Application of dynamic programming approach to aircraft take-off in a windshear

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Outline

Dynamic programming approach

- Differential game formulation
- Viscosity solutions of H.-J. equations
- Numerical method: direct and upwind operators

Application to aircraft take-off

- Dynamic and kinematic equations
- Microburst models
- Notes on previous results
- Three different problem statements
- Simulation results
Differential game

Dynamics

$$\dot{x} = f(t, x, \alpha, \beta), \quad \alpha \in A \subset \mathbb{R}^p, \quad \beta \in B \subset \mathbb{R}^q, \quad t_0 \in [0, t_f]$$

Cost functional:

$$J(x(\cdot)) = \int_{t_0}^{t_f} \sigma(t, x(t)) dt \rightarrow \min_{\alpha[\cdot]} \max_{\beta(\cdot)}$$

State constraint:

$$N := \{(t, x) : \theta(t, x) \leq 1\}$$

Hamiltonian:

$$H(t, x, p) = \min_{\alpha \in A} \max_{\beta \in B} \left\langle p, f(t, x, \alpha, \beta) \right\rangle + \sigma(t, x)$$


Hamilton-Jacobi-Bellman-Isaacs equation:

$$c_t + H(t, x, c_x) = 0, \quad c(t_f, x) = 0$$
Viscosity solutions

A Lipschitz function $c(t, x)$ is the value function of (1) if and only if

(i) for any $(t, x) \in [0, t_f] \times \mathbb{R}^n$; $c(t_f, x) = 0$ and

$$c(t, x) \geq \theta(t, x)$$

(ii) for any point $(s_0, y_0) \in (0, t_f) \times \mathbb{R}^n$ and function $\varphi \in C^1$ s.t.

$c - \varphi$ attains a local minimum at $(s_0, y_0)$ the following inequality holds

$$\frac{\partial \varphi}{\partial t}(s_0, y_0) + H(s_0, y_0, \frac{\partial \varphi}{\partial y}(s_0, y_0)) \leq 0$$

(iii) for any point $(s_0, y_0) \in (0, t_f) \times \mathbb{R}^n$ s.t. $c(s_0, y_0) > \theta(s_0, y_0)$ and function $\varphi \in C^1$ s.t. $c - \varphi$ attains a local maximum at $(s_0, y_0)$ the following inequality holds

$$\frac{\partial \varphi}{\partial t}(s_0, y_0) + H(s_0, y_0, \frac{\partial \varphi}{\partial y}(s_0, y_0)) \geq 0$$
Numerical method

\(\tau, h_1, \ldots, h_n\) are time and space discretization steps

1. Direct operator defined on continuous functions

\[
\mathcal{F}(c; t, \tau)(x) = \min_{\alpha \in A} \max_{\beta \in B} c(x + \tau f(t, x, \alpha, \beta))
\]

Introduce grid functions

\[
c^\ell(x_{i_1}, \ldots, x_{i_n}) = c(t_\ell, i_1 h_1, \ldots, i_n h_n)
\]

\[
\theta^\ell(x_{i_1}, \ldots, x_{i_n}) = \theta(t_\ell, i_1 h_1, \ldots, i_n h_n)
\]

\[
t_\ell = \ell \tau, \quad \ell = 0, \ldots, N; \quad N = \frac{t_f - t_0}{\tau} + 1
\]
Let $\mathcal{L}_h$ be an interpolation operator that maps grid functions into continuous functions and satisfies the condition

$$
\| \mathcal{L}_h[\tilde{\phi}] - \phi \| \leq C \max\{h_1, \ldots, h_n\}^2 \|D^2 \phi\|
$$

for any smooth function $\phi$, where $\tilde{\phi}$ is the restriction of $\phi$ to the grid.

Finite difference scheme

$$
c^{\ell-1} = \max \left\{ \mathcal{F}(\mathcal{L}_h[c^{\ell}]; t_\ell, \tau), \theta^{\ell} \right\}
$$

(2)

$$
c^N = 0, \quad \ell = N, N - 1, \ldots, 1
$$

Theorem The grid function obtained by (2) converges point-wise to the value function of the differential game (1) as $\tau, h_1, \ldots, h_n \to 0$.

The convergence rate is $\max(\sqrt{\tau}, \max_i \sqrt{h_i})$. 

2. Upwind operator

\[
F(c; t_{\ell}, \tau, h_1, \ldots, h_n)(x) = c(x) + \tau \max_{\beta \in B} \min_{\alpha \in A} \sum_{i=1}^{n} (p_i^R f_i^+ + p_i^L f_i^-)
\]

where \( x = (x_{i_1}, \ldots, x_{i_n}) \), \( f_i^+ = \max \{f_i, 0\} \), \( f_i^- = \min \{f_i, 0\} \)

\( p_i^L \), \( p_i^R \), \( i = 1, n \), are the right and left divided differences with spatial steps \( h_i \)

\( f_i = f_i(t_{\ell}, x, \alpha, \beta) \) are the RHS of the control system

Finite difference scheme

\[
c^{\ell-1} = \max \left\{ F(c^{\ell}; t_{\ell}, \tau, h_1, \ldots, h_n), \theta^{\ell} \right\},
\]

\[
c^{N} = 0, \quad \ell = N, N - 1, \ldots, 1.
\]

Theorem  Let $M$ be a bound of the RHS of the control system. If \[ \frac{\tau}{h_i} \leq \frac{1}{M \sqrt{n}} \], then the grid function obtained by the procedure (3) converges pointwise to the value functions of problem (1) as $\tau \to 0, h_i \to 0$, and the convergence rate is $\sqrt{\tau}$.

