

Propagation of Disturbances in Inhomogeneous Anisotropic Media

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1 Introduction

The study of questions related to the propagation of acoustic waves in anisotropic media is very important for many applications such as the development of acoustic sensors whose operating principle is based on the excitation and detection of acoustic waves in crystals. Propagation of acoustic waves in isotropic media is well investigated and mathematically described by means of geometric optics. The Fermat principle holds in this case, which expresses the fact that the rays, energy propagation trajectories, are minimizers of a functional which is the curve integral of some Lagrangian. In the case of short waves in anisotropic media, one can also utilize geometric optics using WKB approximations. This leads to Hamilton-Jacobi equations describing dynamics of the wave phase. However, one can not find a proper Lagrangian that would realize the Fermat principle in this case because of nonconvexity of the corresponding Hamiltonians in the impulse variable.

The paper studies relations between the Lagrange formalism, the Pontryagin maximum principle, and Hamilton-Jacobi equations for media whose Hamiltonians constructed on the base of the phase velocity field are convex. Note that all isotropic materials satisfy this property. In the case of convex-concave Hamiltonians typical for anisotropic media, we consider the possibility to approximate the Hamiltonians through conflict-controlled systems where the first player chooses the current velocity from an admissible set of velocities with the goal to minimize the signal propagation time whenever the second, opposite, player strives to maximize this time. Although such an approach might look artificial, the point is that solutions to problems with close Hamiltonians must be also close each to other. Thus, the value function of the associated time-optimal differential game satisfies the eikonal equation of the wave propagation problem. Therefore level sets of the value function represent wave fronts (first arrival) propagating in the anisotropic medium. Like in the Fermat principle, optimal trajectories in the associated differential game can apparently deliver information about the rays representing the energy flow. Note, that the structure of optimal trajectories in differential games is rather complicated because of the so-called singular surfaces that can attract, repulse and break the trajectories. In the paper, a preliminary analysis of possible type of singular surfaces is performed for Hamiltonians that correspond to some typical phase velocity contours related to anisotropic crystals.

2 Description of properties of anisotropic media

In the related literature, the following three characteristic surfaces describing anisotropic properties of a medium [1], [6], [9] are being considered.

Wave Surface. This surface is also called Ray Surface or Group Velocity Surface. A scalar function $V = V(x, e)$ is given, where the vector $x = (x_1, x_2)$ determines the coordinates of a medium point, the unit vector $e = (e_1, e_2)$ shows the direction of propagation. The number $V = V(x, e)$ is the velocity value with which the ray propagates in the direction e from the point x . The ray surface (see Fig. 1(a)) will be denoted by the calligraphic letter \mathcal{V} . For each unit vector $e \in B_1$, the dilated vector $V(x, e)e$ defines a point of the surface \mathcal{V} , i.e. the surface consists of the points

$$v = (v_1, v_2) = (V(x, e)e_1, V(x, e)e_2), \quad e \in B_1$$

where B_1 is the unit ball. Thus, in (v_1, v_2) -plane, the equation of the surface can be written as

$$\mathcal{V} : \quad V \left(x, \frac{v}{|v|} \right) = |v| \quad \left(= \sqrt{v_1^2 + v_2^2} \right).$$

Slowness Surface. This surface is also called Wave Vector Surface. It is defined using the surface \mathcal{V} as follows. For a given $e \in B_1$ let the vector $q = q(x, e)$, $q \in B_1$, be the unit normal vector to the surface \mathcal{V} at the point $(V(x, e)e_1, V(x, e)e_2)$. We put

$$N(x, q) = \frac{1}{V(x, e) \langle q, e \rangle}, \quad e = e(x, q).$$

The surface \mathcal{N} , see Fig. 1(b), consists of the vectors

$$n = (n_1, n_2) = (N(x, q)q_1, N(x, q)q_2), \quad q \in B_1$$

so that $\langle n, v \rangle = 1$. The physical meaning of the vector n is

$$n = \frac{k}{\omega}$$

where k is the wave vector, and ω the wave frequency which is fixed in our considerations. Vector n is an analog of the refractive index in optics.

Similarly, the equation of the surface \mathcal{N} can be written as

$$\mathcal{N} : \quad N \left(x, \frac{n}{|n|} \right) = |n|.$$

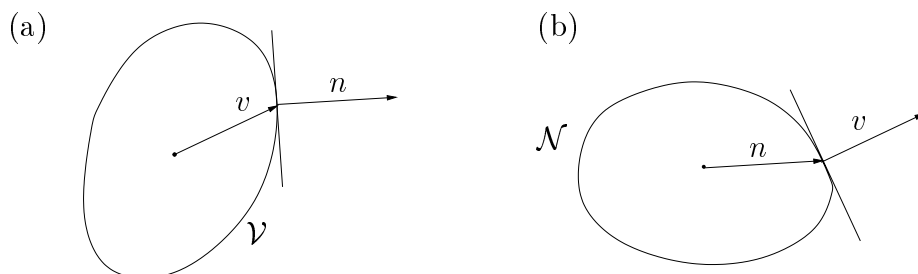


Figure 1: (a) Wave surface; (b) Slowness surface.

