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Approximation schemes for solving disturbed control problems with non-terminal time and state constraints

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Summary: A finite difference scheme for finding viscosity solutions of specifically stated problems for Hamilton–Jacobi equations is proposed. Solutions of such problems satisfy two differential inequalities involved into the definition of viscosity solutions. The specifics is that the inequalities must hold only in the region where solutions are less than a given function or greater than another given function. From the point of view of optimal control theory and the theory of differential games, these results can be applied to finding value functions in problems with nonterminal payoff functionals and state constraints.

1 Introduction

In optimal control theory and the theory of differential games [13, 14], a new approach related to a generalization of the Bellman–Isaacs equation has been developed during the last several decades. It was proposed in [21, 22] to replace the Bellman–Isaacs equation by a pair of directional derivative inequalities. These inequalities express in infinitesimal form the u-stability and v-stability properties of the value function [14, 23]. A function satisfying these inequalities is called a minmax solution of the Bellman–Isaacs equation.

In [8, 9], the concept of viscosity solutions for Hamilton–Jacobi equations has been proposed. Further investigations [24] showed that minmax solutions of Bellman–Isaacs equations coincide with viscosity solutions. Therefore, the advanced theory of viscosity solutions and corresponding numerical methods can be used for solving differential games, see [10, 19]. Paper [10] suggests grid methods based on vanishing viscosity techniques for finding viscosity solutions of Hamilton–Jacobi equations. In [19], an abstract operator that generates approximate solutions is introduced. It is proved that, under certain conditions posed on this operator, the uniform convergence of approximate solutions to a viscosity solution holds. It is also shown that the operator can be represented

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as an explicit or implicit finite difference scheme. In [20], another representation of the above mentioned operator is given in terms of differential games theory. In [4], the approach of [19] and [20] is extended to Bellman–Isaacs equations arising from differential games with payoff functionals of the form:

$$\gamma_\chi(x(\cdot)) = \min_{t\in[t_0,T]} \chi(t, x(t)),$$

where t_0 is the starting time, T the termination time, $x(\cdot)$ a trajectory of the controlled system, and χ a given function. In [2], a discrete scheme for solving minimum time games with discontinuous value functions is proposed. Papers [25] and [26] propose a finite difference method that requires the computation of generalized gradients of local convex hulls of approximate solutions. In [6], an approach based on the approximation of viability kernels (see [1]) is presented. A discrete version of the dynamic programming method is developed in [3] and [7]. Thesis [17] describes application of the Lax-Friedrichs approximation to the computation of viscosity solutions of Bellman–Isaacs equations arising from the level set method (see [18]).

Paper [11] considers pursuit-evasion differential games with state constraints. The payoff functional is the first (if any) time of arrival of the state vector at a given target set provided that the state vector does not leave a given state constraint. The authors derive a steady-state implicit equation that describes time-discretized Kružkov's transform of the value function. A fully-discrete spatial approximation is then proposed. The convergence is proved, and some numerical examples are presented. Work [12] develops a backward procedure for computing solvability sets in differential games of approach with state constraints. The authors introduce a backward time step operator that maps compact sets in compact sets and prove that such an operator has the properties of u and v-stability. From the numerical point of view, the procedure is based on union and intersection operations of non-convex compact sets. Examples of computations in two dimensions are given.

The present paper deals with Bellman–Isaacs equations arising from differential games with payoff functionals of the form:

$$\gamma(x(\cdot)) = \max\{ \min_{t\in[t_0,T]} \chi(t, x(t)), \max_{t\in[t_0,T]} \theta(t, x(t))\},$$

where χ and θ are given functions satisfying the relation $\chi(t, x) \geq \theta(t, x)$ for all t, x. Note that the first part, $\min_{t\in[t_0,T]} \chi(t, x)$, is responsible for the quality of the process, and the second part, $\max_{t\in[t_0,T]} \theta(t, x(t))$, accounts for a state constraint. To see that, consider a differential game of approach with the target set $M := \{(t, x) : t \in [t_0, T], \chi(t, x) \leq 1\}$ and the state constraint set $N := \{(t, x) : t \in [t_0, T], \theta(t, x) \leq 1\}$. Obviously $M \subset N$ because χ dominates θ. If the value function of the game is less than or equal to 1 at the starting position, then there exists a strategy of the first player such that, for all strategies of the second player and all trajectories $x(\cdot)$, two conditions hold:

(a) $\min_{t\in[t_0,T]} \chi(t, x(t)) \leq 1$ (the position $(t, x(t))$ arrives at the target set at some time instant $t \leq T$),

(b) $\max_{t\in[t_0,T]} \theta(t, x(t)) \leq 1$ (the position $(t, x(t))$ remains in the state constraint set for all $t \in [t_0, T]$).

In this paper, differential inequalities defining viscosity solutions of Bellman–Isaacs equations arising from differential games with the modified payoff functional are formulated. A modified abstract operator which is roughly the maximum of the operator considered in [4] and the function θ is introduced. It is shown that such a modified operator generates approximations of viscosity solutions (value functions) for differential games with the modified payoff functional. The convergence rate of these approximations is found. The modified operator can be represented as a fully discrete finite difference scheme or, by analogy with [20], as minmax (maxmin) backward procedure. Numerical examples are given.

2 Notation

$\mathbb{R}$ and $\mathbb{R}^n$ are the set of real numbers and the canonical Euclidean space, respectively.

$\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^n$.

$|\cdot|$ denotes the absolute value of a scalar, the Euclidean norm of a vector in $\mathbb{R}^n$, the operator norm of a $n \times n$ matrix, or the length of the largest interval of a partition.

$C^k(Q)$ is the space of k times continuously differentiable functions defined on Q.

$C_b^k(Q)$ is the space of k times continuously differentiable functions defined on Q and bounded together with their k derivatives.

$C_0^k(Q)$ is the space of k times continuously differentiable functions with compact supports.

$C_b^{0,1}(Q)$ is the space of bounded real valued Lipschitz functions defined on Q.

$\|v\| = \sup_{x\in Q} |v(x)|$ for $v \in C_b^{0,1}(Q)$.

$\|Dv\|$ is the Lipschitz constant of $v \in C_b^{0,1}(Q)$.

$Dv(x) = \left(\frac{\partial v}{\partial x_1}(x), \frac{\partial v}{\partial x_2}(x), \ldots, \frac{\partial v}{\partial x_n}(x)\right)$ is the gradient of a differentiable function.

$\|D^k v\| = \sup_{x\in Q} |D^k v(x)|$ for $v \in C_b^k(Q)$, $k = 1, 2$.

$\mathrm{supp}(v)$ is the support of v.

$B_n(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| \le r\}$.

3 Differential games

Consider a differential game with the dynamics

$$\dot{x} = f(t, x, \alpha, \beta), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad \alpha \in A \subset \mathbb{R}^a, \quad \beta \in B \subset \mathbb{R}^b, \tag{3.1}$$

where t is time; x the state vector; α, β are control parameters of the players; and A, B are given compacts. The game starts at $t = t_0$ and finishes at $t = T$. The payoff functional defined on the trajectories of system (3.1) is given by

$$\gamma(x(\cdot)) = \max\{\min_{t\in[t_0,T]} \chi(t, x(t)), \max_{t\in[t_0,T]} \theta(t, x(t))\}, \tag{3.2}$$

where $\chi : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ and $\theta : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are some given function such that $\chi(t, x) \geq \theta(t, x)$ for all $t \in [0, T],\ x \in \mathbb{R}^n$. The first player uses the control parameter α to minimize payoff functional (3.2). The aim of the second player dealing with the control parameter β is opposite.

The game is formalized as in [14, 23]. That is, the players use feedback strategies which are arbitrary functions

$$\mathcal{A} : [0, T] \times \mathbb{R}^n \to A, \quad \mathcal{B} : [0, T] \times \mathbb{R}^n \to B.$$

For any initial position $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ and any strategies $\mathcal{A}$ and $\mathcal{B}$, two functional sets $X_1(t_0, x_0, \mathcal{A})$ and $X_2(t_0, x_0, \mathcal{B})$ are defined. These sets consist of the limits of the step-by-step solutions of (3.1) generated by the strategies $\mathcal{A}$ and $\mathcal{B}$, respectively (see [14, 23]).

Assume that the following conditions are fulfilled:

(f1) The function f is uniformly continuous on $[0, T] \times \mathbb{R}^n \times A \times B$.

(f2) f is bounded, i.e. $|f(t, x, \alpha, \beta)| \leq M$ for all $(t, x, \alpha, \beta) \in [0, T] \times \mathbb{R}^n \times A \times B$.

