Drag on spheres in micropolar fluids with nonzero boundary conditions for microrotations

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The Stokes formula for the resistance force exerted on a sphere moving with constant velocity in a fluid is extended to the case of micropolar fluids. A nonhomogeneous boundary condition for the microrotation vector is used: the microrotation on the boundary of the sphere is assumed proportional to the rotation rate of the velocity field on the boundary.

1. Introduction

The hydrodynamics of classical fluids is based on the assumption that the fluid particles do not have any internal structure. This results in the well known Navier-Stokes equations which describe a lot of hydrodynamical phenomena. Nevertheless, fluid particles may exhibit some microscopical effects such as rotation, shrinking, and etc. for some fluids like polymeric suspensions, animal blood, and etc. Therefore, the internal structure should be taken into account for fluids whose particles have complex shapes. Moreover, the internal structure plays a role even for ordinary fluids like water in models with small scales (see e.g. Papautsky et al. (1999)). A well accepted theory that accounts for internal structures of fluids is *micropolar fluid theory* by Eringen (see Eringen (1964), Eringen (1966), Stokes (1984), Straughan (2004)). Here, individual particles can rotate independently from the rotation and movement of the fluid as whole. Therefore, new variables which represent angular velocities of fluid particles and new equations governing this variables should be added to the conventional model.

The aim of this paper is to calculate the resistant force exerted on a sphere moving with a constant velocity in a micro-polar fluid and to compare the result as well with the conventional Stokes force derived from the classical hydrodynamics (see e.g. Landau and Lifshitz (1995)) as with similar results of other authors that have obtained such a formula in the case of homogeneous boundary conditions for the micro-rotation (compare Lakshmana (1970), Erdogan (1971), Ramkissoon (1976), Ramkissoon (1985), Hayakawa (2000)). This allows us to estimate the influence of the micro-rotation on the motion of rigid bodies in micropolar fluids for various boundary conditions posed on the variables describing micro-rotations.

The paper is structured as follows. First, a mathematical model of micropolar fluids is discussed and its most important features are outlined. Then, a formula for the resistent force is derived. Finally, the comparison with the conventional Stokes force is given for water and blood. Comparison with results of other authors are given in the conclusion.

2. Micropolar field equation

The most important feature of micropolar fluid theory is utilizing a non-symmetric stress tensor so that the conservation of angular momentum results in new equations describing rotation of fluid particles on the micro-scale. Such a stress tensor is given as follows (see e.g. Lukaszewicz (1999)):

$$T_{ij} = (-p + \zeta\, v_{k,k})\delta_{ij} + \nu(v_{i,j} + v_{j,i}) + \nu_R(v_{j,i} - v_{i,j}) - 2\nu_R \varepsilon_{mij}\omega_m. \tag{2.1}$$

In some papers, micropolar fluids are defined as being governed by such a tensor. Here, commas followed by indices denote differentiation with respect to the corresponding coordinates, δ_{ij} and ε_{mij} are the Christoffel and Levi-Civita symbols, respectively, and the summation over repeating indices is assumed. The meaning of the other variables and constants is explained below.

In the most general form, the micropolar field equations for incompressible and viscous fluids read (see Lukaszewicz (1999)):

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho = 0,\\
&\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} \right) = (\nu + \nu_r)\Delta\, \boldsymbol{v} - \nabla p + 2\nu_r \operatorname{curl} \boldsymbol{\omega} + \rho \boldsymbol{f},\\
&\rho I \left(\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{\omega} \right) = (c_a + c_d)\Delta \boldsymbol{\omega} + (c_0 + c_d - c_a)\nabla \operatorname{div} \boldsymbol{\omega} + 2\nu_r \left(\operatorname{curl} \boldsymbol{v} - 2\boldsymbol{\omega}\right) + \rho \boldsymbol{g},\\
&\operatorname{div} \boldsymbol{v} = 0,
\end{aligned}$$

where

ρ is the density,
$\boldsymbol{v}$ the velocity field,
$\boldsymbol{\omega}$ the microrotation field,
I the microinertia coefficient,
$\boldsymbol{f}$ body forces per unit mass,
$\boldsymbol{g}$ microrotation driving forces per unit mass,
p the hydrostatical pressure,
ν the classical viscosity coefficient,
ν_r the vortex viscosity coefficient,
c_a, c_d, c_0 are spin gradient viscosity coefficients.

The first equation represents the conservation of mass, the second and third ones describe the conservation of impulse and angular momentum, respectively. The last equation accounts for the incompressibility of the fluid. If $\nu_r = 0$, the conservation of impulse becomes independent on the microrotation. The system reduces to the classical Navier-Stokes equation, if ν_r, c_0, c_a, c_d and g vanish. Note that the choice of boundary conditions for micropolar fluids is not obvious. The boundary condition for the velocity field is the same as in the classical case. As for the microrotation, there is no general agreements in the literature (see Sec. 1.5(3) of Lukaszewicz (1999) and papers cited there). Very often, the Dirichlet boundary condition $\boldsymbol{\omega} = 0$ is used. Some authors propose the following dynamic boundary condition: $\boldsymbol{\omega} = \frac{\alpha}{2}\,\mathrm{curl}\,\boldsymbol{v}$ with $0 \le \alpha \le 1$ (see Lukaszewicz (1999)). This boundary condition for the microrotation and the no-slip boundary condition for the velocity field will be used in the paper presented.