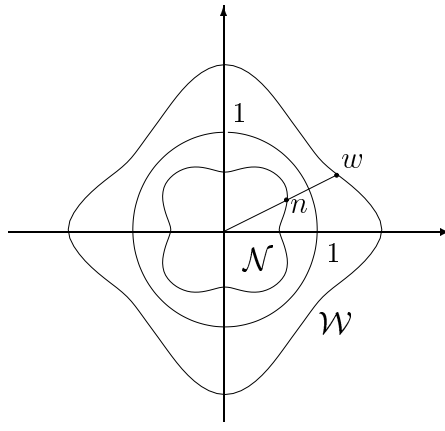


Figure 2: Velocity surface (\mathcal{W}) compared with the slowness surface (\mathcal{N}).

Velocity Surface. It is also called Phase Velocity Surface and will be denoted as \mathcal{W} . For a given $e \in B_1$ the vectors $n(x, e)$ and $w(x, e)$ have the same direction, see Fig. 2, while their lengths $N(x, e)$ and $W(x, e)$ are connected as

$$N(x, e) W(x, e) = 1$$

so that

$$W(x, e) = \frac{1}{N(x, e)}.$$

The equation for the surface \mathcal{W} has the form

$$\mathcal{W}: \quad W\left(x, \frac{w}{|w|}\right) = |w|.$$

Thus, given one of the surfaces $\mathcal{V}, \mathcal{N}, \mathcal{W}$, the other two can be constructed using the above definitions.

Remark. Considerations of this and the next sections involve two-dimensional vectors x, e, v, n, w because surface waves are under study. Many relations hold also true for three-dimensional bulk media, if the norm and the inner product of R^3 are used:

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}, \quad \langle q, v \rangle = q_1 v_1 + q_2 v_2 + q_3 v_3.$$

3 Variational principle for propagation of rays

The vectogram (surface) \mathcal{V} defines the set of admissible tangent vectors for rays. This can be expressed through differential inclusion in one of the forms

$$\dot{x} \in \mathcal{V}, \quad \text{or} \quad \dot{x} = v, \quad v \in \mathcal{V},$$

the latter one being traditional for the optimal control setting.

A ray is a curve (trajectory) along which the excitation (energy) propagates with the group velocity. Among all the trajectories $x(t)$, $t_0 \leq t \leq t_1$, running from a point A to a point B of the medium considered, the feasible one minimizes the optical distance (length) given by

$$\int_A^B \langle n(x, q), dx \rangle = \int_A^B (n_1(x, q) dx_1 + n_2(x, q) dx_2) = \int_A^B \langle n(x, q), \dot{x} \rangle dt \quad (q = q(x, e), \quad e = \dot{x}/|\dot{x}|).$$

Here we use the vector arc element dx of a curve:

$$dx = (dx_1, dx_2) = \dot{x}dt.$$

Another criterium for a trajectory being feasible, Fermat's principle, says that the time of the signal propagation from A to B has to be minimal:

$$T = \int_A^B dt = \int_A^B \frac{ds}{V(x, e)} \rightarrow \min.$$

Here $ds = Vdt$ is the arc length. Taking into account that

$$\dot{x} = v(x, e), \quad ds = |dx|, \quad dx = ds \frac{v}{|v|}, \quad \langle n, v \rangle = 1, \quad n = Nq, \quad v = Ve$$

enables to convert the last formulation to Fermat's principle and vice versa.

Consider a parametrization of a ray, $x(\tau)$, $\tau_0 \leq \tau \leq \tau_1$, where the parameter τ is not necessarily the time variable. With $ds = |\dot{x}|d\tau$, Fermat's principle assumes the form:

$$T = \int_{\tau_0}^{\tau_1} \frac{|\dot{x}|}{V(x, \dot{x}/|\dot{x}|)} d\tau =: \int_{\tau_0}^{\tau_1} L(x, \dot{x}) d\tau \rightarrow \min.$$

Therefore, feasible rays are solutions of the Euler equation

$$L_x - \frac{d}{d\tau} L_{\dot{x}} = 0.$$

This is a system of second order ODEs where the number of equations is equal to the dimension of the vector x . Note that the Lagrangian L is homogeneous in \dot{x} , which implies linear dependence of the equations of the Euler system.

Really, taking the full time derivative of both sides of the representation

$$L(x, \dot{x}) = \langle \dot{x}, L_{\dot{x}}(x, \dot{x}) \rangle$$

(this holds because of the homogeneity of L , see Section 4) yields

$$\langle L_x, \dot{x} \rangle + \langle L_{\dot{x}}, \ddot{x} \rangle = \left\langle \frac{d}{dt} L_{\dot{x}}, \dot{x} \right\rangle + \langle L_{\dot{x}}, \ddot{x} \rangle.$$

Cancelling of equal terms on both sides results in

$$\left\langle L_x - \frac{d}{dt} L_{\dot{x}}, \dot{x} \right\rangle = 0,$$

which means linear dependence of the equations. Thus, one of the equations can be omitted. Therefore, only one equation is valuable, if x is two dimensional.