Proof  1. Monotonicity of the operator $F$:

$$c \leq d \Rightarrow F(c; t, \tau, h_1, ..., h_n) \leq F(d; t, \tau, h_1, ..., h_n)$$

2. Generator property of the operator $F$:

$$\left| \frac{F(\phi; t, \tau, h_1, ..., h_n)(x) - \phi(x)}{\tau} - H(t, x, D\phi(x)) \right| \leq C \left( 1 + \|D\phi\| + \|D^2\phi\| \right) \tau$$

for any $\varphi \in C^b_2(R^n), \ x \in R^n$, and fixed $\tau/h_i$.
Application to aircraft take-off in a windshear
Aircraft dynamics

\[ \begin{align*}
\dot{x} &= V \cos \gamma + W_x \\
\dot{h} &= V \sin \gamma + W_h \\
\dot{V} &= T \cos(\alpha + \delta) - D - mg \sin \gamma - mW_x \cos \gamma - mW_h \sin \gamma \\
V \dot{\gamma} &= T \sin(\alpha + \delta) + L - mg \cos \gamma + mW_x \sin \gamma - mW_h \cos \gamma
\end{align*} \]
Forces

Thrust: \[ T = A_0 + A_1 V + A_2 V^2 \]

Drag: \[ D = \frac{1}{2} C_D \rho S V^2, \quad C_D = B_0 + B_1 \alpha + B_2 \alpha^2 \]

Lift: \[ L = \frac{1}{2} C_L \rho S V^2 \]

\[ C_L = \begin{cases} C_0 + C_1 \alpha, & \alpha \leq \alpha_{**} \\ C_0 + C_1 \alpha + C_2 (\alpha - \alpha_{**})^2, & \alpha \in [\alpha_{**}, \alpha_*] \end{cases} \]
Wind (microburst)
Microburst: Model 1

\[ W_x = \begin{cases} -k, & x \leq a \\ -k + 2k(x-a)/(b-a), & a \leq x \leq b \\ k, & x \geq b, \end{cases} \]

\[ W_h = \begin{cases} 0, & x \leq a \\ -k(h/h_*)(x-a)/(c-a), & a \leq x \leq c \\ -k(h/h_*)(b-x)/(b-c), & c \leq x \leq b \\ 0, & x \geq b, \end{cases} \]

\[
\dot{W}_x = \frac{\partial W_x}{\partial x} (V \cos \gamma + W_x) + \frac{\partial W_x}{\partial h} (V \sin \gamma + W_h)
\]

\[
\dot{W}_h = \frac{\partial W_h}{\partial x} (V \cos \gamma + W_x) + \frac{\partial W_h}{\partial h} (V \sin \gamma + W_h)
\]

Microburst: Model 2

\[ V_\theta = \begin{cases} 
V_0 r / R, & 0 \leq r \leq R \\
V_0 R / r, & r > R 
\end{cases} \]


Take-off control providing $h(t) \geq 0$ for any $t$

Full 4D dynamics

Cost functional:

\[
J = \int_{0}^{t_f} \left( V(t) \sin \gamma(t) + W_h(x(t), h(t)) \right) dt \rightarrow \max_{\alpha[\cdot]}
\]

State constraint: $h(t) \geq 0$, $0 \leq t \leq t_f$
Take-off with tracking reference trajectory

\[ V_r - \text{reference velocity} \]

\[ \gamma_r - \text{reference path inclination} \]

\[ x = V_r \cos \gamma_r \cdot t \]

\[ h = V_r \sin \gamma_r \cdot t \]

Substitute \( x \) and \( h \) into (2) to arrive at the differential game:

\[ \dot{V} = f_1(t, V, \gamma, \alpha, \gamma_\nu), \quad \dot{\gamma} = f_2(t, V, \gamma, \alpha, \gamma_\nu) \]

Control \( \alpha : 0 \leq \alpha \leq \alpha^* \)

Disturbance \( \gamma_\nu : |\gamma_\nu - \gamma_r| \leq \gamma^* \)

\[
J = \int_0^{t_f} \left( V(t) \sin \gamma(t) - V_r \sin \gamma_r \right)^2 dt \quad \longrightarrow \quad \min_{\alpha[\cdot]} \ \max_{\gamma_\nu(\cdot)}
\]
Substitute \( x \) and \( h \) into (2) to arrive at the differential game:

\[
\begin{align*}
V_r & \quad \text{reference velocity} \\
\gamma_r & \quad \text{reference path inclination}
\end{align*}
\]

\[
x = V_r \cos \gamma_\nu \cdot t \\
h = V_r \sin \gamma_\nu \cdot t \\
|\gamma_\nu - \gamma_r| \leq \gamma_*
\]