3. Calculation of the resistance force

Assume equivalently that a sphere of radius R is immovable, whereas the fluid exhibits a steady-state flow with velocity $\boldsymbol{u}$ at infinity. The velocities and microrotations are assumed to be small so that the field equations become linear:

$$-(\nu+\nu_r)\Delta\,\boldsymbol{v} + \nabla p = 2\nu_r \mathrm{curl}\,\boldsymbol{\omega} \tag{3.1}$$

$$-(c_a+c_d)\Delta\boldsymbol{\omega} - (c_0+c_d-c_a)\nabla\,\mathrm{div}\,\boldsymbol{\omega} + 4\nu_r\boldsymbol{\omega} = 2\nu_r \mathrm{curl}\,\boldsymbol{v} \tag{3.2}$$

$$\mathrm{div}\,\boldsymbol{v} = 0, \tag{3.3}$$

Conditions at infinity read:

$$\boldsymbol{v} = \boldsymbol{u}, \tag{3.4}$$

$$\boldsymbol{\omega} = 0. \tag{3.5}$$

The boundary condition for the sphere are:

$$\boldsymbol{v} = 0, \tag{3.6}$$

$$\boldsymbol{\omega} = \frac{\alpha}{2}\mathrm{curl}\,\boldsymbol{v}, \quad \text{with} \quad 0 \leqq \alpha \leqq 1, \quad \text{at} \quad |x| = R. \tag{3.7}$$

The calculation of the resistance force is based on the explicit analytical representation of solutions to equations (3.1)-(3.3).

3.1. *Calculation of the velocity, microrotation, and pressure*

Equation (3.3) implies that $\mathrm{div}\,(\boldsymbol{v}-\boldsymbol{u}) = 0$. Hence, $\boldsymbol{v}$ can be expressed as follows:

$$\boldsymbol{v} = \boldsymbol{u} + \mathrm{curl}\,\boldsymbol{A},$$

where $\boldsymbol{A}$ is a vector field such that $\mathrm{curl}\,\boldsymbol{A}$ vanishes at infinity. Analogously to Landau and Lifshitz (1995), observe that $\boldsymbol{A}$ is a polar vector and take into account the symmetry of the sphere to conclude that $\boldsymbol{A} = f'(r)\boldsymbol{n}\times\boldsymbol{u}$, where f is a function of $r = (x^2+y^2+z^2)^{1/2}$. Therefore, the velocity $\boldsymbol{v}$ is of the form :

$$\boldsymbol{v} = \boldsymbol{u} + \mathrm{curl}(\nabla f \times \boldsymbol{u}) = \boldsymbol{u} + \mathrm{curl}\,\mathrm{curl} f\boldsymbol{u} \tag{3.8}$$

Taking the curl of $\boldsymbol{v}$ yields:

$$\mathrm{curl}\,\boldsymbol{v} = \mathrm{curl}\,\mathrm{curl}\,\mathrm{curl} f\boldsymbol{u} = (\nabla\,\mathrm{div} - \Delta)\mathrm{curl} f\boldsymbol{u} = -\Delta\mathrm{curl} f\boldsymbol{u} \tag{3.9}$$

Applying the curl operator to (3.2) and using the well-known formula $\mathrm{curl}\,\mathrm{curl} = \nabla\mathrm{div} - \Delta$ gives:

$$-(c_a+c_d)\Delta\mathrm{curl}\,\boldsymbol{\omega} + 4\nu_r\mathrm{curl}\,\boldsymbol{\omega} = 2\nu_r\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v} \tag{3.10}$$

Combining equations (3.1) and (3.10) results in:

$$\frac{(c_a+c_d)(\nu+\nu_r)}{2\nu_r}\Delta^2\boldsymbol{v} - \frac{(c_a+c_d)}{2\nu_r}\Delta\nabla p + 2(-(\nu+\nu_r)\Delta\,\boldsymbol{v} + \nabla p) = 2\nu_r\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v}$$

Applying now the curl operator to the last equation yields:

$$\frac{(c_a+c_d)(\nu+\nu_r)}{2\nu_r}\Delta^2\mathrm{curl}\,\boldsymbol{v} - 2(\nu+\nu_r)\Delta\mathrm{curl}\,\boldsymbol{v} = 2\nu_r\mathrm{curl}\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{v}$$

Using again the formula curl curl $= \nabla\text{div} - \Delta$ results in:

$$\frac{(c_a + c_d)(\nu + \nu_r)}{2\nu_r}\Delta^2 \text{curl}\,\boldsymbol{v} - 2\nu\Delta\text{curl}\,\boldsymbol{v} = 0$$