(f3) f is Lipschitzian in t, x, i.e.

$$|f(t_1, x_1, \alpha, \beta) - f(t_2, x_2, \alpha, \beta)| \leq N(|t_1 - t_2| + |x_1 - x_2|),$$

for all $(t_i, x_i, \alpha, \beta) \in [0, T] \times \mathbb{R}^n \times A \times B,\ i = 1, 2$.

(f4) χ and θ are bounded and Lipschitzian in t, x, i.e.

$$\begin{aligned} |\chi(t, x)| &\leq C_{00}, \qquad\qquad |\theta(t, x)| \leq C_{01}, \\ |\chi(t_1, x_1) - \chi(t_2, x_2)| &\leq L_{00}(|t_1 - t_2| + |x_1 - x_2|), \\ |\theta(t_1, x_1) - \theta(t_2, x_2)| &\leq L_{01}(|t_1 - t_2| + |x_1 - x_2|), \end{aligned}$$

for all $(t, x),\ (t_i, x_i) \in [0, T] \times \mathbb{R}^n,\ i = 1, 2$.

(f5) The function f satisfies the saddle point condition:

$$\min_{\alpha\in A} \max_{\beta\in B} \langle p, f(t, x, \alpha, \beta)\rangle = \max_{\beta\in B} \min_{\alpha\in A} \langle p, f(t, x, \alpha, \beta)\rangle,$$

for any $p \in \mathbb{R}^n,\ (t, x) \in [0, T] \times \mathbb{R}^n$.

Proposition 3.1 ([14, 23]) *Under assumptions (f1)–(f5), the differential game (3.1)–(3.2) has a value function $c : (t, x) \to c(t, x)$ defined by the relation*

$$c(t, x) = \min_{\mathcal{A}} \max_{\substack{x(\cdot)\in \\ X_1(t,x,\mathcal{A})}} \gamma(x(\cdot)) = \max_{\mathcal{B}} \min_{\substack{x(\cdot)\in \\ X_2(t,x,\mathcal{B})}} \gamma(x(\cdot)).$$

Thus, the upper value of the game coincides with the lower one for all $(t, x) \in [0, T] \times \mathbb{R}^n$. The value function is bounded and Lipschitzian in t, x, i.e.

$$|c(t, x)| \leq C, \quad \textit{and} \quad |c(t_1, x_1) - c(t_2, x_2)| \leq L(|t_1 - t_2| + |x_1 - x_2|),$$

for any $(t, x),\ (t_i, x_i) \in [0, T] \times \mathbb{R}^n,\ i = 1, 2.$

Remark 3.2 The following relations hold:

$$\theta(t, x) \leq c(t, x) \leq \chi(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n. \tag{3.3}$$

The first inequality follows immediately from the definition of the functional γ. The second inequality is not obvious. It will be obtained below from the same inequality for approximating functions and from the point-wise convergence.

4 Differential inequalities

In [23], a pair of differential inequalities determining, under some additional assumptions, the value function of the game (3.1) with payoff functional γ_χ was introduced (see Theorem 6.5.2, p. 269). In particular, the directional differentiability of the value function was required. In [22], this requirement was relaxed, and the results were stated in terms of upper and lower directional derivatives (see Theorem 7.1, p. 130). Later, it was shown in [24] that the inequalities for the upper and lower directional derivatives are equivalent to the inequalities defining viscosity solutions. This equivalence is local, i.e. if the inequalities for the directional derivatives are fulfilled in some neighborhood of a point, then the corresponding inequalities for viscosity solutions are fulfilled at this point. Note that all these arguments can be easily extended to the game (3.1) with the payoff functional (3.2). To this end, it is sufficient to show that the υ-stability property of the value function should be checked only at positions (t, x) where $c(t, x) > \theta(t, x)$. The proof is the same as in Lemma 3.6.2 of [23]. Thus, the following proposition is valid.

Proposition 4.1 ([8, 22, 24]) *A continuous function $c : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is the value function of the game (3.1)–(3.2) iff the functions*

$$u(\tau, x) := c(T - \tau, x), \quad u_0(\tau, x) := \chi(T - \tau, x), \quad u_1(\tau, x) := \theta(T - \tau, x)$$

satisfy the following conditions:

a) For any point $(\tau, x) \in [0, T] \times \mathbb{R}^n$,

$$u(\tau, x) \geq u_1(\tau, x),\ u(0, x) = u_0(0, x).$$

b) For any point $(s_0, y_0) \in (0, T) \times \mathbb{R}^n$ *such that* $u(s_0, y_0) > u_1(s_0, y_0)$ *and any function* $\varphi \in C^1((0, T) \times \mathbb{R}^n)$ *such that* $u - \varphi$ *attains a local maximum at* (s_0, y_0)*, the following inequality holds:*

$$\frac{\partial \varphi}{\partial \tau}(s_0, y_0) + H(s_0, y_0, D\varphi(s_0, y_0)) \leq 0. \tag{4.1}$$

c) For any point $(s_0, y_0) \in (0, T) \times \mathbb{R}^n$ *such that* $u(s_0, y_0) < u_0(s_0, y_0)$ *and any function* $\varphi \in C^1((0, T) \times \mathbb{R}^n)$ *such that* $u - \varphi$ *attains a local minimum at* (s_0, y_0)*, the following inequality holds:*

$$\frac{\partial \varphi}{\partial \tau}(s_0, y_0) + H(s_0, y_0, D\varphi(s_0, y_0)) \geq 0. \tag{4.2}$$

Here

$$H(s, y, p) = -\max_{\beta \in B} \min_{\alpha \in A} \langle p, f(T - s, y, \alpha, \beta) \rangle$$

is the Hamiltonian of the differential game.

5 Approximation scheme

In this section, an abstract time step operator possessing certain properties is introduced, approximate solutions are constructed, and a convergence result is proved.

5.1 Abstract time step operator

Set $u(\tau, x) := c(T - \tau, x)$, $u_0(\tau, x) := \chi(T - \tau, x)$, $u_1(\tau, x) := \theta(T - \tau, x)$. Then, for u, u_0, u_1, assertions a)–c) of Proposition 4.1 are valid, and

$$\begin{aligned} |u_0(t_1, x_1) - u_0(t_2, x_2)| &\leq L_{00}(|t_1 - t_2| + |x_1 - x_2|), |u_0(t, x)| \leq C_{00}, \\ |u_1(t_1, x_1) - u_1(t_2, x_2)| &\leq L_{01}(|t_1 - t_2| + |x_1 - x_2|), |u_1(t, x)| \leq C_{01}, \\ |u(t_1, x_1) - u(t_2, x_2)| &\leq L(|t_1 - t_2| + |x_1 - x_2|), \quad |u(t, x)| \leq C \end{aligned}$$

for any (t, x), $(t_i, x_i) \in [0, T] \times \mathbb{R}^n$, $i = 1, 2$.

Let us introduce an operator (see [19]) $F(\tau, \rho, \cdot) : C_b^{0,1}(\mathbb{R}^n) \to C_b^{0,1}(\mathbb{R}^n)$, where $0 \leq \tau \leq T$, $\rho \geq 0$, satisfying the following conditions:

(F1) $F(\tau, 0, v) = v$.

(F2) The map $(\tau, \rho) \to F(\tau, \rho, v)$ is continuous w.r.t. the norm $\|\cdot\|$.

(F3) $F(\tau, \rho, v + k) = F(\tau, \rho, v) + k$ for any $k \in \mathbb{R}$.

(F4) $\|F(\tau, \rho, v) - v\| \leq C_1 \rho$, where C_1 may depend on $\|v\|$ and $\|Dv\|$.

(F5) If $v_1(x) \leq v_2(x)$ for every $x \in \mathbb{R}^n$, then

$$F(\tau, \rho, v_1)(x) \leq F(\tau, \rho, v_2)(x)$$

for every $x \in \mathbb{R}^n$.

(F6) There exists a constant $C_2 > 0$ such that

$$\|F(\tau, \rho, v)\| \leq e^{\rho C_2}(\|v\| + \rho C_2)$$

provided that $\|Dv\| \leq \bar{L} := 2L + 1$.

(F7) There exist constants $C_3, C_4 > 0$ such that

$$e^{T(C_3+C_4)}(L_{00} + TC_4) \leq \bar{L}$$

and

$$\|DF(\tau, \rho, v)\| \leq e^{\rho(C_3+C_4)}(\|Dv\| + \rho C_4)$$

provided that $\|v\| \leq e^{TC_2}(C_{00} + TC_2)$ and $\|Dv\| \leq \bar{L}$.

