

$$\begin{aligned}
p_1^R &= [V^m(x_{i_1+1}, x_{i_2}, \dots, x_{i_n}) - V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n})]/\Delta_{x_1},\\
p_2^R &= [V^m(x_{i_1}, x_{i_2+1}, \dots, x_{i_n}) - V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n})]/\Delta_{x_2},\\
&\dots\\
p_n^R &= [V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n+1}) - V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n})]/\Delta_{x_n},\\
p_1^L &= [V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n}) - V^m(x_{i_1-1}, x_{i_2}, \dots, x_{i_n})]/\Delta_{x_1},\\
p_2^L &= [V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n}) - V^m(x_{i_1}, x_{i_2-1}, \dots, x_{i_n})]/\Delta_{x_2},\\
&\dots\\
p_n^L &= [V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n}) - V^m(x_{i_1}, x_{i_2}, \dots, x_{i_n-1})]/\Delta_{x_n}.
\end{aligned}$$

Lemma 3.5. *If* $H(t, x_1, \dots, x_n, p_1, \dots, p_n)$ *decreases (resp., increases) in* $p_1, \dots, p_n$*, then the approximation* $V_{x_1}^m = p_1^R, \dots,$ $V_{x_n}^m = p_n^R$ *(resp.,* $V_{x_1}^m = p_1^L, \dots,$ $V_{x_n}^m = p_n^L$*) yields the monotonicity of the operator* $\Pi(\cdot; t, \tau, a_1\tau, \dots, a_n\tau)$*, provided that* $a_i \ge \Lambda\sqrt{\tau}$*, where* Λ *is the Lipschitz constant of* H *in* $p_1, \dots, p_n$*.*

The monotone decrease of H can be achieved through the following transformation of variables in the controlled system:

$$\hat{x} = x + C(t_f - t),$$

where $C \ge \Lambda$ is a constant. Note that such a variable transformation changes the Lipschitz constant of the Hamiltonian in the impulse variables. Therefore, the condition $C \ge \Lambda'$, where Λ' is the new Lipschitz constant, can be violated, so that the desired monotonicity cannot be guaranteed. It should be stressed that such a situation is typical, and, therefore, this technique is applicable for small values of t_f only, which makes treatment of realistic tasks impossible. We omit the proof because our method does not rely on this lemma.

3.1. Upwind solution operator. Let us investigate the solution operator proposed in [14]. We will prove that this operator is originally monotone and does not require any transformation of variables. Unfortunately, the convergence arguments given in [14] are very sketchy and not strong. They are solely based on topological considerations and do not take into account the nature of viscosity solutions. Nevertheless, the idea of the operator proposed is brilliant. Denote

$$a^+ = \max(a, 0), \quad a^- = \min(a, 0).$$

In [14], the following approximations of the spatial derivatives are suggested:

$$V^m_{x_i} \cdot f_i = p_i^R \cdot f_i^+ + p_i^L \cdot f_i^-,$$

where $f_i = f_i(t_m, x_{i_1}, \dots, x_{i_n}, u, v)$, $i = \overline{1,n}$, are the right-hand sides of the controlled system computed at $(t_m, x_{i_1}, \dots, x_{i_n}, u, v)$, and p_i^R and p_i^L the right and the left divided differences, respectively. Here and in the following, the arguments $t_m, x_{i_1}, \dots, x_{i_n}, u$, and v of the functions f_i are omitted for brevity. Finally, the operator is given by

$$\Pi(V^m; t_m, \tau, \Delta_{x_1}, \dots, \Delta_{x_n})(x_{i_1}, \dots, x_{i_n}) = V^m(x_{i_1}, \dots, x_{i_n}) + \tau \max_{v \in Q} \min_{u \in P} \sum_{i=1}^{n} \big(p_i^R \cdot f_i^+ + p_i^L \cdot f_i^-\big). \tag{3.1}$$

LEMMA 3.6 (monotonicity). *Let M be the bound of the right-hand side of the controlled system. If $a_i \ge M\sqrt{n}$, $i = \overline{1,n}$, then the operator $\Pi(\cdot; t, \tau, a_1\tau, \dots, a_n\tau)$ given by* (3.1) *is monotone.*

Proof. Suppose $V \le W$. Let us show that

$$\Pi(V; t, \tau, a_1\tau, \dots, a_n\tau) \le \Pi(W; t, \tau, a_1\tau, \dots, a_n\tau).$$