Introducing the notation $c_1 = (c_a + c_d)(\nu + \nu_r)/2\nu_r$, $c_2 = 2\nu$ and combining the last equations with (3.9) gives:

$$c_1\Delta^3 \text{curl}\, f\boldsymbol{u} - c_2\Delta^2 \text{curl}\, f\boldsymbol{u} = 0, \tag{3.11}$$

which can be rewritten as

$$(c_1\Delta - c_2\text{Id})\Delta^2(\nabla f \times \boldsymbol{u}) = 0$$

or

$$(c_1\Delta - c_2\text{Id})\Delta^2\nabla f \equiv \nabla(c_1\Delta - c_2\text{Id})\Delta^2 f = \lambda u,$$

where λ is a scalar function, Id denotes the identity operator. Considering spherical reference system r, θ, ϵ whose polar axis has the direction of $\boldsymbol{u}$ and taking into account that the function $\psi = (c_1\Delta - c_2\text{Id})\Delta^2 f$ depends only on r, one concludes that $(\nabla\psi)_\theta = (\nabla\psi)_\epsilon = 0$ at each point (r, θ, ϵ) in the local reference system formed by the tangents to the coordinate lines. On the other hand, $(\lambda\boldsymbol{u})_\theta = -\lambda|\boldsymbol{u}|\sin\theta$ in this local reference system, which implies that $\lambda \equiv 0$. Thus,

$$(c_1\Delta - c_2\text{Id})\Delta^2 f = \text{const.} \tag{3.12}$$

Equation (3.8) considered in the spherical reference system shows that only the second derivatives of f in r describe $\boldsymbol{v} - u$ at infinity and vanish there because $\boldsymbol{v} - \boldsymbol{u} \to 0$ as $r \to \infty$. Therefore all higher derivatives of f are expected to vanish at infinity too. Thus, the constant on the right-hand-side may assumed to be zero. Note, that this argumentation is not a strict proof but some physically reasonable consideration. It is shown below that this assumption leads to a unique solution of the problem.

Thus, equation (3.12) reduces to the following one

$$c_1\Delta g - c_2 g = 0,$$

where $g := \Delta^2 f$. Since $\Delta g = r^{-2}\frac{d}{dr}\left(r^2\frac{d}{dr}g\right)$ in spherical coordinates, the general solution of the last equation is of the form:

$$g(r) = \frac{\mathcal{A}e^{kr} + Be^{-kr}}{r},$$

where $k = \sqrt{\frac{c_2}{c_1}}$. Choose $\mathcal{A} = 0$ because g vanishes at infinity and integrate the equation

$$\Delta^2 f = \frac{Be^{-kr}}{r}$$

bearing in mind that the Laplace operator is considered in spherical coordinates. This yields:

$$f = \frac{1}{k^4}\frac{Be^{-kr}}{r} + ar + \frac{b}{r}. \tag{3.13}$$

Substitution of (3.13) into (3.8) yields:

$$\begin{aligned} \boldsymbol{v} = \boldsymbol{u}&\left(1 - \frac{a}{r} - \frac{b}{r^3} - Be^{-kr}\left(\frac{1}{k^4r^3} + \frac{1}{k^3r^2} + \frac{1}{k^2r}\right)\right) + \\ \boldsymbol{n}(\boldsymbol{un})&\left(-\frac{a}{r} + \frac{3b}{r^3} + Be^{-kr}\left(\frac{3}{k^4r^3} + \frac{3}{k^3r^2} + \frac{1}{k^2r}\right)\right). \end{aligned} \tag{3.14}$$

It is easily seen that the components of v in spherical coordinates (the polar axis has the direction of $\boldsymbol{u}$) read:

$$v_r = u\cos\theta\left(1-\frac{2a}{r}+\frac{2b}{r^3}+Be^{-kr}\left(\frac{2}{k^4r^3}+\frac{2}{k^3r^2}\right)\right) \tag{3.15}$$

$$v_\theta = -u\sin\theta\left(1-\frac{a}{r}-\frac{b}{r^3}-Be^{-kr}\left(\frac{1}{k^4r^3}+\frac{1}{k^3r^2}+\frac{1}{k^2r}\right)\right) \tag{3.16}$$

$$v_\epsilon = 0. \tag{3.17}$$

One can prove immediately (see also Landau and Lifshitz (1995)) that only the ϵ-component of $\operatorname{curl}\boldsymbol{v}$ is different from zero. Applying the curl operator to (3.1) yields:

$$-(\nu+\nu_r)\Delta\operatorname{curl}\boldsymbol{v} = 2\nu_r\operatorname{curl}\operatorname{curl}\omega \tag{3.18}$$

which implies that only the ϵ-component of $\operatorname{curl}\operatorname{curl}\boldsymbol{\omega}$ can be different from zero. Due to the symmetry of the sphere, $\boldsymbol{\omega}$ depends only on r and θ. Taking this into account and performing calculations in polar coordinates on equation (3.2) shows that only the ϵ-component of $\boldsymbol{\omega}$ can be nonzero.
Setting now $\overline{\omega} = ((\nu+\nu_r)\operatorname{curl}\boldsymbol{v} - 2\nu_r\boldsymbol{\omega}$, we obtain from (3.18):

$$\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\omega}} = 0.$$

The solution of such an equation under the condition that $\overline{\boldsymbol{\omega}}$ has just one nonzero component (the ϵ-one) and depends only on r and θ is well-known (see e.g. Loitsyanskii (1996)). It reads:

$$\overline{\omega}_\epsilon = \frac{A\sin\theta}{r^2}. \tag{3.19}$$

Computing the ϵ-component of $\operatorname{curl}\boldsymbol{v}$ in spherical coordinates, we obtain:

$$\omega_\epsilon = -\frac{A\sin\theta}{2\nu_r r^2} - \frac{\nu+\nu_r}{2\nu_r}\frac{u\sin\theta\left(2ak^2+Be^{-kr}(1+kr)\right)}{k^2r^2}. \tag{3.20}$$