4 Properties of homogeneous Lagrangians

One can see that the Lagrangian L is positively homogeneous of degree one in \dot{x} , that is

$$L(x, \lambda \dot{x}) = \lambda L(x, \dot{x}), \quad \lambda > 0.$$

Let us outline briefly why Hamilton-Jacobi theory for such Lagrangians differs from the traditional case.

Differentiating both sides of the above equality with respect to λ and letting then $\lambda = 1$ yields the representation formula

$$L(x, \dot{x}) = \langle \dot{x}, L_{\dot{x}}(x, \dot{x}) \rangle,$$

which is a particular case of the Euler theorem on forms. Differentiating the above equality with respect to \dot{x} and cancelling out λ gives

$$L_{\dot{x}}(x, \lambda \dot{x}) = L_{\dot{x}}(x, \dot{x}), \quad \lambda > 0,$$

which means that $L_{\dot{x}}$ is homogeneous of degree zero in \dot{x} .

Differentiating the representation formula with respect to \dot{x} yields successively:

$$L_{\dot{x}} = L_{\dot{x}} + L_{\dot{x}\dot{x}}\dot{x}, \quad L_{\dot{x}\dot{x}}\dot{x} = 0.$$

The last equation (identity in \dot{x}) is only feasible if the determinant $\det L_{\dot{x}\dot{x}}$ vanishes. This, particularly, means that the equation

$$p = L_{\dot{x}}(x, \dot{x})$$

is not solvable with respect to \dot{x} . Note that such a degeneration does not happen as a rule in the case of a non-homogeneous Lagrangian, therefore, \dot{x} can be expressed as follows $\dot{x} = \omega(x, p)$, where ω is an single-valued function. The Hamiltonian $H(x, p)$ is defined as

$$\bar{H}(x, \dot{x}) = \langle \dot{x}, L_{\dot{x}} \rangle - L(x, \dot{x})$$

where $L_{\dot{x}}$ has to be replaced by p and \dot{x} by $\omega(x, p)$. The function $\bar{H}(x, \dot{x})$ appears in the first variation formula as a coefficient on the variation of the time (independent) variable. The representation formula implies that $\bar{H}(x, \dot{x})$ is identically zero in the case of homogeneous L .

Therefore, Hamiltonians for problems with homogeneous Lagrangians should have some special structure that excludes the degeneration. Such a structure can be derived using a new representation formula for Lagrangians. Introduce a function $R(x, \dot{x})$ as follows:

$$R(x, \dot{x}) = \frac{1}{2}L^2(x, \dot{x})$$

Obviously, the function is positively homogeneous of degree 2 in \dot{x} :

$$R(x, \lambda \dot{x}) = \frac{1}{2}\lambda^2 L^2(x, \dot{x}).$$

Differentiating both sides with respect to λ yields

$$\langle R_{\dot{x}}(x, \lambda \dot{x}), \dot{x} \rangle = \lambda L^2(x, \dot{x}).$$

Another differentiation with respect to λ leads to the following equation, which holds true for any $\lambda > 0$:

$$\langle R_{\dot{x}\dot{x}}(x, \lambda \dot{x})\dot{x}, \dot{x} \rangle = L^2(x, \dot{x})$$

Since the right hand side does not depend on λ , the matrix $R_{\dot{x}\dot{x}}$ is homogeneous of degree zero in \dot{x} . This conclusion can also be obtained directly by twice differentiating the equality $R(x, \lambda \dot{x}) = \lambda^2 L^2(x, \dot{x})/2$ with respect to \dot{x} and cancelling out λ^2 .

Now, substituting $\lambda = 1$ in the above formula for $R_{\dot{x}\dot{x}}$ yields a new representation formula for L :

$$L(x, \dot{x}) = \sqrt{\langle G(x, \dot{x})\dot{x}, \dot{x} \rangle}, \quad G(x, \dot{x}) = R_{\dot{x}\dot{x}}(x, \dot{x}),$$

where $G(x, \dot{x})$ is a symmetric, homogeneous of degree zero in \dot{x} matrix:

$$G(x, \lambda\dot{x}) = G(x, \dot{x}), \quad \lambda > 0.$$

The sign of the square root depends on the sign of the original lagrangian $L(x, \dot{x})$.

In many problems, $G(x, \dot{x})$ is positively definite and independent on \dot{x} , $G = G(x)$, having sense of a metric tensor. The case $G = G(x)$ will be called ellipsoidal case.

