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Citation: AIP Conf. Proc. 1479, 1226 (2012); doi: 10.1063/1.4756373
View online: http://dx.doi.org/10.1063/1.4756373
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=1479&Issue=1
Published by the American Institute of Physics.

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Application of Dynamic Programming Approach to Aircraft Take-Off in a Windshear

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Abstract. Application of dynamic programming method to the problem of aircraft control during take-off in a windshear is considered. A simplified four dimensional model of the aircraft dynamics is used, and stable numerical algorithms for solving Hamilton-Jacobi-Bellman-Isaacs equations arising from differential games with state constraints are utilized for the control design.

Keywords: Optimal control, State constraints, Hamilton-Jacobi-Bellman-Isaacs equations, Viscosity solutions, Finite difference schemes

PACS: 47.11.-j, 47.56.+r, 91.65.My

MODEL EQUATIONS

Many aircraft accidents occur due to severe windshears, e.g. microburst. The microburst appears when a descending air flow hits the earth surface. It is especially dangerous for aircrafts passing the microburst zone during the landing or take-off, because quick changes of the wind velocity occur at relatively low altitudes. In this paper, a simplified model of the aircraft dynamics is considered. A feedback control that is effective against a microburst is designed using dynamic programming techniques.

Consider the motion of the aircraft in the vertical plane. The aircraft dynamics, see [1], is described by four ordinary differential equations governing the aircraft relative velocity $V$, the relative path inclination $\gamma$, the horizontal distance $x$, and the altitude $h$:

$$m\ddot{V} = T\cos(\alpha + \delta) - D - mg\sin\gamma - mW_x\cos\gamma - mW_z\sin\gamma$$

$$m\ddot{\gamma} = T\sin(\alpha + \delta) + L - mg\cos\gamma + mW_x\sin\gamma - mW_z\cos\gamma$$

$$\dot{x} = V\cos\gamma + W_x$$

$$\dot{h} = V\sin\gamma + W_h.$$  \hspace{1cm} (1)

Here, $\alpha$ is the attack angle, $W_x$ and $W_h$ are the horizontal and vertical components of the wind velocity, $g$ is the acceleration of gravity, $m$ the aircraft mass, $\delta$ the thrust inclination; $T, D,$ and $L$ are the thrust, the drag, and the lift force, respectively, defined as follows:

$$T = A_0 + A_1V + A_2V^2,$$

$$D = \frac{1}{2}C_D\rho SV^2,$$

$$L = \frac{1}{2}C_L\rho SV^2,$$

$$C_D = B_0 + B_1\alpha + B_2\alpha^2,$$

$$C_L = \begin{cases} C_0 + C_1\alpha, & \alpha \leq \alpha_* \\ C_0 + C_1\alpha + C_2(\alpha - \alpha_*)^2, & \alpha \in [\alpha_*, \alpha_*] \end{cases}.$$  \hspace{1cm} (2)

For simplicity, the coefficients $A_i, B_i, C_i, i = 0, 1, 2,$ are assumed to be constants; $\alpha_*$ and $\alpha_*$ are given constants, $\rho$ is the air density, and $S$ is the reference surface area.

The attack angle $\alpha$ restricted by the condition $0 \leq \alpha \leq \alpha_*$ is considered as the control. The longitudinal and vertical wind disturbances, $W_x$ and $W_h$, are simulated using a model of microburst described in [1]:

$$W_x = \begin{cases} -k, & x \leq a \\ -k + 2k(x-a)/(b-a), & a \leq x < b \\ k, & x \geq b, \end{cases}$$

$$W_h = \begin{cases} 0, & x \leq a \\ -k(h/h_*)/(c-a), & a \leq x \leq c \\ -k(h/h_*)(b-x)/(b-c), & c \leq x \leq b \\ 0, & x \geq b, \end{cases}$$
where \( c = \frac{(a + b)}{2} \), and \( h_* \) is a fixed constant. The parameter \( k \) defines the intensity of the microburst.

**STATEMENT OF THE PROBLEM**

Two variants of the problem statement are considered.