Consider first the conditions on the boundary of the sphere to determine the unknown constants.

$$\boldsymbol{v}|_{r=R} = 0, \tag{3.21}$$

$$\boldsymbol{\omega}|_{r=R} = \frac{\alpha}{2}\operatorname{curl}\boldsymbol{v}|_{r=R} \quad \text{with} \quad 0 \leqq \alpha \leqq 1. \tag{3.22}$$

Since $\boldsymbol{n}$ is arbitrary, relations (3.14) and (3.21) imply:

$$1-\frac{a}{R}-\frac{b}{R^3}-Be^{-kR}\left(\frac{1}{k^4R^3}+\frac{1}{k^3R^2}+\frac{1}{k^2R}\right) = 0, \tag{3.23}$$

$$-\frac{a}{R}+\frac{3b}{R^3}+Be^{-kR}\left(\frac{3}{k^4R^3}+\frac{3}{k^3R^2}+\frac{1}{k^2R}\right) = 0. \tag{3.24}$$

Moreover, relations (3.20) and (3.22) yield:

$$-\frac{A\sin\theta}{2\nu_r R^2} - \frac{\nu+\nu_r}{2\nu_r}\frac{u\sin\theta\left(2ak^2+Be^{-kR}(1+kR)\right)}{k^2R^2} = \frac{\alpha}{2}\frac{u\sin\theta\left(2ak^2+Be^{-kR}(1+kR)\right)}{k^2R^2}. \tag{3.25}$$

The system (3.23)-(3.25) defines a,b, and A as functions of B as follows:

$$\begin{aligned} a &= \frac{3}{4}R - \frac{Be^{-kR}}{2k^2} \\ b &= \frac{1}{4}R^3 - \frac{Be^{-kR}}{4k^4}\left(2k^2R^2 + 4kR + 4\right) \\ A &= -\frac{uR}{2k}\left(3k + 2Be^{-kR}\right)\left(\nu + (1-\alpha)\nu_r\right). \end{aligned}$$

To determine B we substitute these expressions in (3.15)-(3.17) and (3.20) and then apply the result to (3.2). The value

$$B = \frac{3k^2R\nu_r(1-\alpha)}{2e^{-kR}\left[kR\left((\alpha-1)\nu_r - \nu\right) - \nu\right]}$$

satisfies the resulting relation independently on the variables r, θ, and ϵ, which means that the desired solution is constructed.

Compute now the pressure. Equations (3.1), the definition of $\overline{\boldsymbol{\omega}}$, and the identity $\operatorname{curl}\operatorname{curl} = \nabla\operatorname{div} - \Delta$ imply that

$$\nabla p = -\operatorname{curl}\overline{\boldsymbol{\omega}}.$$

Considering the last relation in spherical coordinates and taking into account (3.19), we obtain

$$p = p_0 + \frac{A}{r^2}\cos\theta. \tag{3.26}$$

3.2. *Calculation of the resistance force*

The force is given by the the formula (see Landau and Lifshitz (1995))

$$F = \oint \left(-p\cos\theta + T'_{rr}\cos\theta - T'_{r\theta}\sin\theta\right) df, \tag{3.27}$$

where $T' = T + p\delta_{ij}$, and T is the tensor introduced in (2.1). Expressing the components of this tensor through the calculated velocity and microrotation fields leads to the formulae:

$$T'_{rr} = 2\nu\frac{\partial v_r}{\partial r}, \quad T'_{r\theta} = \nu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) + \nu_r\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) - 2\nu_r\omega_\epsilon.$$

on the boundary of the sphere, we have:

$$T'_{rr} = 0, \qquad T'_{r\theta} = \frac{A}{R^2}\sin\theta$$

$$p = p_0 + \frac{A}{R^2}\cos\theta$$

Therefore, the integral (3.27) reduces to

$$F = -\frac{A}{R^2}\oint df.$$

Finally, we get

$$F = -4\pi A = 6\pi\nu uR + \frac{6\pi\nu uR\nu_r(\alpha-1)}{kR\left((\alpha-1)\nu_r - nu\right) - \nu}. \tag{3.28}$$

It is very interesting to compare the calculated value with the classical Stokes force given by $F_S = 6\pi\nu u R$. We have

$$F = F_S\left(1 + \frac{(1-\alpha)\nu_r}{(1-\alpha)kR\nu_r + (1+kR)\nu}\right). \tag{3.29}$$

Remembering that $k = \sqrt{\frac{4\nu_r\nu}{(\nu+\nu_r)(c_a+c_d)}}$, we obtain:

$$F = F_S\left(1 + \frac{(1-\alpha)(c_a+c_d)\nu_r}{(1-\alpha)\sqrt{\frac{\nu\nu_r(c_a+c_d)}{\nu+\nu_r}}\nu_r R + \nu\left(c_a + c_d + 2\sqrt{\frac{\nu\nu_r(c_a+c_d)}{\nu+\nu_r}}R\right)}\right),$$

which implies:

i) $F = F_s$, if $\alpha = 1$ (nonsymmetric part of the stress tensor vanishes),
ii) $F = F_s$, if $\nu_r \longrightarrow 0$, whereas $c_a + c_d$ remains bounded,
iii) $F = F_s$, if $c_a + c_d \longrightarrow 0$, whereas $\nu_r$remains bounded.