(F8) For every $\phi \in C_b^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ such that $|D\phi(x)| < 2L + 1$,

$$\left| \frac{F(\tau, \rho, \phi)(x) - \phi(x)}{\rho} + H(\tau, x, D\phi(x)) \right| \leq C_5 \cdot (1 + \|D\phi\| + \|D^2\phi\|)\rho.$$

5.2 Example of an appropriate time step operator

An operator satisfying conditions (F1)–(F8) is proposed in [15]. It originally possesses the monotonicity property (F5) and does not require any transformation of variables to ensure this property. Although the idea of the operator proposed is brilliant, the convergence arguments given in [15] are not reliable. They are solely based on topological considerations and do not take into account the nature of viscosity solutions. The operator is given by

$$F(\tau, \rho, v)(x) = v(x) + \rho \max_{\beta \in B} \min_{\alpha \in A} \sum_{i=1}^{n} \left(p_i^L \cdot f_i^+ + p_i^R \cdot f_i^-\right), \tag{5.1}$$

where

$$\begin{aligned} f_i &= f_i(T - \tau, x_1, \ldots, x_n, \alpha, \beta), \\ p_i^L &= \frac{v(x_1, \ldots, x_i, \ldots, x_n) - v(x_1, \ldots, x_i - \Delta x_i, \ldots, x_n)}{\Delta x_i}, \\ p_i^R &= \frac{v(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - v(x_1, \ldots, x_i, \ldots, x_n)}{\Delta x_i}, \end{aligned}$$

$\Delta x_i = a_i \rho$, $f_i^+ = \max(f_i, 0)$, and $f_i^- = \min(f_i, 0)$, $i = \overline{1, n}$. Let M be the bound of the right-hand side of the controlled system. If $\Delta x_i / \rho \equiv a_i \geq \sqrt{n} \cdot M$, $i = \overline{1, n}$, then the operator $F(\tau, \rho)$ given by (5.1) is monotone (see [5]).

Remark 5.1 Note that, if ρ is fixed, the time step operator can be restricted to functions defined on rectangular grids with the step Δx_i in ith coordinate, $i = \overline{1, n}$. Therefore, this operator will yield fully discrete finite difference schemes when used in the approximation procedure considered below.

Remark 5.2 If the function f is linear in α and β for all fixed (t, x), then the operations "min" and "max" in the definition of the operator (5.1) can be computed over ext(A) and ext(B), respectively, where "ext" returns the set of the extremal points. In particular, "max min" can be computed over the set of vertices, if A and B are polyhedrons (see [5]). Note that this remark is very important for numerical implementations of the operator (5.1) because the operation "max min" is applied to functions that are nonlinear and non convex (concave) in α and β.

5.3 Approximation of the value function

Let $P = \{0 = t_0 < t_1 < \cdots < t_{m(P)} = T\}$ be a partition of the time interval. Define an approximate solution $u_P : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\begin{aligned} u_P(0, x) &= \max\{u_0(0, x), u_1(0, x)\} = u_0(0, x), \\ u_P(\tau, x) &= \max\left\{\min\left\{u_0(\tau, x), F(\tau, \tau - t_{i-1}, u_P(t_{i-1}, \cdot))(x)\right\}, u_1(\tau, x)\right\}, \end{aligned} \tag{5.2}$$

where $\tau \in (t_{i-1}, t_i]$ for some $i = 1, 2, \ldots, m(P)$.

Lemma 5.3 *The following assertions are true:*

(a) For every $\tau \in [0, T]$,

$$\|u_P(\tau, \cdot)\| \le e^{\tau C_2}(C_{00} + \tau C_2),$$

and $u_P(\tau, \cdot) \in C_b^{0,1}(\mathbb{R}^n)$ with

$$\|Du_P(\tau, \cdot)\| \le e^{\tau(C_3+C_4)}(L_{00} + \tau C_4).$$

Moreover, if $t_i > \tau \ge t_{i-1}$ for some $i = 1, \ldots, m(P)$, then

$$\|u_P(\tau, \cdot) - u_P(t_{i-1}, \cdot)\| \le \max\{L_{00},\ \bar{C}_1\}(\tau - t_{i-1}),$$

where $\bar{C}_1 = C_1(e^{TC_2}(C_{00} + TC_2),\ \bar{L})$.

(b) u_P is bounded and uniformly continuous on $[0, T] \times \mathbb{R}^n$.

(c) For any $(\tau, x) \in [0, T] \times \mathbb{R}^n$, the function $u_P(\tau, x)$ satisfies the following inequality

$$u_1(\tau, x) \le u_P(\tau, x) \le u_0(\tau, x). \tag{5.3}$$

The proof can be easily obtained from the properties of the operator F and the definition of u_P.

5.4 Point-wise convergence of the approximation

Theorem 5.4 *Under assumptions (f1)–(f5) and (F1)–(F8), the following estimate holds:*

$$|u(\tau, x) - u_P(\tau, x)| \le K|P|^{1/2}, \quad (\tau, x) \in [0, T] \times \mathbb{R}^n.$$

The constant K depends only on the constants involved into conditions (F1)–(F8) and the constants M, N, C_{00}, L_{00}, C_{01}, and L_{01} characterizing the functions f, u_0, and u_1.

Proof: As it was mentioned in the introduction, this paper generalizes the results of [19] and [4]. Many arguments in the proof of Theorem 5.4 are close to these used in [19] and [4]. Nevertheless, we will not avoid possible repetitions for the sake of self-completeness of the paper.

It suffices to show that there exists a constant K_1 depending on the above mentioned constants such that

$$M_P = \sup_{(\tau,x)\in[0,T]\times\mathbb{R}^n} (u_P(\tau,x) - u(\tau,x))^{\pm} \le K_1|P|^{1/2},$$

where $r^+ = \max(r, 0)$ and $r^- = \max(-r, 0)$.

Note that, when writing "$\pm$", we bear in mind that the cases "$+$" and "$-$" should be considered separately. We use the notation "$\pm$" whenever definitions or derived results hold in both cases. For brevity, we do not supply expressions (e.g. M_P) which appear in the both cases with "+" or "–".

Without any loss of generality, we may assume

$$M_P > 0.$$

By Lemma 5.3, there exists an independent on the partition P constant $\Re_1$ such that

$$|u_P(\tau, x)| \le \Re_1, \quad \forall(\tau, x) \in [0, T] \times \mathbb{R}^n.$$

With $\Re := \max(\Re_1, C)$ and $\varepsilon := |P|^{1/4}$, consider a function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T] \to \mathbb{R}$ defined as follows:

$$\Phi(x, y, \tau, s) = (u_P(\tau, x) - u(s, y))^{\pm} + 4(\Re+1)\beta_\varepsilon(x-y) + 4(\Re+1)\gamma_\varepsilon(\tau-s) - \frac{\tau+s}{4T} M_P.$$

Here, $\beta_\varepsilon(\cdot) = \beta(\cdot/\varepsilon)$ and $\gamma_\varepsilon(\cdot) = \gamma(\cdot/\varepsilon)$, where β and γ are functions satisfying the conditions:

$$\begin{gathered}
\beta \in C_0^2(\mathbb{R}^n), \qquad \beta(0) = 1, \qquad 0 \le \beta \le 1, \\
\|D\beta\| \le 2, \qquad \|D^2\beta\| \le 15, \qquad \beta(w) = 0 \quad \text{for} \quad |w| > 1, \\
\beta(w) = 1 - |w|^2 \quad \text{for} \quad |w| \le \sqrt{\frac{3}{4}}, \qquad \beta(w) < \frac{1}{4} \quad \text{for} \quad |w| > \sqrt{\frac{3}{4}}, \\
\gamma \in C_0^2(R), \qquad \gamma(0) = 1, \qquad 0 \le \gamma \le 1, \\
|D\gamma| \le 2, \qquad |D^2\gamma| \le 15, \qquad \gamma(t) = 0 \quad \text{for} \quad |t| > 1, \\
\gamma(t) = 1 - t^2 \quad \text{for} \quad |t| \le \sqrt{\frac{3}{4}}, \qquad \gamma(t) < \frac{1}{4} \quad \text{for} \quad |t| > \sqrt{\frac{3}{4}}.
\end{gathered}$$

Note that the restrictions on the first and the second derivatives of β and γ are compatible with the other required properties of these functions.