Denote $\mathfrak{h}_1 = (a_1, 0, \dots, 0)$, $\mathfrak{h}_2 = (0, a_2, \dots, 0), \dots, \mathfrak{h}_n = (0, 0, \dots, a_n)$. We have

$$\begin{aligned} &\Pi(V; t, \tau, a_1\tau, \dots, a_n\tau)(x) - \Pi(W; t, \tau, a_1\tau, \dots, a_n\tau)(x) = V(x) - W(x) \\ &+ \tau\Big[\max_{v \in Q} \min_{u \in P} \sum_{i=1}^{n} \Big(\frac{V(x + \mathfrak{h}_i\tau) - V(x)}{a_i\tau} \cdot f_i^+ + \frac{V(x) - V(x - \mathfrak{h}_i\tau)}{a_i\tau} \cdot f_i^-\Big) \\ &- \max_{v \in Q} \min_{u \in P} \sum_{i=1}^{n} \Big(\frac{W(x + \mathfrak{h}_i\tau) - W(x)}{a_i\tau} \cdot f_i^+ + \frac{W(x) - W(x - \mathfrak{h}_i\tau)}{a_i\tau} \cdot f_i^-\Big)\Big]. \end{aligned}$$

By rearranging terms and using the obvious relations $f_i^+ - f_i^- = |f_i|$ and $\max_v \min_u g_1(u,v) - \max_v \min_u g_2(u,v) \le \max_v \max_u [g_1(u,v) - g_2(u,v)]$, one obtains

$$\begin{aligned} &\Pi\,(V; t, \tau, a_1\tau, \dots, a_n\tau)(x) - \Pi(W; t, \tau, a_1\tau, \dots, a_n\tau)(x) \le W(x) - V(x) \\ &+ \tau \max_{v \in Q} \max_{u \in P} \sum_{i=1}^{n} \Big[\Big(\frac{V(x + \mathfrak{h}_i\tau) - W(x + \mathfrak{h}_i\tau)}{a_i\tau} - \frac{V(x) - W(x)}{a_i\tau}\Big) f_i^+ \\ &+ \Big(\frac{V(x - \mathfrak{h}_i\tau) - W(x - \mathfrak{h}_i\tau)}{a_i\tau} - \frac{V(x) - W(x)}{a_i\tau}\Big)(-f_i^-)\Big] \\ &\le V(x) - W(x) - \tau \sum_{i=1}^{n} \frac{|f_i|}{a_i\tau}(V(x) - W(x)) = \Big(1 - \sum_{i=1}^{n} \frac{|f_i|}{a_i}\Big)(V(x) - W(x)). \end{aligned}$$

With $\sum_{i=1}^{n} |f_i| \le \sqrt{n \sum_{i=1}^{n} f_i^2}$ one obtains $1 - \sum_{i=1}^{n} \frac{|f_i|}{a_i} \ge 0$, which finally implies the required inequality. □

LEMMA 3.7 (generator property). *The generator property holds.*

Proof. Let $\phi \in C_b^2(R^n)$ and $(x_1, \dots, x_n) \in R^n$. Denote $\mathfrak{d}_1 = (\Delta_{x_1}, 0, \dots, 0), \mathfrak{d}_2 = (0, \Delta_{x_2}, \dots, 0), \dots, \mathfrak{d}_n = (0, 0, \dots, \Delta_{x_n})$. We have

$$\begin{aligned}
&\Pi(\phi; t, \tau, \Delta_{x_1}, \dots, \Delta_{x_n})(x) = \phi(x) \\
&\quad + \tau \max_{v \in Q} \min_{u \in P} \sum_{i=1}^{n} \Big(\frac{\phi(x + \mathfrak{d}_i) - \phi(x)}{\Delta_{x_i}} \cdot f_i^+ + \frac{\phi(x) - \phi(x - \mathfrak{d}_i)}{\Delta_{x_i}} \cdot f_i^- \Big).
\end{aligned}$$

Estimate

$$\begin{aligned}
&\Big| \frac{\Pi(\phi; t, \tau, \Delta_{x_1}, \dots, \Delta_{x_n})(x) - \phi(x)}{\tau} - \max_{v \in Q} \min_{u \in P} \langle D\phi(x), f \rangle \Big| \\
&= \Big| \max_{v \in Q} \min_{u \in P} \sum_{i=1}^{n} \Big(\frac{\phi(x + \mathfrak{d}_i) - \phi(x)}{\Delta_{x_i}} f_i^+ + \frac{\phi(x) - \phi(x - \mathfrak{d}_i)}{\Delta_{x_i}} f_i^- \Big) \\
&- \max_{v \in Q} \min_{u \in P} \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} (f_i^+ + f_i^-) \Big| \le \max_{u \in P} \max_{v \in Q} \sum_{i=1}^{n} \Big| \Big(\frac{\phi(x + \mathfrak{d}_i) - \phi(x)}{\Delta_{x_i}} - \frac{\partial \phi}{\partial x_i}(x) \Big) \cdot f_i^+ \\
&+ \Big(\frac{\phi(x) - \phi(x - \mathfrak{d}_i)}{\Delta_{x_i}} - \frac{\partial \phi}{\partial x_i}(x) \Big) \cdot f_i^- \Big| \le M \, \| D^2 \phi \| \sum_{i=1}^{n} \Delta_{x_i}.
\end{aligned}$$