Substitute \( x \) and \( h \) into (2) to arrive at the differential game:

\[
\begin{align*}
\dot{V} &= f_1(t, V, \gamma, \alpha, \gamma_\nu), \\
\dot{\gamma} &= f_2(t, V, \gamma, \alpha, \gamma_\nu)
\end{align*}
\]

Control \( \alpha : 0 \leq \alpha \leq \alpha_* \)

Disturbance \( \gamma_\nu : |\gamma_\nu - \gamma_r| \leq \gamma_* \)

\[
J = \int_{0}^{t_f} \left( V(t) \sin \gamma(t) - V_r \sin \gamma_r \right)^2 dt \quad \longrightarrow \quad \min_{\alpha[\cdot]} \max_{\gamma_\nu(\cdot)}
\]
Take-off control using the climb rate

\[
\ddot{h} = \frac{T}{m} [\cos (\alpha + \delta) \sin \gamma + \sin (\alpha + \delta) \cos \gamma] - \frac{D}{m} \sin \gamma + \frac{L}{m} \cos \gamma - g
\]

\[
\sin \gamma = \frac{\dot{h} - W_h}{V}, \quad \cos \gamma = \ldots \quad \rightarrow \quad \ddot{h} = \mathcal{Z}(\dot{h}, \alpha, V, W_h)
\]

Differential game

\[
\dot{h} = z
\]

\[
\dot{z} = \mathcal{Z}(z, \alpha, V, W_h)
\]

Control \quad \alpha : \quad 0 \leq \alpha \leq \alpha_*

Disturbances \quad W_h : \quad W_h \in [-100, 0]
\quad V : \quad V \in [V_r - 50, V_r]

\[
J = \int_0^{t_f} (z - V_r \sin \gamma_r)^2 dt \quad \rightarrow \quad \min_{\alpha[]} \max_{V,W_h(\cdot)}
\]

The structure of the wind velocity field is supposed to be unknown.

Example of computed value function (P3)

\[ c(0, h, \dot{h}) \]

Contour lines
Control design

\[
\min_{\alpha \in A} \max_{\beta \in B} c(x_{i_1 i_2 \ldots i_n} + \tau f(t_\ell, x_{i_1 i_2 \ldots i_n}, \alpha, \beta))
\]

\[\alpha_{i_1 i_2 \ldots i_n}\] is stored in an array \(A[\ell, i_1 \ldots i_n]\)

\[
\begin{align*}
x_{i_1 i_2 + 1} & \quad \alpha_{i_1 i_2 + 1} \\
\alpha_{i_1 i_2 + 1} & \quad \alpha_{i_1 + 1i_2 + 1} \\
x_{i_1 i_2} & \quad \alpha_{i_1 + 1i_2} \\
\alpha_{i_1 i_2} & \quad \alpha_{i_1 + 1i_2}
\end{align*}
\]

\[x \big|_{t = t_\ell} = L_h [\alpha^\ell](x)\]
Simulation of the original nonlinear system

P1 (4d, control prob.)
P2 (2d, diff. game)

Microburst model 1 (k=60)
Microburst model 1 (k=60)

- P1 (4d, control prob.)
- P2 (2d, diff. game)

Path inclination

Angle of attack
Microburst model 1 (k=50)

- P1 (4d, control prob.)
- P2 (2d, diff. game)

Altitude

\[ h \text{ (ft)} \]

Angle of attack

\[ \alpha \text{ (deg)} \]
Microburst model 2

P3 (2d, diff. game, no wind information)

Altitude

Velocity

\[ h \ (\text{ft}) \]

\[ V_0 = 100 \]

\[ V_0 = 140 \]
Wind velocity

Control (averaging of bang-bang)
# Numerical characteristics

Linux SMP-Computer with 8xQuad-Core AMD Opteron processors (2.7 GHz)  
Shared 64 Gb memory  
Efficiency of parallelization up to 80%

<table>
<thead>
<tr>
<th>Grid size</th>
<th>Run time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>P1</strong> 250 x 200 x 100 x 20</td>
<td>8 min</td>
</tr>
<tr>
<td><strong>P2</strong> 2000 x 400 (200 x 40)</td>
<td>1.5 min (10 s)</td>
</tr>
<tr>
<td><strong>P3</strong> 400 x 200 (200 x 80)</td>
<td>40 s (20 s)</td>
</tr>
</tbody>
</table>
Conclusion

What we can do:

• Nonlinear dynamics
• State constraints
• Fixed/nonfixed termination time, integral objective functionals
• Dimensions: 2, 3, 4

Future:

Higher dimensions using sparse tensor representations of functions*

*Tensor train format, or simply TT-format and Quantized-TT (QTT)- format.

*Sergey Dolgov, Boris N. Khoromskij, and Ivan V. Oseledets, Fast solution of multi-dimensional parabolic problems in the TT/QTT-format with initial application to the Fokker-Planck equation, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig, Preprint no.: 80, 2011. Dimension = 9!