Consider a perturbed function

$$\Psi(x, y, \tau, s) = \Phi(x, y, \tau, s) + 2\delta\xi_\delta(x, y),$$

where $\delta > 0$ is an arbitrary small parameter, and $\xi_\delta \in C_0^2(\mathbb{R}^n \times \mathbb{R}^n)$ is a function such that the function Ψ assumes its maximum value on $\mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]$. Such a function exists. Really, since Φ is bounded on $\mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]$, for any $\delta > 0$ there exists a point $(x_1, y_1, \tau_1, s_1) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]$ such that

$$\Phi(x_1, y_1, \tau_1, s_1) > \sup\{\Phi(x, y, \tau, s) : (x, y, \tau, s) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]\} - \delta.$$

Let us choose $\xi_\delta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the conditions:

$$0 \le \xi_\delta \le 1,\ \xi_\delta(x_1, y_1) = 1,\ \|D\xi_\delta\| \le 1,\ \|D^2\xi_\delta\| \le 1.$$

Obviously, $\Psi = \Phi$ off the support of ξ_δ, and

$$\begin{aligned}\Psi(x_1, y_1, \tau_1, s_1) &= \Phi(x_1, y_1, \tau_1, s_1) + 2\delta \\ &> \sup\{\Phi(x, y, \tau, s) : (x, y, \tau, s) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]\} + \delta \\ &> \sup\{\Psi(x, y, \tau, s) : (x, y, \tau, s) \in (\mathbb{R}^n \times \mathbb{R}^n \setminus \mathrm{supp}(\xi_\delta)) \times [0, T] \times [0, T]\}.\end{aligned}$$

Therefore, there is a point $(x_0, y_0, \tau_0, s_0) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]$ such that

$$\Psi(x_0, y_0, \tau_0, s_0) \ge \Psi(x, y, \tau, s) \tag{5.4}$$

for every $(x, y, \tau, s) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \times [0, T]$. Note that, although the maximizing point depends on δ, the next considerations do not use this fact. Therefore, the maximizing point is not supplied by the index δ.

The idea to consider a function of the duplicate set of variables (the function Ψ in our case) was proposed in [8] to estimate difference of two viscosity solutions. This idea was used in [19] to obtain error estimates for finite difference schemes. With the use of the function Ψ, we will construct some function satisfying inequality (4.2) of Proposition 4.1 at (s_0, y_0). Using Ψ, we will construct another function which approximately satisfies inequality (4.1) of Proposition 4.1 at (τ_0, x_0). These two functions will provide the desired estimate of $(u_P - u)^+$. The estimate of $(u_P - u)^-$ will be obtained in the similar way.

To continue the proof of the theorem, we have to estimate the distance between the points (τ_0, x_0) and (s_0, y_0).

Lemma 5.5 *For $\delta < \min\{\frac{3}{4}, \frac{1}{8}M_P\}$, the following inequalities hold:*

$$\begin{aligned} &|x_0 - y_0| \le \sqrt{3/4}\,\varepsilon, && |\tau_0 - s_0| \le \sqrt{3/4}\,\varepsilon, \\ &|x_0 - y_0| \le \frac{L + 2\delta}{4(\Re + 1)}\varepsilon^2, && |\tau_0 - s_0| \le \frac{L + \Re/T}{4(\Re + 1)}\varepsilon^2, \end{aligned} \tag{5.5}$$

$$0 < \frac{1}{4}M_P \le (u_P(\tau_0, x_0) - u(s_0, y_0))^{\pm}. \tag{5.6}$$

Proof: Let us estimate $|x_0 - y_0|$. We have

$$\begin{aligned}
&\Psi(x_0, y_0, \tau_0, s_0) - \Psi(x_0, x_0, \tau_0, s_0)\\
&\quad = (u_P(\tau_0, x_0) - u(s_0, y_0))^{\pm} - (u_P(\tau_0, x_0) - u(s_0, x_0))^{\pm}\\
&\qquad + 2\delta\xi_\delta(x_0, y_0) - 2\delta\xi_\delta(x_0, x_0) + 4(\Re + 1)\beta_\varepsilon(x_0 - y_0) - 4(\Re + 1)\\
&\quad \le |u(s_0, y_0) - u(s_0, x_0)| + 2\delta(\xi_\delta(x_0, y_0) - \xi_\delta(x_0, x_0))\\
&\qquad + 4(\Re + 1)\beta_\varepsilon(x_0 - y_0) - 4(\Re + 1).
\end{aligned}$$

If $|x_0 - y_0| > \sqrt{3/4}\varepsilon$, then $\beta_\varepsilon(x_0 - y_0) < 1/4$, and with $\delta < 3/4$ we obtain

$$\Psi(x_0, y_0, \tau_0, s_0) < \Psi(x_0, x_0, \tau_0, s_0)$$

due to the choice of $\Re$. The last inequality contradicts to (5.4). Thus, we have $|x_0 - y_0| \le \sqrt{3/4}\varepsilon$, and hence

$$\begin{aligned}
&\Psi(x_0, y_0, \tau_0, s_0) - \Psi(x_0, x_0, \tau_0, s_0)\\
&\quad \le (L + 2\delta)|x_0 - y_0| - 4(\Re + 1)\frac{|x_0 - y_0|^2}{\varepsilon^2}\\
&\quad = |x_0 - y_0|\frac{4(\Re + 1)}{\varepsilon^2}\left(\frac{L + 2\delta}{4(\Re + 1)}\varepsilon^2 - |x_0 - y_0|\right).
\end{aligned}$$

The last formula shows that

$$|x_0 - y_0| \le \frac{L + 2\delta}{4(\Re + 1)}\varepsilon^2,$$

since otherwise $\Psi(x_0, y_0, \tau_0, s_0) < \Psi(x_0, x_0, \tau_0, s_0)$, which contradicts to (5.4).

Let us estimate $|\tau_0 - s_0|$. We have

$$\begin{aligned}
&\Psi(x_0, y_0, \tau_0, s_0) - \Psi(x_0, y_0, \tau_0, \tau_0)\\
&\quad = (u_P(\tau_0, x_0) - u(s_0, y_0))^{\pm} - (u_P(\tau_0, x_0) - u(\tau_0, y_0))^{\pm}\\
&\qquad - \frac{\tau_0 + s_0}{4T}M_P + \frac{\tau_0 + \tau_0}{4T}M_P + 4(\Re + 1)\gamma_\varepsilon(\tau_0 - s_0) - 4(\Re + 1)\\
&\quad \le |u(\tau_0, y_0) - u(s_0, y_0)| + \frac{|\tau_0 - s_0|}{4T}M_P\\
&\qquad + 4(\Re + 1)\gamma_\varepsilon(\tau_0 - s_0) - 4(\Re + 1).
\end{aligned}$$

If $|\tau_0 - s_0| > \sqrt{3/4}\varepsilon$, then $\gamma_\varepsilon(\tau_0 - s_0) < 1/4$, and we get

$$\Psi(x_0, y_0, \tau_0, s_0) < \Psi(x_0, y_0, \tau_0, \tau_0)$$

due to the choice of $\Re$, which contradicts to (5.4). Thus, we have $|\tau_0 - s_0| \le \sqrt{3/4}\varepsilon$, and therefore

$$\begin{aligned}
\Psi(x_0, y_0, \tau_0, s_0) - \Psi(x_0, y_0, \tau_0, \tau_0) &\le (L + \Re/T)|\tau_0 - s_0| - 4(\Re + 1)\frac{|\tau_0 - s_0|^2}{\varepsilon^2}\\
&= |\tau_0 - s_0|\frac{4(\Re + 1)}{\varepsilon^2}\left(\frac{L + \Re/T}{4(\Re + 1)}\varepsilon^2 - |\tau_0 - s_0|\right).
\end{aligned}$$

The last formula implies that

$$|\tau_0 - s_0| \le \frac{L + \Re/T}{4(\Re + 1)}\varepsilon^2,$$

since otherwise $\Psi(x_0, y_0, \tau_0, s_0) < \Psi(x_0, y_0, \tau_0, \tau_0)$, which contradicts to (5.4).