Choosing $\Delta_{x_i} = a_i \tau$ yields

$$\Big| \frac{\Pi(\phi; t, \tau, \Delta_{x_1}, \dots, \Delta_{x_n})(x) - \phi(x)}{\tau} - H(t, x, D\phi(x)) \Big| \le M \, C \, \| D^2 \phi \| \, \tau,$$

where $C = \sum_{i=1}^{n} a_i$. □

Thus, the operator considered satisfies the conditions of Theorems 3.3.

3.2. Direct solution operator. We describe now a solution operator related to the successive direct computation of the programming max-min (see [13], [24]). Such an operator was, for example, considered and effectively implemented in [6] in the case of the absence of the second player. The last restriction was caused by the complexity of the computation of the operation $\max_{v \in Q} \min_{u \in P}$ applied to a nonlinear function. We will show that if (f5a) holds, the computation can be reduced to $\max_{v \in \text{ext}\, Q} \min_{u \in \text{ext}\, P}$, where "ext" returns the set of the extremal points, i.e., vertices, if P and Q are polyhedrons. Moreover, the convergence holds for any relation between the spatial and time step lengths.

First of all, an interpolation technique is required to extend grid functions to continuum ones. The multilinear interpolation of grid functions will be used.

Let $k \in \overline{1, 2^n}$ be an integer, and $(j_1^k, \dots, j_n^k)$ is the binary representation of k so that j_i^k is either 0 or 1. Thus, each multi-index $(j_1^k, \dots, j_n^k)$ represents a vertex of the unit cube in R^n, and k numbers them. Introduce the following functions:

$$\omega_k(x_1, \dots, x_n) = \prod_{i=1}^{n} (1 - x_i)^{1 - j_i^k} x_i^{j_i^k}, \quad k = 1, \dots, 2^n. \tag{3.2}$$

Note that the member with the number i in the product (3.2) is either $1 - x_i$ or x_i depending on the value of j_i^k. Assume that every axis x_i, $i = \overline{1, n}$, is divided with the spatial step Δ_{x_i}. Consider a point $x = (x_1, \dots, x_n) \in R^n$. Denote by $\underline{x}_i$ the lower and by $\overline{x}_i = \underline{x}_i + \Delta_{x_i}$ the upper grid points of the axis x_i such that $\underline{x}_i \le x_i \le \overline{x}_i$. Let V_k, $k = 1, \dots, 2^n$, be the values of a grid function in the vertices of the n-brick $\prod_{i=1}^n [\underline{x}_i, \overline{x}_i]$ (the vertices are ordered in the same way as the vertices of the unit n-cube above). The multilinear interpolation of V at $(x_1, \dots, x_n)$ is

$$\mathcal{L}_\Delta[V](x) = \sum_{k=1}^{2^n} V_k \cdot \omega_k\Big(\frac{x_1 - \underline{x}_1}{\Delta_{x_i}}, \dots, \frac{x_n - \underline{x}_n}{\Delta_{x_n}}\Big).$$

Thus, an interpolation operator $\mathcal{L}_\Delta$ is defined.

The "direct" grid operator is defined as follows:

$$\begin{aligned} \Pi(V^m; t_m, \tau, \Delta_{x_1}, \dots, \Delta_{x_n})(x_{i_1}, \dots, x_{i_n}) \\ = \max_{v\in Q} \min_{u \in P} \mathcal{L}_\Delta[V^m](x_{i_1} + \tau f_1, \dots, x_{i_n} + \tau f_n). \end{aligned} \tag{3.3}$$

LEMMA 3.8. *The operator* $\Pi(\cdot; t, \tau, a_1\tau, \dots, a_n\tau)$ *given by* (3.3) *is monotone for any parameter set and possesses the generator property.*

Proof. The monotonicity of Π follows evidently from its definition. The generator property can be proved similarly to Lemma 3.7 utilizing the observation that

$$\|\mathcal{L}_\Delta[\tilde{\phi}] - \phi\| \le C \max\{\Delta_{x_1}, \dots, \Delta_{x_n}\}^2 \, \|D^2\phi\|$$

for any $\phi \in C_b^2(R^n)$. Here, $\tilde{\phi}$ is the restriction of ϕ to the grid, $\|\cdot\|$ is the pointwise maximum norm, $D^2\phi$ is the Hessian matrix of ϕ, and C is an independent constant. ◻

Thus, the operator considered satisfies the conditions of Theorem 3.3.