These results are in agreement with the expected behavior of the force when varying material parameters related to the microrotation.

4. Numerical results

The following examples present the calculation of the modified resistance force for water and blood. The results are compared with the classical Stokes force. Unfortunately, there is very little information concerning the values of micropolar viscosity coefficients and boundary constant α in the literature. We refer to Kolpashchikov et al. (1983) where formulae for the calculation of material constants for water on the base of experimental data are given. Unfortunately, results for a simplified model (2D-Poiseuille flow), for which only two material constants are needed, are given. To calculate the other material constants, the guess $\alpha = 0.5$ was done. In this way, the following values for the micropolar viscosity constants were found: $\nu_r = 1.448275862 \cdot 10^{-3}$, $(c_a + c_d) = 4.828973844 \cdot 10^{-19}$. Figure 1 shows the dependence of the modified resistance force F (solid line), the Stokes force F_S (dashed line), and the difference $F - F_S$ (dotted line) on the radius R. In the second figure, F/F_S versus R is shown. The velocity $u = 1m/s$ was used. The difference $F - F_S$ is very small for macro- and mesoscopic values of R. The difference is observable for very small radii only. This is not surprising because water is a classical Newton fluid so that effects of the inner structure of its particles are only important on very small scales.

As for blood, some values can be found in the literature (see e.g. Papautsky et al. (1999)). The values $\nu = 2.9 \cdot 10^{-3}$, $\nu_r = 2.32 \cdot 10^{-4}$, $(c_a + c_d) = 10^{-6}$ are declared there for the Dirichlet boundary condition ($\alpha = 0$). In figure 3, the same curves as in the case of water are shown. As expected, a higher influence of the microrotation is present.

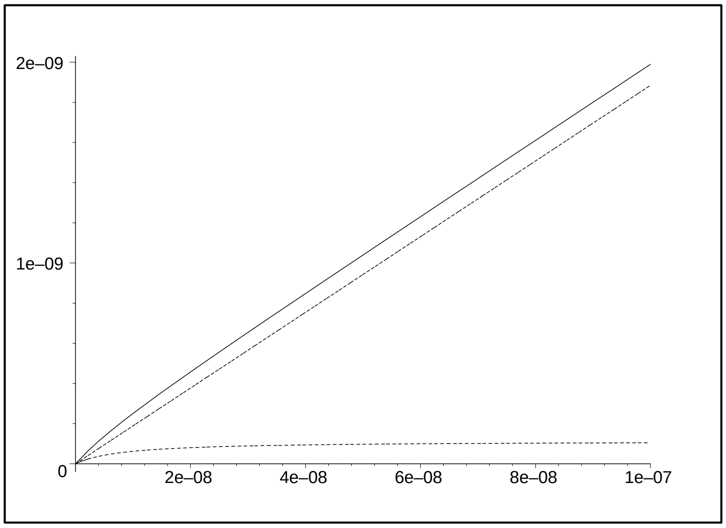

FIGURE 1. The modified force F (solid line), the Stokes force F_S (dashed line), and the difference $F - F_S$ (dotted line) versus R in the case of water.

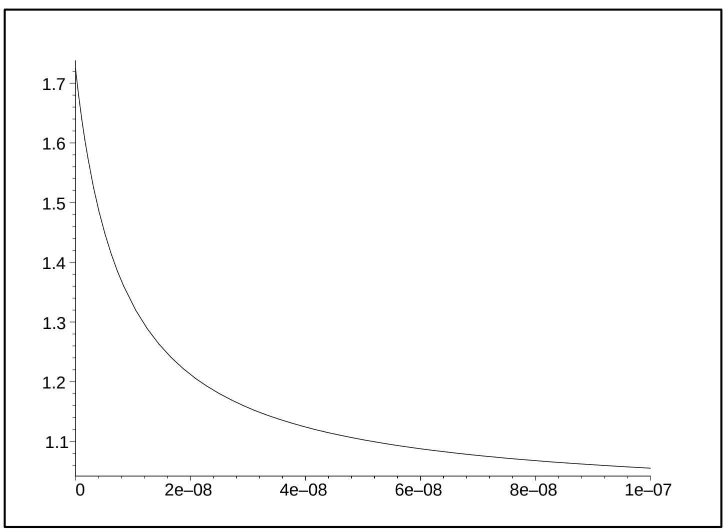

FIGURE 2. The ratio F/F_S versus R in the case of water.

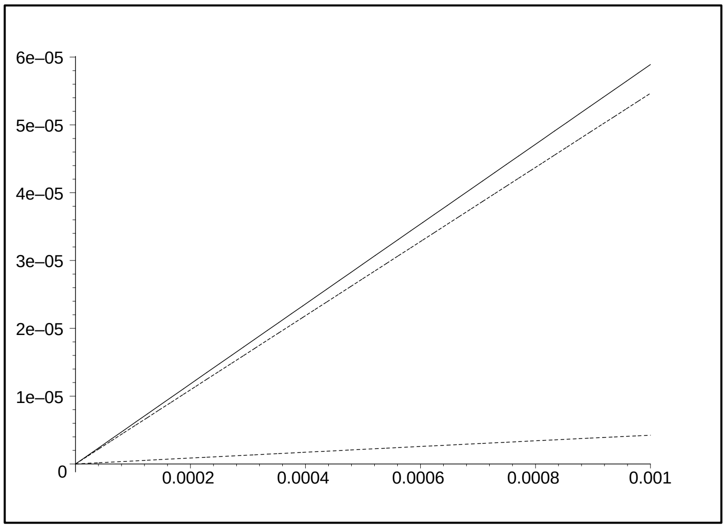

FIGURE 3. The modified force F (solid line), the Stokes force F_S (dashed line), and the difference $F - F_S$ (dotted line) versus R in the case of blood.