Moreover, from the definition of Ψ and (5.4), we have

$$\begin{aligned}(u_P(\tau_0, x_0) - u(s_0, y_0))^{\pm} &+ 8(\Re + 1) + 2\delta \\ &\ge \Psi(x_0, y_0, \tau_0, s_0) \ge \Psi(x, x, \tau, \tau) \\ &\ge (u_P(\tau, x) - u(\tau, x))^{\pm} + 8(\Re + 1) - \frac{1}{2}M_P\end{aligned}$$

for any $(\tau, x) \in [0, T] \times \mathbb{R}^n$. Hence

$$(u_P(\tau_0, x_0) - u(s_0, y_0))^{\pm} + 2\delta \ge \frac{1}{2}M_P,$$

and, for $\delta < \frac{1}{8}M_P$, we have

$$(u_P(\tau_0, x_0) - u(s_0, y_0))^{\pm} \ge \frac{1}{4}M_P > 0,$$

which proves Lemma 5.5 and allows us to continue the proof of the theorem. □

1. Consider the case "+", that is, estimate the value

$$M_P = \sup_{(\tau,x)\in[0,T]\times\mathbb{R}^n} (u_P(\tau, x) - u(\tau, x))^+ .$$

We begin with the case $\underline{\tau_0 = 0,\, s_0 \ge 0}$. By Lemma 5.5, we have

$$\begin{aligned}0 &< \frac{1}{4}M_P \le (u_P(0, x_0) - u(s_0, y_0))^+ = u_P(0, x_0) - u(s_0, y_0) \\ &= u_0(0, x_0) - u(0, y_0) + u(0, y_0) - u(s_0, y_0) \\ &\le L_{00}|x_0 - y_0| + L|\tau_0 - s_0| \le \left(L_{00}\frac{L + 2\delta}{4(\Re + 1)} + L\frac{L + \Re/T}{4(\Re + 1)}\right)|P|^{1/2}.\end{aligned}$$

Note that the constants C and L are functions of $M, N, C_{00}, C_{01}, L_{00}, L_{01}$, and T (see [23]). Hence, $\Re$ is a function of $M, N, C_{00}, C_{01}, L_{00}, L_{01}$, and T, and therefore the assertion of the theorem is true in this case.

Now, consider the main case $\underline{\tau_0 > 0,\, s_0 \ge 0}$. Note that the relations $u_1 \le u$ and $u_1 \le u_P \le u_0$, see (3.3) and (5.3), restrict the consideration to the following subcases:

(a) Assume $u(s_0, y_0) > u_0(s_0, y_0)$. Lemma 5.3 provides that $u_P(\tau_0, x_0) \le u_0(\tau_0, x_0)$, which yields

$$0 < u_P(\tau_0, x_0) - u(s_0, y_0) \le u_0(\tau_0, x_0) - u_0(s_0, y_0),$$

and with Lemma 5.5 we obtain the required estimate

$$M_P \le 4L_{00}\left(\frac{L+2\delta}{4(\Re+1)}+\frac{L+\Re/T}{4(\Re+1)}\right)|P|^{1/2}.$$

Taking into account the above remark on the constants C, L, and $\Re$, we get the proof.

(b) If $u_P(\tau_0, x_0) = u_1(\tau_0, x_0)$, then from (5.6) and (3.3) we obtain

$$u_1(\tau_0, x_0) = u_P(\tau_0, x_0) > u(s_0, y_0) \ge u_1(s_0, y_0).$$

Therefore,

$$0 > u(s_0, y_0) - u_P(\tau_0, x_0) \ge u_1(s_0, y_0) - u_1(\tau_0, x_0),$$

and with Lemma 5.5 we obtain the required estimate

$$M_P \le 4L_{01}\left(\frac{L+2\delta}{4(\Re+1)}+\frac{L+\Re/T}{4(\Re+1)}\right)|P|^{1/2}.$$

Taking into account the above remark on the constants C, L, and $\Re$, we get the proof.

(c) If $u(s_0, y_0) = u_0(s_0, y_0)$, then from (5.6) and (5.3) we obtain

$$u_0(s_0, y_0) = u(s_0, y_0) < u_P(\tau_0, x_0) \le u_0(\tau_0, x_0).$$

Therefore,

$$0 < u_P(\tau_0, x_0) - u(s_0, y_0) \le u_0(\tau_0, x_0) - u_0(s_0, y_0),$$

and with Lemma 5.5 we obtain the required estimate

$$M_P \le 4L_{00}\left(\frac{L+2\delta}{4(\Re+1)}+\frac{L+\Re/T}{4(\Re+1)}\right)|P|^{1/2}.$$

Taking into account the above remark on the constants C, L, and $\Re$, we get the proof.

(d) Assume now $u(s_0, y_0) < u_0(s_0, y_0)$ and $u_P(\tau_0, x_0) > u_1(\tau_0, x_0)$, which in particular implies $s_0 > 0$. The treatment of this subcase is divided into two steps.

Step 1. Consider the function

$$\begin{aligned}\varphi(s, y) &= u_P(\tau_0, x_0) + 4(\Re+1)\beta_\varepsilon(x_0 - y) + 4(\Re+1)\gamma_\varepsilon(\tau_0 - s)\\ &\quad + 2\delta\xi_\delta(x_0, y) - \frac{\tau_0 + s}{4T} M_P.\end{aligned}$$

It is clear that $u(s, y) - \varphi(s, y) \ge -\Psi(x_0, y, \tau_0, s)$ and $u(s_0, y_0) - \varphi(s_0, y_0) = -\Psi(x_0, y_0, \tau_0, s_0)$. Therefore, $u - \varphi$ attains a minimum at (s_0, y_0). Substitution of φ in (4.2) yields

$$0 \le -4(\Re+1)\gamma_\varepsilon'(\tau_0 - s_0) - \frac{1}{4T}M_P + H(s_0, y_0, D\varphi(s_0, y_0)),$$

and therefore

$$\frac{1}{4T}M_P \leq -4(\Re+1)\gamma_\varepsilon'(\tau_0 - s_0) + H(s_0, y_0, D\varphi(s_0, y_0)). \tag{5.7}$$

Step 2. Consider now the function

$$\psi(\tau, x) = u(s_0, y_0) - 4(\Re+1)\beta_\varepsilon(x - y_0) - 4(\Re+1)\gamma_\varepsilon(\tau - s_0) \\ - 2\delta\xi_\delta(x, y_0) + \frac{\tau + s_0}{4T}M_P.$$

Lemma 5.6 *The following estimates hold:*

$$|D\psi(\tau, x)| \leq 8(\Re+1)\left(\frac{1}{\varepsilon} + \delta\right), \quad |D^2\psi(\tau, x)| \leq 60(\Re+1)\left(\frac{1}{\varepsilon^2} + \delta\right), \tag{5.8}$$

$$|D\psi(\tau, x_0)| \leq 2L + 6\delta. \tag{5.9}$$

Proof: By the definition of ψ, we have

$$D\psi(\tau, x) = -4(\Re+1)D\beta_\varepsilon(x - y_0) - 2\delta D\xi_\delta(x, y_0),$$

where the right hand side does not depend on τ. From the choice of the functions β_ε and ξ_δ, we have (5.8). Moreover, using Lemma 5.5, we obtain

$$|D\psi(\tau, x_0)| = |-4(\Re+1)D\beta_\varepsilon(x_0 - y_0) - 2\delta D_x\xi_\delta(x_0, y_0)| \\ \leq \varepsilon^{-2}8(\Re+1)|x_0 - y_0| + 2\delta \leq 2L + 6\delta.$$

□

Continue the consideration of *Step 2*. Observe that $u_P(\tau, x) - \psi(\tau, x) \leq \Psi(x, y_0, \tau, s_0)$ and $u_P(\tau_0, x_0) - \psi(\tau_0, x_0) = \Psi(x_0, y_0, \tau_0, s_0)$. Hence $u_P - \psi$ attains a maximum at (τ_0, x_0). Therefore, for t_{i-1} such that $\tau_0 \in (t_{i-1}, t_i]$, the inequality

$$(u_P - \psi)|_{(t_{i-1}, x)} \leq (u_P - \psi)|_{(\tau_0, x_0)}$$

holds for any $x \in \mathbb{R}^n$. In other words:

$$u_P(t_{i-1}, x) \leq \psi(t_{i-1}, x) + (u_P(\tau_0, x_0) - \psi(\tau_0, x_0)). \tag{5.10}$$

Using definition (5.2) and bearing in mind the assumption $u_P(\tau_0, x_0) > u_1(\tau_0, x_0)$ yields

$$u_P(\tau_0, x_0) = \min\{u_0(\tau_0, x_0), F(\tau_0, \tau_0 - t_{i-1}, u_P(t_{i-1}, \cdot))(x_0)\}.$$

Taking into account properties (F3) and (F5) of the operator F and inequality (5.10), we obtain

$$u_P(\tau_0, x_0) \leq F(\tau_0, \tau_0 - t_{i-1}, u_P(t_{i-1}, \cdot))(x_0) \\ \leq F(\tau_0, \tau_0 - t_{i-1}, \psi(t_{i-1}, \cdot))(x_0) + (u_P(\tau_0, x_0) - \psi(\tau_0, x_0)).$$