In the three-dimensional case, a linear tetrahedron-based interpolation of grid functions can also be used. We divide every brick of the rectangular grid partition of R^3 into tetrahedrons $\mathcal{T}_k$, $k = \overline{1, 6}$, that have no common interior points (see Figure 3.1). For $(x_1, x_2, x_3) \in \mathcal{T}_k$, let $\mathcal{L}_\Delta[V](x)$ be the linear interpolation of V spanned on the vertices of $\mathcal{T}_k$. Thus, the interpolation operator $\mathcal{L}_\Delta$ uniquely extends a grid function to a continuum piecewise linear function. The "direct" grid operator is defined as before by (3.3).

In two dimensions, the linear approximation can be achieved by dividing each grid rectangle into two triangles.

Remark 3.9. If condition (f5a) is fulfilled, then the operation $\max_{v\in Q}\min_{u\in P}$ in the definitions of the operators (3.1) and (3.3) can be replaced by $\max_{v\in \operatorname{ext} Q}\min_{u\in \operatorname{ext} P}$, where "ext" returns the set of the extremal points. In particular, "max-min" can be computed over the set of vertices if P and Q are polyhedrons. The proof of the monotonicity is the same because the structure of the sets P and Q appearing in the definition of the operators is not used. To prove the generator property, it is sufficient to observe that the Hamiltonian is equal to that computed using the sets ext P

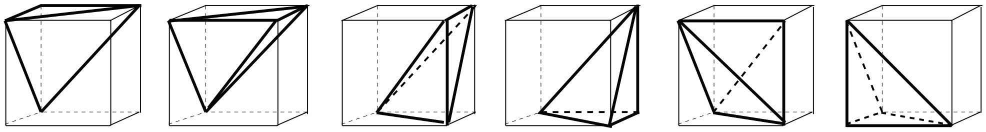

FIG. 3.1. *Tetrahedron partition of a cube.*

and $\operatorname{ext} Q$ whenever condition (f5a) holds. Note that this remark is very important for numerical implementations of the operators (3.1) and (3.3) because the operation "max-min" is applied to functions that are nonlinear and nonconvex/concave in u and v.

4. Test examples. We have verified the above-described approximation schemes using well-known examples of differential games and control problems. Grid sizes, time step lengths, and run times are given for the method based on the upwind operator. It should be noted that the method based on the direct operator delivers the same solutions. The comparison of the efficiencies of these methods is discussed in the conclusion.

4.1. Acoustic version of the homicidal chauffeur game. The homicidal chauffeur game was introduced in [11]. In this game, a car whose minimum turn radius is bounded from below strives to capture an inertia-less pedestrian as soon as possible. The linear velocity of the car is constant, and the magnitude of the pedestrian's velocity does not exceed some given value. The car is controlled by changing the rate of turn of the linear velocity, and the pedestrian changes his velocity arbitrarily within the restriction. The original system of five dynamic equations can be reduced [11] to the following two-dimensional one:

$$\begin{aligned} &\dot{x} = -yu + v_1, \\ &\dot{y} = xu + v_2 - 1, \\ &|u| \le 1, \quad \sqrt{v_1^2 + v_2^2} \le \nu \end{aligned} \tag{4.1}$$

if a movable reference coordinate system with y-axis directed along the car velocity vector is used (see Figure 4.1). A target set T containing the origin is given. The pursuer (the car) strives to provide the inclusion $(x(\tau), y(\tau)) \in T$ for some $\tau \in [0, t_f]$, whereas the evader (the pedestrian) aims to ensure $(x(\tau), y(\tau)) \notin T$ for any $\tau \in [0, t_f]$. The game terminates as soon as $(x(\tau), y(\tau)) \in T$.