**P1.** The objective of the control \( \alpha \) in system (1)–(2) is to maximize the performance index

\[
J = \int_0^{t_f} \left( V(t) \sin \gamma(t) + W_h(x(t), h(t)) \right) dt,
\]

where \( t_f \) is the final time of the process, and to satisfy the following state constraint

\[
h(t) \geq 0, \quad 0 \leq t \leq t_f.
\]

In this variant, the full four dimensional nonlinear optimal control problem (1)–(4) is numerically solved. That is, a grid approximation of the optimal result function is computed, and optimizing values of the attack angle at each grid node and at each sampling time instant are stored on a hard disk. In the simulation process, the control is computed as a weighted linear combination of the control values stored in the nodes of the grid cell in which the current four dimensional state vector lies. The weights are computed on the base of relative coordinates of the state vector in the cell.

**P2.** In the second variant, a two dimensional differential game is designed in the following manner. The horizontal distance and the altitude are determined from the assumption that the aircraft moves with a constant velocity \( V_r \) along a straight line with an inclination angle \( \gamma_v \):

\[
x = V_r \cos \gamma_v \cdot t, \quad h = V_r \sin \gamma_v \cdot t.
\]

The expressions (5) are substituted into equations (1), which yields a two-dimensional differential game where the angle \( \gamma_v \) is considered as a disturbance controlled by the opposite player. It is assumed that \( |\gamma_v - \gamma_r| \leq \gamma_* \), where \( \gamma_r \) is a reference value of the path inclination, and \( \gamma_* \) is a given bound. The objective of the control \( \alpha \) is to minimize the functional

\[
J = \int_0^{t_f} \left( V(t) \sin \gamma(t) - V_r \sin \gamma_r \right)^2 dt,
\]

whereas the objective of the disturbance is the opposite.

A grid approximation of the value function of this differential game is computed. When simulating trajectories of the original four dimensional system, the control is being chosen using current values of the relative velocity and the relative path inclination by determining which grid cell this couple belongs to.

**NUMERICAL METHOD**

Let us outline the solution method for problems 1 and 2. The description will be given for a general case of a nonlinear differential game with an integral payoff functional and state constraints.

Consider the following differential game (see [2])

\[
\dot{x}_i = f_i(t, x, u, v),
\]

where \( x := (x_1, ..., x_n) \in \mathbb{R}^n \) is the state vector, \( u \) and \( v \) are control parameters of the first and second player restricted as

\[
u \in P \subset \mathbb{R}^p, \quad v \in Q \subset \mathbb{R}^q.
\]

Here, \( P \) and \( Q \) are given compacts. The game starts at \( t_0 \in [0, t_f] \) and finishes at \( t_f \). The objective of the control \( u \) of the first player is to minimize the functional

\[
J(x(\cdot)) = \int_{t_0}^{t_f} \sigma(t, x(t)) dt,
\]
where \( \sigma : [0, t_f] \times \mathbb{R}^n \to \mathbb{R} \) is a given function. The objective of the control \( v \) of the second player is the opposite. Besides, the trajectories should remain in a state constraint set given by

\[
N := \{(t, x) : t \in [0, t_f], \theta(t, x) \leq 1\},
\]

where, \( \theta : [0, t_f] \times \mathbb{R}^n \to \mathbb{R} \) is a given function.

Define the Hamiltonian as follows:

\[
H(t, x, p) = \max_{v \in Q} \min_{w \in P} \{p, f(t, x, u, v)\} + \sigma(t, x)
\]

and consider the Hamilton-Jacobi-Bellman-Isaacs equation

\[
\mathcal{V}_t + H(t, x, \mathcal{V}_x) = 0, \quad \mathcal{V}(t_f, x) = 0.
\]

It is a well-known fact in the theory of differential games that the value function of a differential game (6)–(9) is a viscosity solution of equation (11). To compute viscosity solutions of (11), the following upwind finite difference scheme is applied.