Adding to the both sides the term $\psi(t_{i-1}, x_0)$ yields

$$0 \le \frac{F(\tau_0, \tau_0 - t_{i-1}, \psi(t_{i-1}, \cdot))(x_0) - \psi(t_{i-1}, x_0)}{\tau_0 - t_{i-1}} + \frac{\psi(t_{i-1}, x_0) - \psi(\tau_0, x_0)}{\tau_0 - t_{i-1}}. \quad (5.11)$$

From the definition of ψ, we have

$$\frac{\psi(t_{i-1}, x_0) - \psi(\tau_0, x_0)}{\tau_0 - t_{i-1}} = 4(\Re + 1)\frac{\gamma_\varepsilon(\tau_0 - s_0) - \gamma_\varepsilon(t_{i-1} - s_0)}{\tau_0 - t_{i-1}} - \frac{1}{4T}M_P.$$

Then, from (5.11), (F8), and the last equality, we obtain

$$\begin{aligned} \frac{1}{4T}M_P \le\ & 4(\Re + 1)\frac{\gamma_\varepsilon(\tau_0 - s_0) - \gamma_\varepsilon(t_{i-1} - s_0)}{\tau_0 - t_{i-1}} \\ & - H(\tau_0, x_0, D\psi(t_{i-1}, x_0)) + C_5 \cdot (1 + \|D\psi(t_{i-1}, \cdot)\| \\ & + \|D^2\psi(t_{i-1}, \cdot)\|)|P|, \end{aligned} \quad (5.12)$$

provided that $\delta < \frac{1}{6}$ (see (5.9) and (F8)). Now we add (5.7) and (5.12) to obtain

$$\begin{aligned} \frac{1}{2T}M_P \le\ & 4(\Re + 1)\left(\frac{\gamma_\varepsilon(\tau_0 - s_0) - \gamma_\varepsilon(t_{i-1} - s_0)}{\tau_0 - t_{i-1}} - \gamma_\varepsilon'(\tau_0 - s_0)\right) \\ & + H(s_0, y_0, D\varphi(s_0, y_0)) - H(\tau_0, x_0, D\psi(t_{i-1}, x_0)) \\ & + C_5 \cdot (1 + \|D\psi(t_{i-1}, \cdot)\| + \|D^2\psi(t_{i-1}, \cdot)\|)|P|. \end{aligned}$$

Observing that

$$|D\varphi(s_0, y_0) - D\psi(t_{i-1}, x_0)| \le 4\delta,$$

$$|H(t_1, x_1, p_1) - H(t_2, x_2, p_2)| \le |p_2|N(|t_1 - t_2| + |x_1 - x_2|) + M|p_1 - p_2|,$$

where the constants N and M are defined in (f2) and (f3), and taking into account (5.8), (5.9), and the estimate

$$|D^2\gamma_\varepsilon(\tau)| \le \frac{15}{\varepsilon^2} \quad (5.13)$$

yield

$$\begin{aligned} \frac{1}{2T}M_P \le\ & 60(\Re + 1)\frac{|P|}{\varepsilon^2} + (2L + 6\delta)N\left(\frac{L + 2\delta}{4(\Re + 1)} + \frac{L + \Re/T}{4(\Re + 1)}\right)\varepsilon^2 + 4\delta M \\ & + C_5 \cdot \left(1 + 8(\Re + 1)\left(\frac{1}{\varepsilon} + \delta\right) + 60(\Re + 1)\left(\frac{1}{\varepsilon^2} + \delta\right)\right)|P|. \end{aligned}$$

Using the fact that $\varepsilon = |P|^{1/4}$, assuming $|P| < 1$, and letting δ tend to 0, we obtain

$$M_P \le K_1|P|^{1/2},$$

where

$$K_1 = 2T\left(60(\Re+1) + \frac{2LN(2L+\Re/T)}{4(\Re+1)} + C_5\cdot(1+68(\Re+1))\right).$$

2. Consider the case "–", that is, estimate the value

$$M_P = \sup_{(\tau,x)\in[0,T]\times\mathbb{R}^n} \left(u_P(\tau,x) - u(\tau,x)\right)^- .$$

We begin with the case $\underline{\tau_0 \geq 0,\ s_0 = 0}$. By Lemma 5.5 and with the use of Lemma 5.3, we have

$$\begin{aligned} 0 &< \frac{1}{4}M_P \leq (u_P(\tau_0,x_0) - u(0,y_0))^- = u(0,y_0) - u_P(\tau_0,x_0) \\ &= u_0(0,y_0) - u(0,x_0) + u(0,x_0) - u_P(\tau_0,x_0) \\ &\leq L_{00}|x_0 - y_0| + \max\{L_{00},\ \bar{C}_1\}|\tau_0 - s_0| \\ &\leq \left(L_{00}\frac{L+2\delta}{4(\Re+1)} + \max\{L_{00},\ \bar{C}_1\}\frac{L+\Re/T}{4(\Re+1)}\right)|P|^{1/2}, \end{aligned}$$

which gives the proof in the subcase considered.

Now, we consider the main case $\underline{\tau_0 \geq 0, s_0 > 0}$. Note that the relations $u_1 \leq u$ and $u_1 \leq u_P \leq u_0$, see (3.3) and (5.3), restrict the consideration to the following subcases:

(a) If $u(s_0,y_0) = u_1(s_0,y_0)$, then

$$u_1(s_0,y_0) = u(s_0,y_0) > u_P(\tau_0,x_0) \geq u_1(\tau_0,x_0),$$

and hence

$$0 > u_P(\tau_0,x_0) - u(s_0,y_0) \geq u_1(\tau_0,x_0) - u_1(s_0,y_0).$$

From inequalities (5.5) and (5.6) of Lemma 5.5, we obtain

$$M_P \leq 4L_{01}\left(\frac{L+2\delta}{4(\Re+1)} + \frac{L+\Re/T}{4(\Re+1)}\right)|P|^{1/2},$$

which gives the proof.

(b) Assume $u(s_0,y_0) > u_1(s_0,y_0)$. The treatment of this subcase is divided into two steps.

Step 1. Consider the function

$$\begin{aligned} \varphi(s,y) &= u_P(\tau_0,x_0) - 4(\Re+1)\beta_\varepsilon(x_0-y) - 4(\Re+1)\gamma_\varepsilon(\tau_0-s) \\ &\quad - 2\delta\xi_\delta(x_0,y) + \frac{\tau_0+s}{4T}M_P. \end{aligned}$$

Obviously, $u(s,y)-\varphi(s,y)\le\Psi(x_0,y,\tau_0,s)$, and therefore $u-\varphi$ attains a maximum at (s_0,y_0). Hence we obtain from (4.1)

$$0\ge 4(\Re+1)\gamma_\varepsilon'(\tau_0-s_0)+\frac{1}{4T}M_P+H(s_0,y_0,D\varphi(s_0,y_0)),$$

and therefore

$$\frac{1}{4T}M_P\le -4(\Re+1)\gamma_\varepsilon'(\tau_0-s_0)-H(s_0,y_0,D\varphi(s_0,y_0)). \tag{5.14}$$

Step 2. Consider now the function

$$\psi(\tau,x)=u(s_0,y_0)+4(\Re+1)\beta_\varepsilon(x-y_0)+4(\Re+1)\gamma_\varepsilon(\tau-s_0)+2\delta\xi_\delta(x,y_0)-\frac{\tau+s_0}{4T}M_P.$$

Note that $u_P(\tau,x)-\psi(\tau,x)\ge-\Psi(x,y_0,\tau,s_0)$, and therefore $u_P-\psi$ attains a minimum at (τ_0,x_0). Thus, for the time instant t_{i-1} such that $\tau_0\in(t_{i-1},t_i]$, the inequality

$$(u_P-\psi)|_{(t_{i-1},x)}\ge(u_P-\psi)|_{(\tau_0,x_0)}$$

holds for any $x\in\mathbb{R}^n$, and hence

$$u_P(t_{i-1},x)\ge\psi(t_{i-1},x)+(u_P(\tau_0,x_0)-\psi(\tau_0,x_0)). \tag{5.15}$$

Note that the assertions of Lemma 5.6 hold for the function ψ and

$$u_P(\tau_0,x_0)\ge F(\tau_0,\tau_0-t_{i-1},u_P(t_{i-1},\cdot))(x_0).$$

Using Lemma 5.6 and the properties (F3) and (F5) of the operator F, we obtain from (5.15) that

$$\begin{aligned}u_P(\tau_0,x_0)&\ge F(\tau_0,\tau_0-t_{i-1},u_P(t_{i-1},\cdot))(x_0)\\&\ge F(\tau_0,\tau_0-t_{i-1},\psi(t_{i-1},\cdot))(x_0)+(u_P(\tau_0,x_0)-\psi(\tau_0,x_0)).\end{aligned} \tag{5.16}$$