The case where the restriction on ν depends on the state (x, y),

$$\nu(x, y) = \nu^* \min\left\{1, \sqrt{x^2 + y^2}/s\right\}, \quad s = \text{const} > 0,$$

is known, due to Pierre Bernhard, as the acoustic version of the homicidal chauffeur problem and is considered in [5] and [20]. Here, ν^* and s are parameters of the

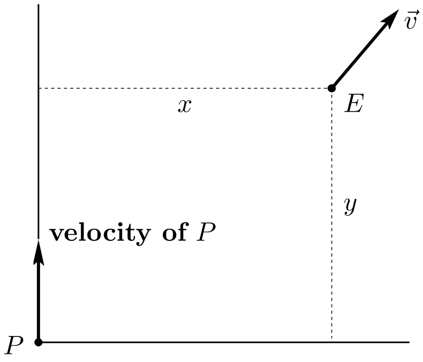

FIG. 4.1. *Movable reference coordinate system in the homicidal chauffeur game.*

problem. Most interesting results are obtained if the target set T is a rectangle $\{(x,y) : -a \le x \le a,\ -b \le y \le 0\}$ stretched along the horizontal axis (i.e., $a \gg b$). The values of the parameters are $\nu^* = 1.5$, $s = 0.9375$, $a = 4.5$, $b = 0.2$, $t_f = 5$. The appropriate functional related to the tasks of the players and to the chosen target set is given by

$$J = \min_{\tau \in [0,t_f]} \sigma(x(\tau), y(\tau)), \quad \sigma(x,y) = \max\{|x|/a, 2|y + b/2|/b\}.$$

Figure 4.2(a) shows the computed level sets $W_i = \{(x,y) :\ V(t_f - i\,\delta, x, y) \le 1\}$, $i = \overline{0,100}$, $\delta = t_f/100$, of the value function. The grid size is equal to 1000×1000, the time step equals 10^{-3}, and the computation time on a computer with 30 threads is about 1 minute. The results obtained are in a very good agreement with those from [21], where the same sets are computed using a characteristics-based method from [20]. These sets are presented in Figure 4.2(b) with the output step $\delta = 0.0625$.

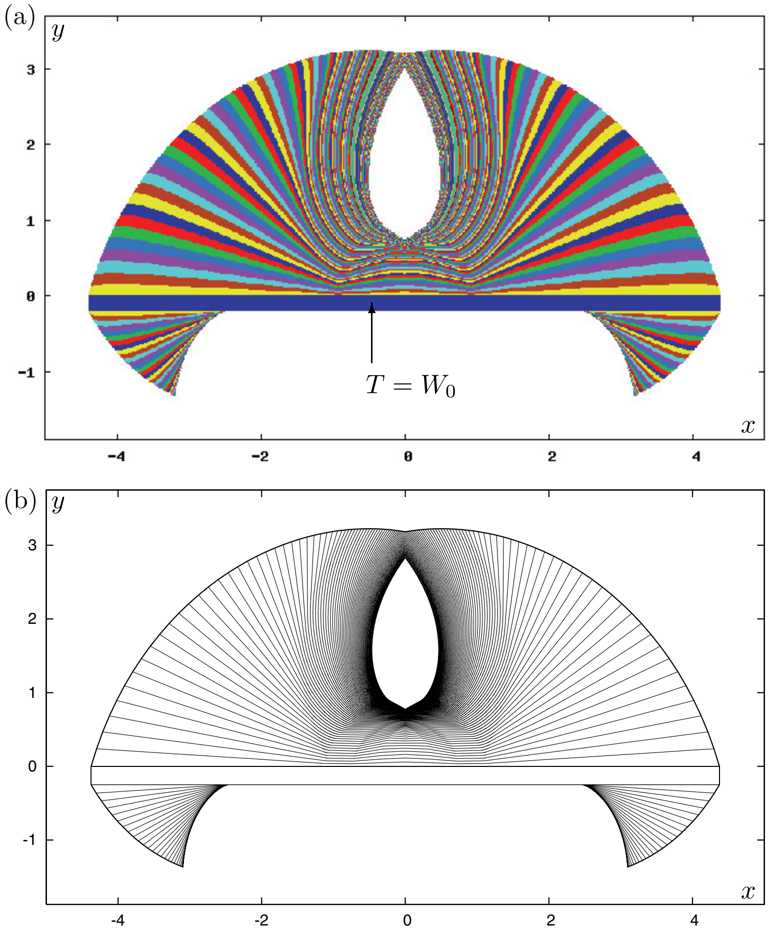

FIG. 4.2. *Level sets of the value function in the acoustic homicidal chauffeur game.* (a) *Computations with the grid method.* (b) *Computations with a characteristics-based method from* [20].