5 Ellipsoidal surfaces \mathcal{V} and \mathcal{N}

Let the wave surface scalar function $V(x, e)$ be defined through the equality

$$L(x, \dot{x}) = \frac{|\dot{x}|}{V(x, \dot{x}/|\dot{x}|)} = \sqrt{\langle G(x)\dot{x}, \dot{x} \rangle}$$

where $G(x)$ is a positive definite symmetric matrix (metric tensor). This means that

$$V(x, e) = \frac{1}{\sqrt{\langle G(x)e, e \rangle}}, \quad v(x, e) = V(x, e)e = \frac{e}{\sqrt{\langle G(x)e, e \rangle}}.$$

Substituting $e = v/V$, one arrives at the equation for the surface \mathcal{V} which is an ellipsoid

$$\langle G(x)v, v \rangle = 1.$$

Differentiating the left hand side with respect to v yields a vector

$$q^* = \frac{Gv}{\sqrt{\langle Gv, v \rangle}} = Gv$$

that is orthogonal to the surface \mathcal{V} . Obviously, $\langle q^*, v \rangle = 1$, and hence $q^* = n$, where n is the vector from the definition of the slowness surface \mathcal{N} . Simple calculations show that the vector $n = q^*$ satisfies the equation

$$\langle G^{-1}(x)n, n \rangle = 1$$

which is the equation of the slowness surface \mathcal{N} . Therefore, \mathcal{N} is also an ellipse. Substituting $n = Ne$ gives the length $N(x, e)$ of the vector n as a function of the direction e :

$$N(x, e) = \frac{1}{\sqrt{\langle G^{-1}(x)e, e \rangle}}.$$

To construct the surface \mathcal{W} , note that the vectors n and w are collinear and $|n||w| = 1$, $|n| = N$, $|w| = W$ by definition. Then

$$w = \frac{n}{N}W = \frac{n}{N^2}.$$

Substituting the vector $n = wN^2$ into the equation of the slowness surface \mathcal{N} yields

$$N^4 \langle G^{-1}(x)w, w \rangle = 1.$$

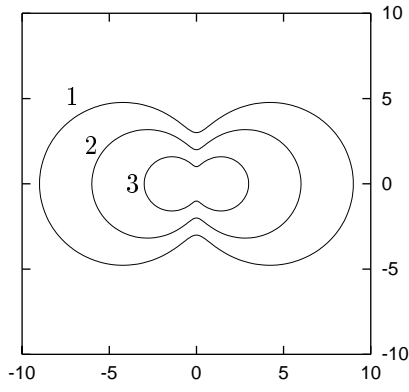


Figure 3: A family of \mathcal{W} (1: $a = 9$, 2: $a = 6$, 3: $a = 3$) for elliptic \mathcal{V} , \mathcal{N} .

Taking the square root and using $N^2 = W^{-2}$ gives the equation of the surface \mathcal{W} in one of the forms:

$$\frac{\sqrt{\langle G^{-1}(x)w, w \rangle}}{W^2} = 1, \quad \sqrt{\langle G^{-1}(x)w, w \rangle} = W^2 = w_1^2 + w_2^2.$$

The curve defined by this equation is not an ellipse, it is a curve of the fourth order. The function $W(x, e)$ is the inverse of $N(x, e)$:

$$W(x, e) = \sqrt{\langle G^{-1}(x)e, e \rangle}.$$

Thus, we end up with the following relations that define the surfaces \mathcal{V} , \mathcal{N} , and \mathcal{W} :

$$\mathcal{V}: \quad V(x, e) = \frac{1}{\sqrt{\langle G(x)e, e \rangle}}, \quad \langle G(x)v, v \rangle = 1,$$

$$\mathcal{N}: \quad N(x, e) = \frac{1}{\sqrt{\langle G^{-1}(x)e, e \rangle}}, \quad \langle G^{-1}(x)n, n \rangle = 1,$$

$$\mathcal{W}: \quad W(x, e) = \sqrt{\langle G^{-1}(x)e, e \rangle}, \quad \sqrt{\langle G^{-1}(x)w, w \rangle} = w_1^2 + w_2^2.$$

For example, if the ellipse \mathcal{V} is given in (x, y) -plane in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equations for the surfaces \mathcal{N} and \mathcal{W} have the following form

$$\mathcal{N}: \quad a^2x^2 + b^2y^2 = 1$$

$$\mathcal{W}: \quad a^2x^2 + b^2y^2 = (x^2 + y^2)^2$$

An example of \mathcal{W} -curves for $\frac{a}{b} = 3$, $a = 3, 6, 9$, is presented in Fig. 3.

6 Hamilton-Jacobi equation

Introduce a function of x as follows

$$S(x) = \min_{x(t)} \int_{\tau_0}^{\tau} L(x, \dot{x}) dt, \quad x(\tau) = x,$$

where the minimum is taken over trajectories ended at x . The first variation computed under assumption of a fixed start point is generally of the form

$$\delta S = \int_{\tau_0}^{\tau} (L_x - \frac{d}{d\tau} L_{\dot{x}}) h(t) dt + \langle \delta x, L_{\dot{x}} \rangle + \bar{H} \delta \tau,$$

where $h(t)$ denotes the variation of the trajectory $x(t)$, $\tau_0 \leq t \leq \tau$, and δx , $\delta \tau$ give the variation of the endpoint. The integral term vanishes due to the Euler equation because $S(x)$ is defined through the minimization. If the Lagrangian is positively homogeneous with respect to \dot{x} , the coefficient \bar{H} , the Hamiltonian in the variables (x, \dot{x}) but not in (x, p) , is identically zero as shown in Section 4. Thus, $S(x)$ does not depend on τ and

$$S_x = L_{\dot{x}}.$$

The last relation simplifies the derivation of the Hamilton-Jacobi equation in the case

$$L(x, \dot{x}) = \sqrt{\langle G(x) \dot{x}, \dot{x} \rangle},$$

where $G(x)$ is symmetric and positively definite so that the inverse $G^{-1}(x)$ exists.