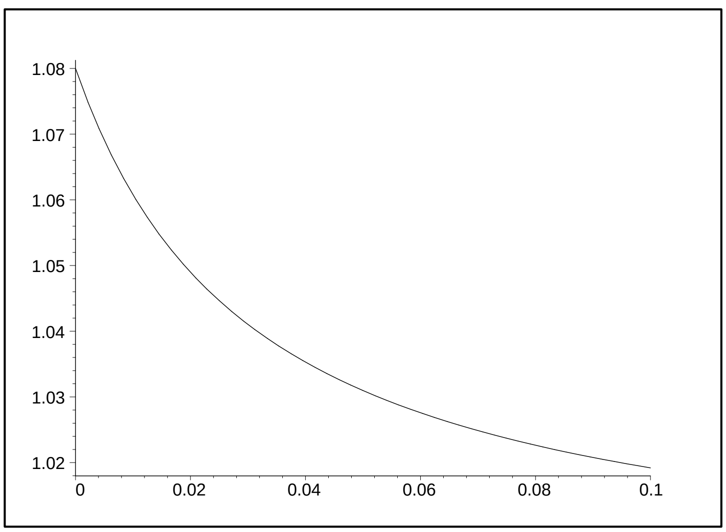

FIGURE 4. The ratio F/F_S versus R in the case of blood.

5. Results of other authors

In this section, results obtained in the papers Lakshmana (1970), Erdogan (1971), Ramkissoon (1976), Ramkissoon (1985), and Hayakawa (2000) on the calculation of the drag force exerted on a sphere by a moving micropolar fluid are compared with the ones of the paper presented. The comparison can be summarized as follows.

1. All off the above cited papers consider the same governing equations.
2. All papers except for the paper presented consider the homogeneous boundary condition for the microrotation field (see (3.7)).
3. The result of the works Lakshmana (1970), Erdogan (1971), and Ramkissoon (1976) is completely identical with the result of the presented paper , if α is equal to zero in (3.7).
4. The computation of the velocity and microrotation fields in Ramkissoon (1985) is not quite correct, which is observed and discussed in Hayakawa (2000).
5. The velocity, pressure, and microrotation fields are found correctly in Hayakawa (2000). They are identical with those found in the paper presented for the case $\alpha = 0$. Nevertheless, the formula for the drag force is derived not correctly in Hayakawa (2000). It is quite different from the formula obtained in Lakshmana (1970), Erdogan (1971), Ramkissoon (1976), and the paper presented although all of these papers consider the same problem.
6. In the sequence of papers: Lakshmana (1970), Erdogan (1971), Ramkissoon (1976), and Hayakawa (2000), each paper does not cite any preceding one.

5.1. *Comparison with Lakshmana (1970), Erdogan (1971), and Ramkissoon (1976)*

Remember the drag force in the paper presented is given by (3.29), i.e.

$$F = 6\pi\nu Ru\left(1 + \frac{(1-\alpha)\nu_r}{(1-\alpha)kR\nu_r + (1+kR)\nu}\right), \text{ where } k = \sqrt{\frac{4\nu_r\nu}{(\nu+\nu_r)(c_a+c_d)}}. \tag{5.1}$$

The formulae for the drag force obtained in Lakshmana (1970) and Ramkissoon (1976) are completely identical. They use the same notation and read:

$$F_{L\&R} = \frac{6\pi au(2\mu+\kappa)(\mu+\kappa)(1+a\chi)}{\kappa+2\mu+2a\chi\mu+2a\chi\kappa}, \text{ with } \chi = \sqrt{\frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}}. \tag{5.2}$$

It holds in the notation of the paper presented:

$$\kappa = 2\nu_r \quad \gamma = c_a + c_d, \quad \mu = \nu - \nu_r, \quad a = R. \tag{5.3}$$

The formula for the drag force obtained in Erdogan (1971) reads:

$$F_E = 6\pi\nu Ru\left[1 - \frac{N}{L}\frac{K_{1/2}(NL)}{K_{3/2}(NL)}\right]^{-1}, \tag{5.4}$$

where $K_{n+1/2}$, $n = 0, 1$ are the modified spherical Bessel functions of the second kind. Remember that

$$K_{1/2}(x) = \sqrt{\pi/2}\frac{e^{-x}}{\sqrt{x}}, \quad K_{3/2}(x) = \sqrt{\pi/2}\frac{e^{-x}(1+1/x)}{\sqrt{x}}.$$