4.2. Dubins' car. A simplified model of a car/aircraft moving in the plane is due to Dubins [10]:

$$\begin{aligned} \dot{x} &= \cos\phi, \\ \dot{y} &= \sin\phi, \\ \dot{\phi} &= u, \\ |u| &\le 1. \end{aligned} \tag{4.2}$$

Here, x, y are the coordinates of the car, and ϕ is the angle between the velocity vector of the car and the axis y. Note that the linear velocity of the car is constant and equal to 1.

The set of points reachable from the origin $(0,0,0)$ at a given time instant t consists of all points $(x,y,z) \in R^3$ such that there exists an admissible control $u(\cdot)$ that transfers the system (4.2) from the state $(0,0,0)$ to the state (x,y,z) at time t.

Figure 4.3 presents the reachable set for $t = 20$. The grid size is equal to $300 \times 300 \times 300$, the time step equals 10^{-3}, and the computation time on a computer with 30 threads is about 10 minutes. The comparison of our results shows a good agreement with the numerical results from [19].

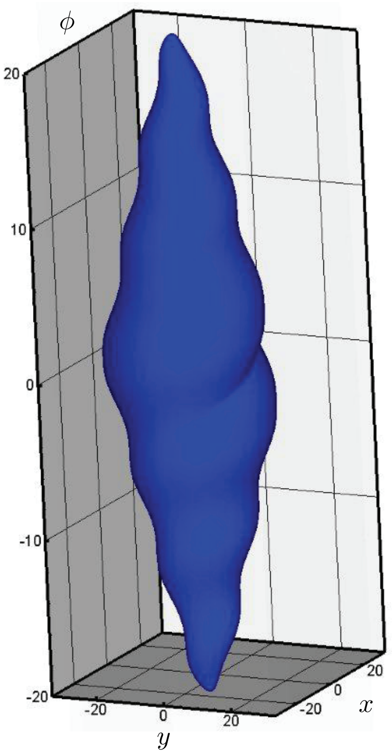

FIG. 4.3. *The set of points reachable from the origin at time* $t = 20$ *in the Dubins' car control problem.*

4.3. Game of two cars. The differential game of two cars was originally introduced in [11]. In this game, the first car pursues the second one, and both cars are moving in the plane. Similar to [15], we consider the case where both cars have the same linear velocities and minimum turn radii. In a movable reference coordinate system, the dynamics of the relative motion can be described as follows:

$$\begin{aligned} \dot{x} &= -yu + \sin\phi, \\ \dot{y} &= \quad xu + \cos\phi - 1, \\ \dot{\phi} &= -u + v, \\ |u| &\le 1, \quad |v| \le 1. \end{aligned} \tag{4.3}$$

The terminal conditions are

$$\sqrt{x^2+y^2} \le r, \quad \cos\left(\phi - \phi_f(x,y)\right) - \cos\phi_f(x,y) \ \le 0 \text{ at } \sqrt{x^2+y^2} = r,$$

$$\text{where} \quad \phi_f(x,y) = \begin{cases} \arccos\left(\ \ x/\sqrt{x^2+y^2}\right), \ y \ge 0, \\ \pi + \arccos\left(-x/\sqrt{x^2+y^2}\right), \ y < 0, \end{cases}$$

which expresses the following capture conditions: the distance between the cars is less than or equal to a given capture radius r and the relative radial velocity on the termination is nonpositive. The terminal time $t_f = 10$.

Figure 4.4 presents the level set $\{(x,y,z) : \ V(0,x,y,z) \le 0.5\}$ of the value function V and the section of this set by the plane $\phi = -2$. The grid size is equal to $300 \times 300 \times 300$, the time step equals 10^{-3}, and the computation time on a computer with 30 threads is about 5 minutes. The results exactly coincide with that analytically computed in [15].