Since

$$S_x = L_{\dot{x}} = \frac{G\dot{x}}{\sqrt{\langle G(x) \dot{x}, \dot{x} \rangle}},$$

we have

$$G^{-1} S_x = \frac{\dot{x}}{\sqrt{\langle G(x) \dot{x}, \dot{x} \rangle}}.$$

Multiplying both sides of this equation by S_x yields

$$\langle G^{-1}(x) S_x, S_x \rangle = \frac{\langle G(x) \dot{x}, \dot{x} \rangle}{\sqrt{\langle G(x) \dot{x}, \dot{x} \rangle} \sqrt{\langle G(x) \dot{x}, \dot{x} \rangle}} = 1.$$

Letting

$$H(x, S_x) = \sqrt{\langle G^{-1}(x) S_x, S_x \rangle},$$

we arrive at the Hamilton-Jacobi equation

$$H(x, S_x) = 1.$$

This equation can also be written in the form

$$|S_x| \sqrt{\left\langle G^{-1}(x) \frac{S_x}{|S_x|}, \frac{S_x}{|S_x|} \right\rangle} = 1$$

which can be represented as follows

$$|S_x| W \left(x, \frac{S_x}{|S_x|} \right) = 1$$

using the function $W(x, e)$ of Section 5.

To derive Hamilton-Jacobi equations for more general Lagrangians being homogeneous of degree one in \dot{x} , the following procedure can be used (see [13]). Assuming that

$$\det \left[\frac{1}{2} (L^2)_{\dot{x}\dot{x}} \right] \neq 0,$$

solve the equation

$$p = \frac{1}{2}(L^2)_{\dot{x}} = LL_{\dot{x}}$$

with respect to \dot{x} and find $\dot{x} = \omega(x, p)$. The vector-function $\omega(x, p)$ is homogeneous in p due to the homogeneity properties of L and $L_{\dot{x}}$. Now the Hamiltonian is defined as

$$H(x, p) = L(x, \omega(x, p)).$$

Substituting $p = LL_{\dot{x}}$ in $H(x, p)$ and utilizing the homogeneity of H in p yields

$$L = H(x, LL_{\dot{x}}) = LH(x, L_{\dot{x}}).$$

Canceling out L , we arrive at $H(x, L_{\dot{x}}) = 1$. Since $S_x = L_{\dot{x}}$, the Hamilton-Jacobi equation has the form

$$H(x, S_x) = 1, \quad \text{or} \quad |S_x|H\left(x, \frac{S_x}{|S_x|}\right) = 1$$

using the homogeneity of H .

Demonstrate how this general procedure works in the ellipsoidal case to repeat the result (Hamiltonian) already obtained above. We have

$$p = LL_{\dot{x}} = \sqrt{\langle G(x)\dot{x}, \dot{x} \rangle} \frac{G(x)\dot{x}}{\sqrt{\langle G(x)\dot{x}, \dot{x} \rangle}} = G(x)\dot{x},$$

$$\dot{x} = G^{-1}(x)p \equiv \omega(x, p),$$

$$H(x, p) = L(x, \omega(x, p)) = \sqrt{\langle G(x)G^{-1}(x)p, G^{-1}(x)p \rangle} = \sqrt{\langle G^{-1}(x)p, p \rangle}.$$

The following example shows another way to treat the homogeneous case using appropriate reductions of original problems to non-homogeneous ones.

Example. Consider the following two-dimensional variational problem whose Lagrangian is homogeneous:

$$J = \int_{t_0}^{t_1} f(x, y) \sqrt{\dot{x}^2 + \dot{y}^2} dt, \quad f(x, y) > 0.$$

This is an ellipsoidal case with the matrix $G = f^2(x, y)I$, with I being the identity matrix. Such a functional can be transformed to a new one with a non-homogeneous Lagrangian by taking one of the components of the original state vector as the new independent variable. Such a trick reduces the dimension of the original problem by one.

According to that, consider x as the new independent variable so that $dx = \dot{x}dt$ and

$$J = \int_{x_0}^{x_1} f(x, y) \sqrt{1 + y'^2} dx = \int_{x_0}^{x_1} F(x, y, y') dx.$$

The problem becomes one-dimensional with the non-homogeneous Lagrangian F . According to the “non-homogeneous technique” y' can be expressed from the equation

$$p = F_{y'} = \frac{fy'}{\sqrt{1 + y'^2}}$$

to obtain

$$y'^2 = \frac{p^2}{f^2 - p^2}, \quad y' = \pm \frac{p}{\sqrt{f^2 - p^2}}.$$

The Hamiltonian H_1 is of the form:

$$H_1(x, y, p) = y'F_{y'} - F = \frac{-f}{\sqrt{1+y'^2}} = -\sqrt{f^2(x, y) - p^2},$$

and the corresponding Hamilton-Jacobi equation reads:

$$\frac{\partial S}{\partial x} + H_1\left(x, y, \frac{\partial S}{\partial y}\right) = 0 \quad \text{or} \quad \frac{\partial S}{\partial x} = \sqrt{f^2(x, y) - \left(\frac{\partial S}{\partial y}\right)^2}.$$