Adding to the both sides of (5.16) the term $\psi(t_{i-1},x_0)$, we get

$$0\ge\frac{F(\tau_0,\tau_0-t_{i-1},\psi(t_{i-1},\cdot))(x_0)-\psi(t_{i-1},x_0)}{\tau_0-t_{i-1}}+\frac{\psi(t_{i-1},x_0)-\psi(\tau_0,x_0)}{\tau_0-t_{i-1}}. \tag{5.17}$$

From the definition of ψ, we have

$$\frac{\psi(t_{i-1},x_0)-\psi(\tau_0,x_0)}{\tau_0-t_{i-1}}=4(\Re+1)\frac{\gamma_\varepsilon(t_{i-1}-s_0)-\gamma_\varepsilon(\tau_0-s_0)}{\tau_0-t_{i-1}}+\frac{1}{4T}M_P.$$

Then, from (5.17), (F8), and the last equality, we obtain

$$\begin{aligned}\frac{1}{4T}M_P&\le 4(\Re+1)\frac{\gamma_\varepsilon(\tau_0-s_0)-\gamma_\varepsilon(t_{i-1}-s_0)}{\tau_0-t_{i-1}}\\&\quad+H(\tau_0,x_0,D\psi(t_{i-1},x_0))+C_5\cdot(1+\|D\psi(t_{i-1},\cdot)\|\\&\quad+\|D^2\psi(t_{i-1},\cdot)\|)|P|,\end{aligned} \tag{5.18}$$

provided that $\delta < \frac{1}{6}$ (see (5.9) and (F8)). Adding (5.14) and (5.18) yields

$$\begin{aligned}\frac{1}{2T}M_P \le\ & 4(\Re+1)\Big(\frac{\gamma_\varepsilon(\tau_0-s_0)-\gamma_\varepsilon(t_{i-1}-s_0)}{\tau_0-t_{i-1}}-\gamma'_\varepsilon(\tau_0-s_0)\Big)\\ & + H(\tau_0,x_0,D\psi(t_{i-1},x_0)) - H(s_0,y_0,D\varphi(s_0,y_0))\\ & + C_5\cdot(1+\|D\psi(t_{i-1},\cdot)\|+\|D^2\psi(t_{i-1},\cdot)\|)|P|.\end{aligned}$$

Observing that

$$|D\psi(t_{i-1},x_0)-D\varphi(s_0,y_0)|\le 4\delta,$$
$$|H(t_1,x_1,p_1)-H(t_2,x_2,p_2)|\le |p_1|N(|t_1-t_2)|+|x_1-x_2|)+M|p_1-p_2|,$$

where the constants N and M are defined in (f2) and (f3), and accounting for (5.8), (5.9), and (5.13) yield

$$\begin{aligned}\frac{1}{2T}M_P \le\ & 60(\Re+1)\frac{|P|}{\varepsilon^2}+(2L+6\delta)N\left(\frac{L+2\delta}{4(\Re+1)}+\frac{L+\Re/T}{4(\Re+1)}\right)\varepsilon^2+4\delta M\\ & + C_5\cdot\left(1+8(\Re+1)\left(\frac{1}{\varepsilon}+\delta\right)+60(\Re+1)\left(\frac{1}{\varepsilon^2}+\delta\right)\right)|P|.\end{aligned}$$

Using the fact that $\varepsilon=|P|^{1/4}$, assuming $|P|<1$, and letting δ tend to 0, we obtain

$$M_P\le K_1|P|^{1/2},$$

where

$$K_1=2T\left(60(\Re+1)+\frac{2LN(2L+\Re/T)}{4(\Re+1)}+C_5\cdot(1+68(\Re+1))\right).$$

Theorem 5.4 is proved. □

Remark 5.7 Note that inequalities (5.3) and the convergence proved show that

$$u_1(\tau,x)\le u(\tau,x)\le u_0(\tau,x),\quad \forall(\tau,x)\in[0,T]\times\mathbb{R}^n,$$

which coincides (up to the notation) with inequalities (3.3) of Remark 3.2.

6 Examples

Examples given in this section illustrate problems associated with functional (3.2) and show the efficiency of the upwind time step operator (5.1). We are going to compute both value functions and optimal trajectories of the players.

Let us describe the construction of optimal strategies of the players. The choice of optimal controls of the player is based on the procedure of extremal aiming proposed in [14]. Let ε be a small positive number, t_n the current time instant. Consider the ε-neighborhood

$$\mathcal{U}_\varepsilon=\{x\in\mathbb{R}^n:\ |x-x(t_n)|\le\varepsilon\}$$

of the current state $x(t_n)$ of system (3.1). By searching through all grid points $x^\# \in \mathcal{U}_\varepsilon$, find a point $x^\#_*$ such that

$$c_P(t_n, x^\#_*) = \min_{x^\# \in \mathcal{U}_\varepsilon} c_p(t_n, x^\#),$$

where c_p is a grid approximation of the value function. The current control $\alpha(t_n)$ ($\beta(t_n)$) which is supposed to be applied on the next time interval $[t_n, t_{n+1})$ is computed from the condition of maximizing (minimizing) the projection of the system velocity f onto the direction of the vector $s = x^\#_* - x(t_n)$. Thus,

$$\alpha(t_n): \quad \max_{\beta \in Q} \langle s,\, f(t_n, x(t_n), \alpha(t_n), \beta)\rangle = \min_{\alpha \in P}\max_{\beta \in Q} \langle s,\, f(t_n, x(t_n), \alpha, \beta)\rangle\,,$$
$$\beta(t_n): \quad \min_{\alpha \in P} \langle s,\, f(t_n, x(t_n), \alpha, \beta(t_n))\rangle = \max_{\beta \in Q}\min_{\alpha \in P} \langle s,\, f(t_n, x(t_n), \alpha, \beta)\rangle\,.$$

Example 6.1 (Nonlinear pendulum) The first example is a nonlinear pendulum with the dynamics

$$\dot{x}_1 = x_2 + \beta, \quad \dot{x}_2 = -gl \sin x_1 - \frac{\alpha}{l}\cos x_1, \quad |\alpha| \le 1, \ |\beta| \le 1. \tag{6.1}$$

Here x_1 is the angle deflection, x_2 the angle velocity, α the horizontal force governed by the first player, β an information disturbance which is at the disposal of the second player, l the length, and g the gravity acceleration. The unit mass is assumed. Let

$$\chi(x_1, x_2) = 10\sqrt{x_1^2 + x_2^2} \quad \text{and} \quad \theta(x_1, x_2) = 2.5|x_1|.$$

We consider the game of approach with the target set $M = \{(x_1, x_2) : \chi(x_1, x_2) \le 1\}$ and the state constraint set $N = \{(x_1, x_2) : \theta(x_1, x_2) \le 1\}$. The starting time is $t_0 = 0$, and the terminal time $T = 5$. According to our technique, the differential game with the dynamics (6.1) and the payoff functional (3.2) is solved. We assume that the starting state $x_0 = (x_{10}, x_{20})$ lies in the solvability set $W = \{(x_1, x_2) : c_P(t_0, x_1, x_2) \le 1\}$, where c_P is the computed approximation of the value function. An optimal trajectory shown in Figure 6.1 reaches the target set and does not violate the state constraint.

Example 6.2 (Material point) The next well-known example is the game of two material points on a line. In the relative coordinates, the dynamics is given by

$$\dot{x}_1 = x_2 + \beta, \quad \dot{x}_2 = \alpha, \quad |\alpha| \le 1, \ |\beta| \le 0.2\,. \tag{6.2}$$

Here x_1 is the relative position of the points, x_2 the relative velocity. The first player uses the control variable α to ensure the condition $\sqrt{x_1(\tau)^2 + x_2(\tau)^2} \le 0.2$ at some time instant τ provided that the state constraint $|x_2(t)| \le 0.5$ holds for all $t \le \tau$. The aim of the second player that uses the control variable β is opposite. According to our technique, we consider the differential game with the dynamics (6.2) and the payoff functional of the form (3.2) with

$$\chi(x_1, x_2) = 5\sqrt{x_1^2 + x_2^2} \quad \text{and} \quad \theta(x_1, x_2) = 2|x_2|.$$

The starting time is $t_0 = 0$, and the terminal time $T = 5$. Let c_P be the computed approximation of the value function.