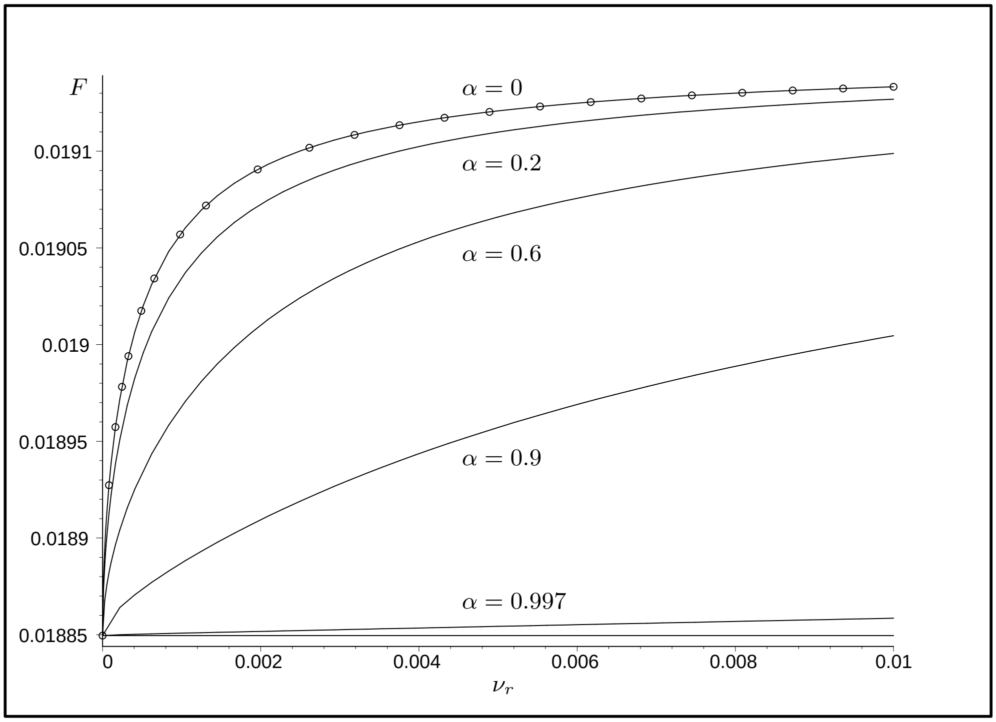

FIGURE 5. The dependence of the drag force F on ν_r for $\nu = 10^{-3}, c_a + c_d = 10^{-6}, u = 1$, and $R = 1$. Thereby, the parameter α assumes the values $0,\ 0.2,\ 0.6,\ 0.9$, and 0.997. The curves are computed using equation (5.1). The horizontal line represents the classical Stokes drag force that is of course independent on ν_r. The circle markers are computed using formula (5.2) (or (5.4)) to emphasis the identity of (5.1), (5.2) and (5.4) as $\alpha = 0$. As it is expected, the drag force tends to the classical Stokes one as α tends to 1.

The quantities N and L are defined as follows:

$$N = \left(\frac{\nu_r}{\nu + \nu_r} \right)^{1/2}, \quad L = R \left(\frac{c_a + c_d}{4\nu} \right)^{-1/2}.$$

Setting $\alpha = 0$ in (5.1) and comparing (5.1) with (5.2) and (5.4) under notation (5.3) shows the identity of F, $F_{L\&R}$ and F_E. The comparison is done using the MAPLE software package for symbolic calculations. Figures 5 and 6 illustrate the dependence of the drag force on some parameters.

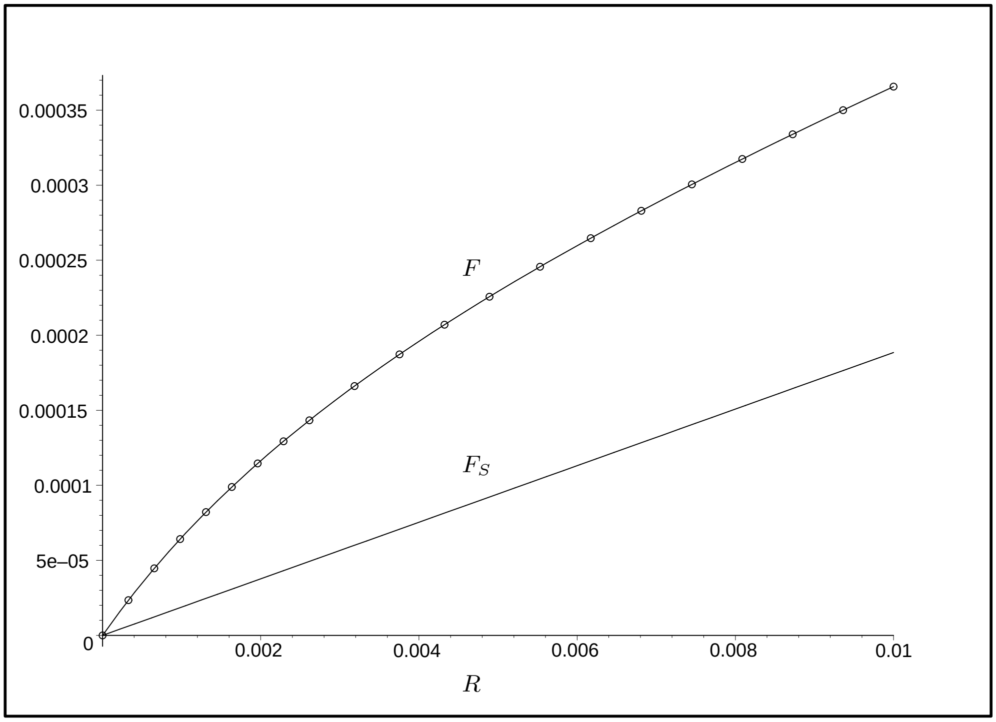

FIGURE 6. The marked curve shows the dependence of the drag force F on R for $\nu = 10^{-3}$, $\nu_r = 3 \cdot 10^{-3}$, $c_a + c_d = 10^{-6}$, $u = 1$, and $\alpha = 0$. The circle markers are computed using equation (5.2) (or (5.4)) to emphasis the identity of (5.1), (5.2) and (5.4) as $\alpha = 0$. The straight line represents the classical Stokes force that is of course linear in R.