Taking square and rearranging terms yields

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = f^2(x, y), \quad \text{or} \quad \frac{1}{f(x, y)}\sqrt{S_x^2 + S_y^2} = 1,$$

which is in agreement with the ‘‘homogeneous technique’’ presented earlier. Really, the ‘‘homogeneous technique’’ yields:

$$H(x, y, p_1, p_2) = \sqrt{\langle G^{-1}p, p \rangle} = \frac{1}{f(x, y)}\sqrt{p_1^2 + p_2^2}, \quad G^{-1} = f^{-2}(x, y)I.$$

Hamilton-Jacobi Equation in Geometrical Optics. The Hamilton-Jacobi equation of the form

$$H(x, p) = |p|W\left(x, \frac{p}{|p|}\right) = 1 \quad (p = S_x),$$

with W being the phase velocity, arises when applying the WKB-approach (see e.g. [2]) to the elasticity equations for anisotropic inhomogeneous media

$$\rho u_{itt} - \frac{\partial}{\partial x_j} \left(C_{ijkl}(x) \frac{\partial u_l}{\partial x_k} \right) = 0, \quad i, j, k, l = 1, 2, 3.$$

Here u_i are the components of the displacement vector, ρ is the density, and C_{ijkl} is the elastic stiffness tensor. The summation over repeated indices is assumed.

We are looking for solutions of the form

$$u_j^\varepsilon(t, x) = \varepsilon e^{iS(t, x)/\varepsilon} v_j^\varepsilon(t, x)$$

with

$$v_j^\varepsilon(t, x) = v_j^0(t, x) + \varepsilon v_j^1(t, x) + \dots,$$

and the initial conditions

$$u_j^\varepsilon(0, x) = \varepsilon e^{iS(0, x)/\varepsilon} \phi_j(x), \quad x \in R^3,$$

$$u_{j,t}^\varepsilon(0, x) = \psi_j(x), \quad x \in R^3.$$

Here ε is a small parameter, $\varepsilon = (\text{length scale} \times \text{wave number})^{-1}$. Assuming that $\varepsilon \ll 1$, restrict the analysis to the first term of the expansion for v_j^ε .

Inserting this ansatz into the elasticity equations and collecting terms of the order $1/\varepsilon$ and 1, respectively, yields the following eikonal and transport equations

$$\det \left| \frac{1}{\rho} C_{ijkl}(x) \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_k} - \left(\frac{\partial S}{\partial t} \right)^2 I \right| = 0,$$

$$\begin{aligned} & \rho S_{tt} v_i^0 + 2\rho S_{it} v_{it}^0 - C_{ijkl}(x) \frac{\partial^2 S}{\partial x_j \partial x_k} v_l^0 - \\ & - 2C_{ijkl}(x) \frac{\partial S}{\partial x_j} \frac{\partial v_l^0}{\partial x_k} - \left[\frac{\partial}{\partial x_j} C_{ijkl}(x) \right] \frac{\partial S}{\partial x_k} v_l^0 = 0, \quad i = 1, 2, 3. \end{aligned}$$

Let functions $c_\alpha(x, n)$, $\alpha = 1, 2, 3$, solve the eigenvalue problem

$$\det \left| \frac{1}{\rho} C_{ijkl}(x) n_j n_k - c^2 I \right| = 0$$

where $|n| = 1$. The functions $c_\alpha(x, n)$ are known to be the phase velocities for three types of waves propagating in anisotropic media in the direction n . The eikonal equation is obviously equivalent to the following three equations

$$S_{\alpha t} - |\nabla S_\alpha| c_\alpha \left(x, \frac{\nabla S_\alpha}{|\nabla S_\alpha|} \right) = 0, \quad \alpha = 1, 2, 3.$$

The substitution $S_\alpha(t, x) = t + T_\alpha(x)$ leads to the equation

$$|\nabla T_\alpha(x)| c_\alpha \left(x, \frac{\nabla T_\alpha(x)}{|\nabla T_\alpha(x)|} \right) = 1, \quad \alpha = 1, 2, 3,$$

which, as we will see in Section 10, determines the propagation times $T_\alpha(x)$ for different wave modes.

7 Method of characteristics for Hamilton-Jacobi equation

Once the Hamilton-Jacobi equation $H(x, S_x) = 1$ is given, the characteristic system (or Hamiltonian system) of the form

$$\dot{x} = H_p, \quad \dot{p} = -H_x \quad (p = S_x)$$

can be considered. As we will see in the sequel, the independent variable here is time t up to a constant shift c .