Let \( \tau, h_1, \ldots, h_n \) be time and space discretization step sizes, and \( F \) be an operator defined on continuous functions as

\[
F(\mathcal{V}', t, \tau)(x) = \max_{v \in Q} \min_{w \in P} \mathcal{V}'(x + \tau f(t, x, u, v)).
\]

Define \( \Lambda = (t_f - t_0)/\tau + 1, \ell = \ell \tau, \ell = 0, \ldots, \Lambda \), and introduce the following notation:

\[
\mathcal{V}^\ell(x_{1_1}, \ldots, x_{n_0}) = \mathcal{V}(t_\ell, i_1 h_1, \ldots, i_n h_n), \quad \theta^\ell(x_{1_1}, \ldots, x_{n_0}) = \theta(t_\ell, i_1 h_1, \ldots, i_n h_n).
\]

Let \( h := (h_1, \ldots, h_n) \), and \( \mathcal{L}_h \) be an interpolation operator which maps grid functions to continuous functions and satisfies the estimate

\[
\|\mathcal{L}_h[\delta] - \phi\| \leq C \max\{h_1, \ldots, h_n\}^2 \|D^2\phi\|
\]

for any smooth function \( \phi \). Here, \( \delta \) is the restriction of \( \phi \) to the grid, \( \| \cdot \| \) the point-wise maximum norm, \( D^2\phi \) the Hessian matrix of \( \phi \), and \( C \) is an independent constant. The construction of a multilinear interpolation operator satisfying the above estimate is described in [3].

The finite-difference scheme being as follows:

\[
\mathcal{V}^{\ell-1} = \max \left\{ F(\mathcal{L}_h[\mathcal{V}^{\ell'}]; t_\ell, \tau), \theta^\ell \right\}, \quad \mathcal{V}^{\Lambda} = 0, \quad \ell = \Lambda, \Lambda - 1, \ldots, 1.
\]

Notice that \( F(\mathcal{L}_h[\mathcal{V}^{\ell'}]; t_\ell, \tau) \) is a continuous function that is assumed to be restricted to the grid and then compared with the grid function \( \theta^\ell \). Thus, the right-hand side of (13) returns a grid function.

**Theorem 1** The grid function obtained by (13) converges point-wise to the value function of the game (6)–(9) as \( \tau, h_1, \ldots, h_n \to 0 \). The convergence rate is \( \max(\sqrt{\tau}, \max\sqrt{h_i}) \).

The proof of the Theorem is similar to that ones given in [3] and [4].

**NUMERICAL RESULTS**

The described grid method is applied both to optimal control problem \( \textbf{P1} \) and differential game \( \textbf{P2} \). The numerical values of the coefficients defining the dynamics of a Boeing-727 aircraft are taken from [5]. The calculations are done on a Linux SMP-computer with 8xQuad-Core AMD Opteron processors (Model 8384, 2.7 GHz) and shared 64 Gb memory. The programming language C with OpenMP (Open Multiprocessing) support is used. The efficiency of parallelization is up to 80%.

For problem \( \textbf{P1} \) where we operate in the four dimensional state space \( (x, h, V, \gamma) \), the grid size is \( 200 \times 200 \times 100 \times 20 \). For problem \( \textbf{P2} \) with only two state variables \( V \) and \( \gamma \), the grid size is \( 2000 \times 400 \). It should be stressed that the reduction of the grid size to \( 200 \times 40 \) does not change the quality of the control designed. However, the runtime is less than one second in this case, which makes possible to develop diverse adaptive in real-time control algorithms.
Figure 1 shows simulation results. The solid lines correspond to problem $P_1$, whereas the dash and gray lines are related to problem $P_2$. The horizontal axes in Figures 1a–f measure time in seconds. The vertical axes in Fig. 1a, b, c, and d measure the altitude $h$ (feet), the aircraft relative velocity $V$ (feet/s), the path inclination $\gamma$ (deg), and the angle of attack $\alpha$ (deg), respectively. The value of the parameter $k$ defining the intensity of the microburst is equal to 60. In Figure 1e and f, the altitude and the angle of attack versus time are presented for a weaker windshear ($k = 50$).

Our simulation results are in a good agreement with those of paper [5] where a robust take-off control based on Lyapunov stability theory is designed for the aircraft dynamics given by (1)–(2). Besides, our results are in conformity with those of report [6] where a control based on the computation of switch lines of an appropriate two-dimensional linear differential game is constructed.

**REFERENCES**