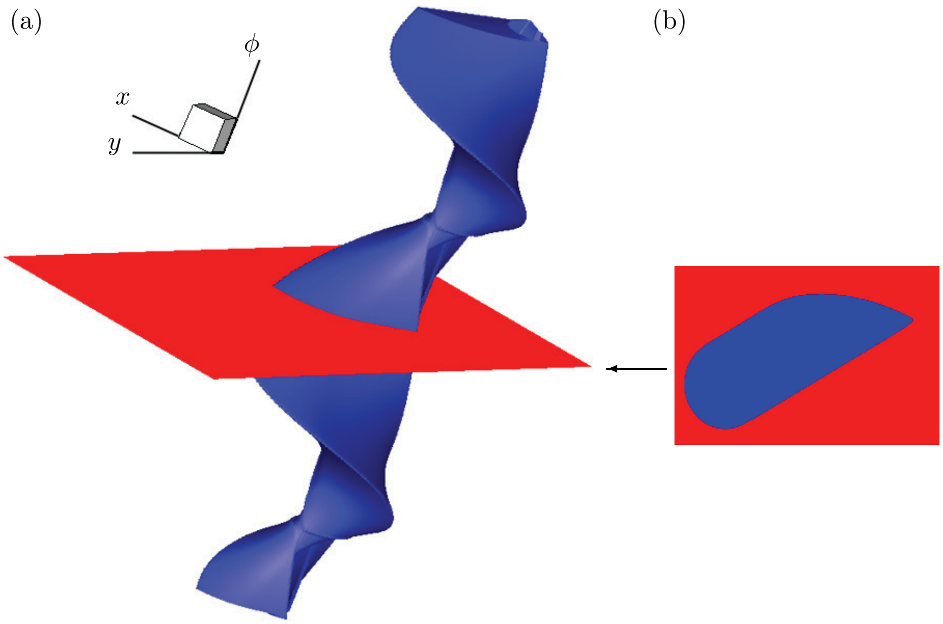

FIG. 4.4. *The game of two cars.* (a) *The level set (the level is equal to* 0.5*) of the value function.* (b) *The section of the level set by the plane* $\phi = -2$.

4.4. Isotropic rockets. In order to verify the feasibility of computations in four dimensions, the game of isotropic rockets investigated by Isaacs (see [11]) is considered. The state equations are the following:

$$\begin{aligned} \dot x_1 &= -x_3 + v_1, \\ \dot x_2 &= -x_4 + v_2, \\ \dot x_3 &= u_1, \\ \dot x_4 &= u_2, \\ |u| &\le \mu, \quad |v| \le \nu. \end{aligned} \tag{4.4}$$

The objective of the control u is to bring the state vector $x = (x_1, x_2, x_3, x_4)$ to the terminal set $T = \{x \in R^4 :\ |x| \le 1\}$; the objective of the control v is opposite.

Additionally, we will consider a reduced two-dimensional differential game

$$\dot x_1 = -x_3 + v, \quad \dot x_3 = u, \quad |u| \le 1, \quad |v| \le 0.2, \tag{4.5}$$

which is the restriction of (4.4) to the plane $x_2 = 0$, $x_4 = 0$.

Level sets of the value function of differential game (4.4) were computed using the following values of parameters and discretization steps: $\mu = 1$, $\nu = 0.2$, $t_f = 2$, $\tau = 5 \cdot 10^{-3}$, $\Delta_{x_1} = \Delta_{x_2} = 1/30$, $\Delta_{x_3} = \Delta_{x_4} = 1/25$ (the grid size was equal to $300\times300\times200\times200$). The run time on 30 threads was about 3 hours. In Figure 4.5, the brown line shows the boundary of the section of the level set $\{x \in R^4 :\ V(0,x) \le 1\}$ by the plane $x_2 = 0$, $x_4 = 0$. The green, blue, and red lines correspond to the computations with the same finite-difference scheme applied to the game (4.5).

The green curve was obtained for the spatial steps $\Delta_{x_1} = 1/30$, $\Delta_{x_3} = 2/75$ (300×300 grid points), the blue curve for $\Delta_{x_1} = 1/80$, $\Delta_{x_3} = 1/100$ (800×800 grid points), and the red one for $\Delta_{x_1} = 1/300$, $\Delta_{x_3} = 1/375$ (3000×3000 grid points). The time step for all variants was $\tau = 0.001$. The run time is about 3 seconds, 15 seconds, and 2 minutes for the first, second, and third variants, respectively. One can see that the precision of the computation increases with the refinement of the spatial steps. In

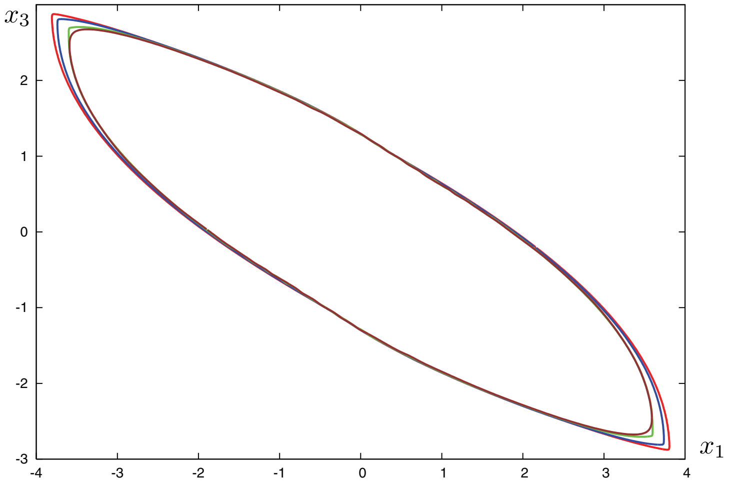

FIG. 4.5. *Two-dimensional section of the four-dimensional level set of the value function in the game of isotropic rockets (brown line). Level sets of the value functions computed for various grid refinements in the two-dimensional reduced game (green, blue, and red lines).*