To state the equivalence of the characteristic system and the Euler equation, investigate relation between partial derivatives of $L(x, \dot{x})$ and $H(x, p)$. Differentiating the identity $H(x, p) = L(x, \omega(x, p))$ with respect to p and multiplying both sides by L yields

$$LH_p = \omega_p LL_{\dot{x}}.$$

Note that ω_p , ω_x , and H_{px} are matrices. Using $p = LL_{\dot{x}}$, $L = H$, and the representation $\omega = \omega_p p$ which is true for the homogeneous function ω , we obtain

$$\omega = HH_p.$$

Differentiating the identity $H(x, p) = L(x, \omega(x, p))$ with respect to x and using above mentioned equalities yields

$$H_x = L_x + L_{\dot{x}} \frac{\partial}{\partial x} \omega = L_x + \frac{p}{L} \frac{\partial}{\partial x} (HH_p).$$

Due to the homogeneity of degree one, we have $H = \langle H_p, p \rangle$ and $H_x = H_{px}p$. Using again $H = L$, one can rewrite the above expression as $H_x = L_x + 2H_x$, which finally gives

$$H_x = -L_x.$$

In variational problems with the objective functional

$$J = \int_{\tau_0}^{\tau_1} L(x, \dot{x}) d\tau$$

where L is homogeneous in \dot{x} , and x is parametrized with some parameter τ , $x = x(\tau)$, a special parameter σ , called natural parameter, can be introduced so that

$$d\sigma = L(x, \dot{x}) d\tau, \quad J = \int_{\sigma_0}^{\sigma_1} d\sigma.$$

Thus, we obtain

$$\dot{x} = \frac{dx}{d\tau} = L(x, \dot{x}) \frac{dx}{d\sigma} = L(x, \dot{x}) x'.$$

This implies the following relation:

$$L(x, x') = L(x, \dot{x}/L(x, \dot{x})) = L(x, \dot{x})/L(x, \dot{x}) = 1$$

being a counterpart of the Hamilton-Jacobi equation $H(x, p) = 1$. Thus,

$$L(x, x') = 1.$$

in the natural parameter form. The transformation to the natural parameter does not change the Euler equation because

$$L_x(x, \dot{x}) - \frac{d}{d\tau} L_{\dot{x}}(x, \dot{x}) = 0 \quad \rightarrow \quad L_x(x, Lx') - L \frac{d}{d\sigma} L_{\dot{x}}(x, Lx') = 0,$$

where $L = L(x, \dot{x})$ is canceled out due to the homogeneity of degree one for L and degree zero for $L_{\dot{x}}$. Substituting Lx' instead of $L_{\dot{x}}$ yields

$$L_x(x, x') - \frac{d}{d\sigma} L_{x'}(x, x') = 0.$$

This equation can not be written directly using the functional and traditional structure of the equation. One should utilize a general parametrization, write the Euler equation and then change to the parameter σ .

With the natural parameter, $p = LL_{x'} = L_{x'}$ since $L = 1$. From here, using Euler equation and the relation $L_x = -H_x$, we obtain

$$p' = L_x = -H_x.$$

Recalling that $\dot{x} = \omega = HH_p$ and $L = H$, we arrive at the equation:

$$x' = \dot{x}/L = H_p$$

Thus, the characteristic system

$$x' = H_p, \quad p' = -H_x$$

corresponds to the Euler equation written in the natural parameter.

We will use the original dot-notation \dot{x}, \dot{p} for the differentiation with respect to σ as well as to t . Which parameter is actually used will be clear from the context.

8 Relations between Lagrangians, Hamiltonians, and Wave/Slowness/Velocity Surfaces

In Section 4, the following representation formula for a homogeneous Lagrangian was obtained

$$L^2(x, \dot{x}) = \langle G(x, \dot{x})\dot{x}, \dot{x} \rangle.$$

Here G is a positively homogeneous of degree zero symmetric matrix. This leads to the following expression for the vector p

$$p = \frac{1}{2}(L^2)_{\dot{x}} = LL_{\dot{x}} = G(x, \dot{x})\dot{x}$$

because $\frac{\partial G}{\partial \dot{x}}\dot{x}$ vanishes due to the homogeneity of degree zero of the matrix G : one has to differentiate the identity $G(x, \dot{x}) = G(x, \lambda\dot{x})$ with respect to λ and then take $\lambda = 1$ to prove that. Solving the equation $p = G(x, \dot{x})\dot{x}$ yields

$$\dot{x} = Q(x, p)p, \quad Q(x, p) = G^{-1}(x, \omega(x, p)).$$

By the definition of the Hamiltonian one has

$$H^2(x, p) = L^2(x, \omega(x, p)) = \langle GQp, Qp \rangle = \langle Q(x, p)p, p \rangle.$$

We assume that

$$\det Q(x, p) \neq 0.$$

Vice versa, if the Hamiltonian, i.e. the matrix Q , is given, one can construct the Lagrangian as follows. Solve $\dot{x} = Q(x, p)p$ with respect to p and find $p = \phi(x, \dot{x})$. Then define the Lagrangian

$$L^2(x, \dot{x}) = \langle G(x, \dot{x})\dot{x}, \dot{x} \rangle, \quad G(x, \dot{x}) = Q^{-1}(x, \phi(x, \dot{x})).$$

By assumption, the matrix G is also nonsingular.