5.2. *Comparison with Hayakawa (2000)*

The result obtained in Hayakawa (2000) reads:

$$F_H = \frac{2\pi(\eta + \eta_r)au(1+\xi)(2-\mu_r)(3-\mu_r)}{2(1+\xi) - \mu_r \xi}, \quad \text{where} \quad \xi = \sqrt{\frac{\mu_B}{2-\mu_r}}. \tag{5.5}$$

In the notation of the paper presented, the parameters read:

$$\eta = \nu, \quad \eta_r = \nu_r, \quad \mu_r = \frac{2\nu_r}{\nu + \nu_r}, \quad \mu_B = \frac{c_a + c_d}{2\nu_r}, \quad a = R. \tag{5.6}$$

Under relation (5.6), the formulae obtained in Hayakawa (2000) and in the paper presented (with $\alpha = 0$) for the velocity, pressure, and microrotation fields are completely identical, which is verified using MAPLE. Nevertheless, final formula (5.5) seems to be not true. First of all, the dependence on the radius R (i.e. on a) is linear, which contradicts with the results of Lakshmana (1970), Erdogan (1971), Ramkissoon (1976), and the paper presented. The linearity means in particular that $(F_H - F_S)/F_S$, where F_S is the Stokes drag force, does not depend on R. This is not correct because the relative effect of the microrotation must decrease as $R \to \infty$. Thus, it would be expected that $(F_H - F_S)/F_S \to 0$ as $R \to \infty$. Such a behavior holds for the dependence given by (5.1), (5.2), and (5.4). Moreover, Figure 7 shows a strange behavior of F_H when varying the parameter ν_r: F_H decrease as ν_r increases.

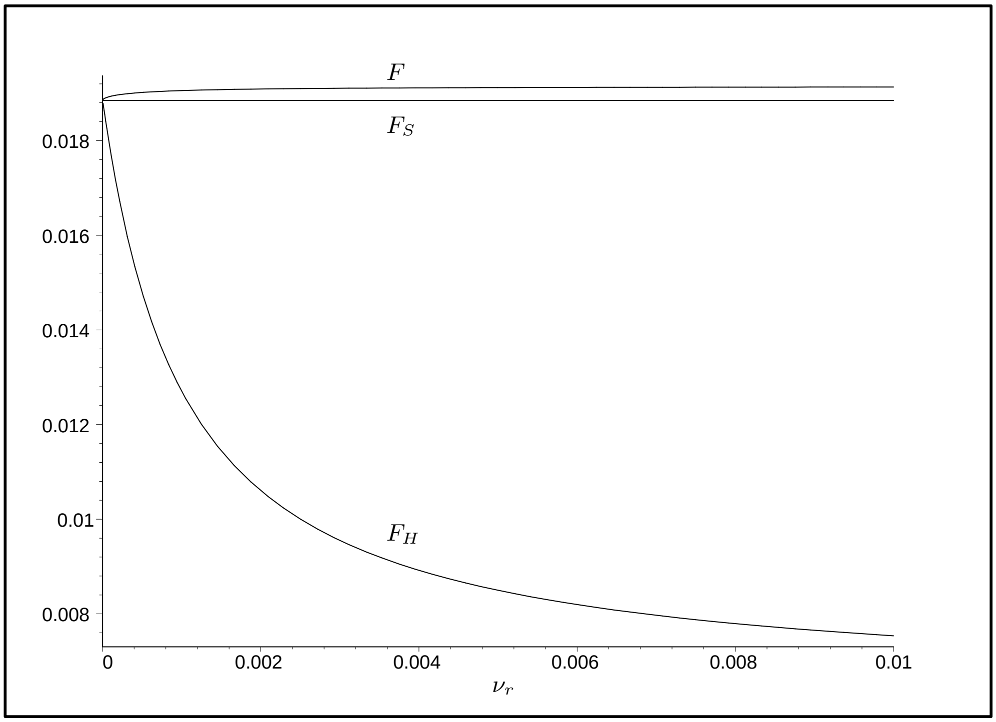

FIGURE 7. The dependence of the function F_H on ν_r for $\nu = 10^{-3}, c_a + c_d = 10^{-6}, u = 1$, and $R = 1$. The horizontal line represents the classical Stokes force that is of course independent on ν_r. The graph of the correct drag force F (see also Fig. 5) is given for comparison.

6. Conclusion

The paper presented can be considered as the extension of the result of Lakshmana (1970), Erdogan (1971), and Ramkissoon (1976) to the case of nonhomogeneous boundary conditions for the microrotation field. Examples show that sufficient deviations from classical results are being observed for very small radii only when considering classical Newtonian fluids like water. This should be expected because the effects of the inner structure of their molecules are important on very small scales only. In the case of fluids with large molecules (e.g. blood), the modified formula shows results that differ from the classical case even for macroscopic radii.

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