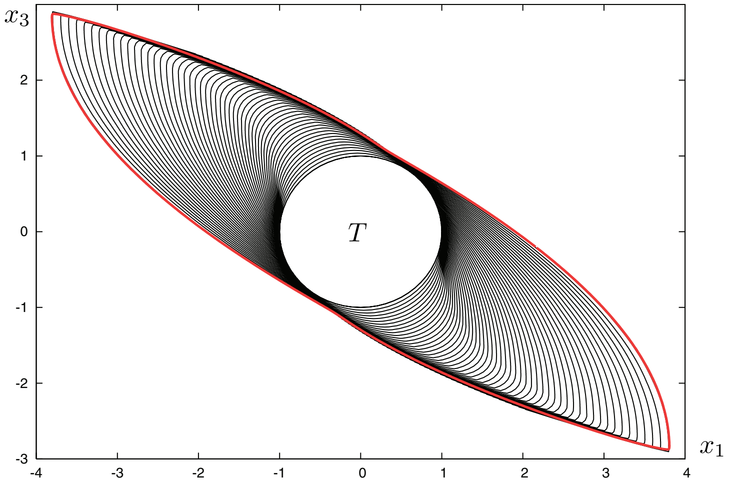

FIG. 4.6. *Reduced game* (4.5). *Comparison of the computation results obtained using the finite difference scheme (red line) and a characteristic-like method (black lines).*

Figure 4.6, the set drawn in red in Figure 4.5 (game (4.5)), which corresponds to the smallest spatial steps, is compared with a very accurate computation performed using the algorithm from [30] (the sets for intermediate times are shown too). A very good agreement of the results is observed. The numerical tests show that the precision of the computations essentially increases with decreasing spatial steps.

5. Conclusion.

Numerical experiments show a very nice property of the method based on the upwind operator: the noise usually coming from the boundary of the grid region is absent so that the grid region may not be too much larger than the region where the solution is searched. The method based on the direct operator does not possess such a property so that larger grid regions are necessary in this case. On the other hand, the direct method admits larger time steps, which can compensate for the necessary extent of the region. In total, the use of the direct operator gives a noticeable advantage regarding the computation time if a proper relation between the time step and the extents of the grid region is found. However, the last task requires some additional numerical efforts and test runs. Therefore, one cannot speak about evident superiority of the direct method.

The grid methods proposed are well appropriate for parallelization on multiprocessor computers. All examples presented in this paper are calculated on a Linux SMP-computer with 8xQuad-Core AMD Opteron processors (Model 8384, 2.7 GHz) and shared 64 GB memory. The programming language is C. The computational performance of the algorithm in two and three dimensions was analyzed. The results obtained are given in Tables 5.1 and 5.2. For two-dimensional computations, the average speed-up value is about 22, and the efficiency is 0.8. For three-dimensional runs, the speed-up (resp., efficiency) decreases from 20 (resp., 0.7) up to 15 (resp., 0.5) with the increase of the number of spatial nodes from 100^3 to 200^3.

Numerical experiments show feasibility of four-dimensional computations with grids of 250^4 nodes and 2000 time steps. The run time here is not terrible (several hours) but the main difficulty consists in a tremendous amount of data (terabytes) needed to process to obtain at least a picture. A rough estimate shows that computations in five dimensions are possible on a computer with 1024 processor cores and 1024 GB memory. Though such a computer is available to the authors, five-dimensional computations are not reasonable objectives at present.

TABLE 5.1
Two-dimensional case.

Time step	Number of steps	Grid size	Time s (30 threads)	Time s (1 thread)	Speed-up
10^{-3}	5100	500×500	35	826	23.6
10^{-3}	5100	1000×1000	146	3321	22.74
10^{-3}	5100	2000×2000	581	13289	22.87
10^{-3}	5100	5000×5000	3726	83347	22.36

TABLE 5.2
Three-dimensional case.

Time step	Number of steps	Grid size	Time s (30 threads)	Time s (1 thread)	Speed-up
10^{-3}	5100	100×100×100	37	750	20.27
10^{-3}	5100	200×200×200	386	6106	15.81
10^{-3}	5100	300×300×300	1400	21033	15.02

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