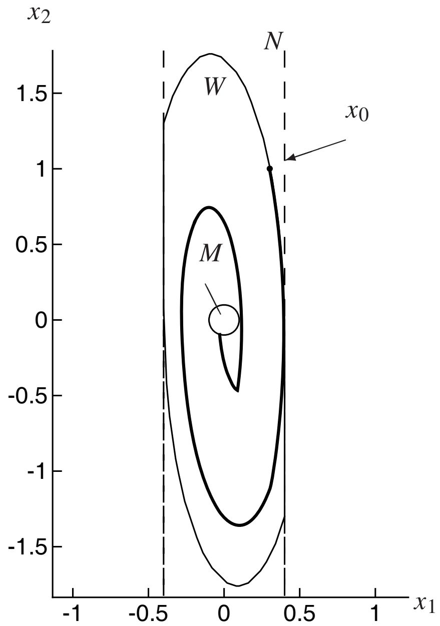

Figure 6.1 The solvability set W, the target set M, the state constraint N, and the optimal trajectory starting from x_0 and arriving at M.

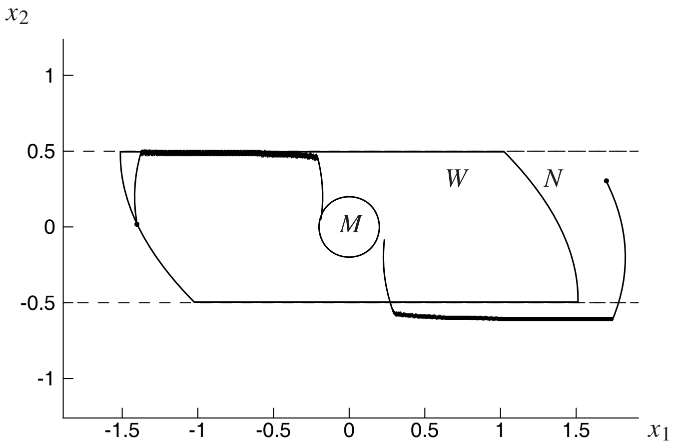

Figure 6.2 The solvability set W, the target set M, the state constraint N, the optimal trajectory for the first player (to the left), and the win trajectory for the second player (to the right).

Figure 6.2 shows a trajectory corresponding to optimal strategies of the players. The trajectory to the left starts from a point of the solvability set $W = \{(x_1, x_2) : c_P(t_0, x_1, x_2) \leq 1\}$. It reaches the target set and remains inside the state constraint. One can see a chattering regime appearing when the trajectory reach the boundary of the state constraint (see Figure 6.3 for the enlarged fragment). The trajectory to the right starts from a false point so that the second player wins.

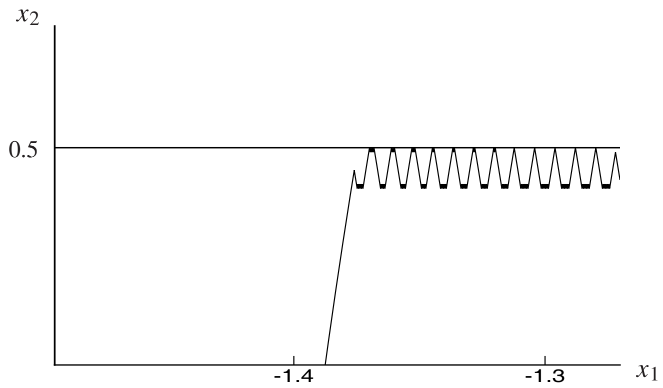

Figure 6.3 The enlarged fragment of Figure 6.2 showing the chattering regime.

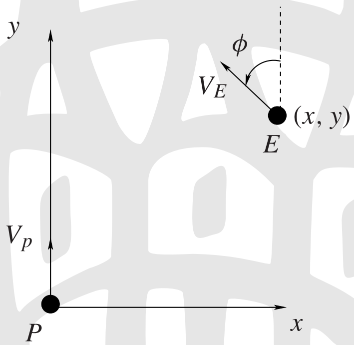

Figure 6.4 Movable coordinate system. Here, y is the axis collinear to the pursuer's velocity, ϕ the angle between the axis y and the evader's velocity, and x is such that $\{x, y, \phi\}$ is the orthogonal right coordinate system.

Example 6.3 (Game of two cars) The differential game of two cars was originally introduced in [13]. In this game, the first car pursuers the second one, both cars are moving in the plane. Similar to paper [16], we consider the case where the both cars have the same linear velocities and minimum turn radii. In a movable reference coordinate system (see Figure 6.4), the dynamics of the relative motion can be described as follows:

$$\dot{x} = -y\alpha + \sin\phi, \quad \dot{y} = x\alpha + \cos\phi - 1, \quad \dot{\phi} = -\alpha + \beta, \quad |\alpha| \leq 1, \ \ |\beta| \leq 1.$$

The termination conditions are

$$\sqrt{x^2+y^2} \le r, \quad \cos(\phi - \phi_f(x,y)) - \cos\phi_f(x,y) \le 0 \quad \text{at} \quad \sqrt{x^2+y^2} = r,$$

$$\text{where} \quad \phi_f(x,y) = \begin{cases} \arccos\left(x/\sqrt{x^2+y^2}\right), & y \ge 0, \\ \pi + \arccos\left(-x/\sqrt{x^2+y^2}\right), & y < 0, \end{cases}$$

which expresses the following capture rules: the distance between the cars is less than or equal to a given capture radius r, and the relative radial velocity on the termination is non-positive. The starting time is $t_0 = 0$, and the terminal time $T = 10$. We extend the game by introducing the state constraint $|\phi + 2| \le s$ on the third component of the state vector. To this end, the payoff functional of the form (3.2) with

$$\begin{aligned} \chi(x,y,\phi) &= \max\left\{\sqrt{x^2+y^2} - r, \ \cos(\phi - \phi_f(x,y)) - \cos\phi_f(x,y)\right\}, \\ \theta(x,y,\phi) &= |\phi + 2| - s \end{aligned}$$

is considered.

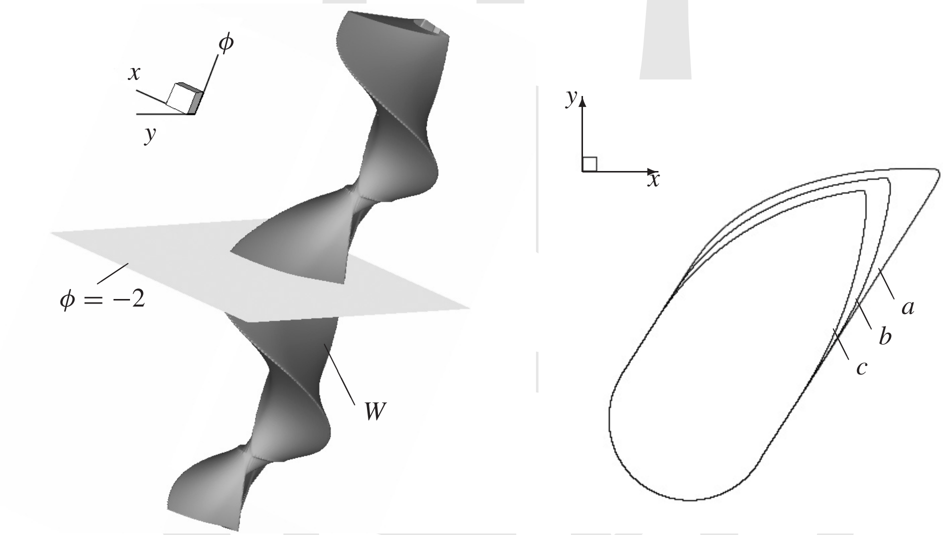

Figure 6.5 To the left. The solvability set W in the case of the absence of the state constraint and the cutting plane $\phi = -2$. To the right. The cross sections of three solvability sets computed: *a*) without the state constraint (the largest one); *b*) with the state constraint corresponding to $s = 1$ (the middle one); and *c*) with the state constraint corresponding to $s = 0.8$ (the smallest one).

Let c_P be a computed approximation of the value function. Figure 6.5 presents the solvability set defined by $\{(x,y,\phi) : c_P(t_0,x,y,\phi) \le 0\}$ and the section of this set by the plane $\phi = -2$. In the computation, the grid size was equal to $300 \times 300 \times 300$, the time step was equal to 10^{-3}, and the run time on a Linux SMP-computer with 30 threads was about 10 minutes. In the case of the absence of the state constraint, the results exactly coincide with those analytically found in [16].

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