In Section 3, the Lagrangian was introduced through the formula

$$L(x, \dot{x}) = \frac{|\dot{x}|}{V(x, \dot{x}/|\dot{x}|)}.$$

Comparing with the equation for the surface \mathcal{V} of the Section 2 one can conclude that this equation can be written in the form:

$$\mathcal{V}: \quad L(x, v) = 1.$$

This representation shows that the vector $L_{\dot{x}}$ is normal to the surface \mathcal{V} , and, since $L_{\dot{x}} = S_x$, it is normal to to the wave front given

$$S(x) = \text{const.}$$

The directions of the vectors \dot{x} and $L_{\dot{x}}$ correspond to unit vectors e and q in the description of the Wave and Slowness surfaces in Section 2.

Thus the surface \mathcal{V} is the unit-level surface of the Lagrangian $L(x, \dot{x})$. One can show that the Hamiltonian $H(x, p)$, as it was introduced in the previous section, plays the same role for the surface \mathcal{N} .

Consider a vector $p = LL_{\dot{x}}$ and the corresponding one $\dot{x} = \omega(x, p)$. The vector $L_{\dot{x}}$ and, hence, the vector $p = LL_{\dot{x}}$, is normal to the surface \mathcal{V} at the point ω . Using the unit direction vector and the corresponding normal vector of the form:

$$e = \frac{\omega}{|\omega|}, \quad q = \frac{p}{|p|},$$

we rewrite the equation $\langle n(x, q), v(x, e) \rangle = 1$ of Section 2 as follows

$$\langle N(x, p/|p|)p/|p|, V(x, \omega/|\omega|)\omega/|\omega| \rangle = 1.$$

Some calculations yield:

$$N(x, p/|p|) = \frac{|\omega||p|}{V(x, \omega/|\omega|)\langle p, \omega \rangle} = \frac{|p|}{L(x, \omega)}.$$

Here, in addition to the representation $L = \langle L_{\dot{x}}, \omega \rangle$, we have used the relations

$$L(x, \omega) = \frac{|\omega|}{V(x, \omega/|\omega|)}, \quad \langle p, \omega \rangle = L \langle L_{\dot{x}}, \omega \rangle = L^2(x, \omega).$$

Finally, we obtain

$$\frac{|p|}{N(x, p/|p|)} = L(x, \omega) = H(x, p).$$

Thus, in addition to the definition $H(x, p) = L(x, \omega(x, p))$, we have another two representations for the Hamiltonian through the functions N and W

$$H(x, p) = \frac{|p|}{N(x, p/|p|)}, \quad H(x, p) = |p|W(x, p/|p|).$$

Now the equation for the surface \mathcal{N} from Section 2 results in the following equation for this surface

$$\mathcal{N} : \quad H(x, n) = 1.$$

9 Rays propagation description in the form of the Maximum Principle

The variational problem of Section 3 in terms of optimal control theory is of the form

$$\dot{x} = u, \quad u \in R^2, \quad T = \int_{\tau_0}^{\tau_1} \frac{|u|}{V(x, u/|u|)} d\tau \rightarrow \min_{u(\tau)}.$$

According to Pontryagin's maximum principle, let p be the adjoint vector. The main requirement of the maximum principle for optimal paths is

$$0 = \max_{u \in R^2} (\langle p, u \rangle - |u|/V(x, u/|u|)) = \max_{u \in B_1} (\langle p, u \rangle - 1/V(x, u)) = \max_{u \in B_1} (\langle p, u \rangle V - 1)/V.$$

Observe now that the conditions

$$\max_{x \in X} \frac{f(x)}{g(x)} = 0, \quad g(x) > 0$$

imply that

$$\max_{x \in X} f(x) = 0.$$

Thus, maximum principle relation is reduced to

$$\max_{u \in B_1} \langle p, uV \rangle = 1.$$

Since the vector $uV(x, u/|u|)$ with $u \in B_1$ belongs to the set \mathcal{V} , the notation $uV = v$ is appropriate. Then the maximum principle requirement can be rewritten as follows

$$\max_{v \in \mathcal{V}} \langle p, v \rangle = 1.$$

Denote

$$H^*(x, p) = \max_{v \in \mathcal{V}} \langle p, v \rangle.$$

One can show that this is the Hamiltonian of Section 6 whenever \mathcal{V} is convex.

Let us calculate H^* in the case of ellipsoidal \mathcal{V} . We have

$$\max_{v \in \mathcal{V}} \langle p, v \rangle, \quad \langle Gv, v \rangle = 1.$$

The necessary optimality condition and the optimal v^* are

$$p + 2\lambda Gv = 0, \quad v^* = -\frac{1}{2\lambda} G^{-1}p,$$

where λ is a Lagrange multiplier. The condition $\langle Gv^*, v^* \rangle = 1$ yields the relation $4\lambda^2 = \langle G^{-1}p, p \rangle$ for the multiplier, which leads to the following formulas for the maximizer v^* and the Hamiltonian:

$$v^* = \frac{G^{-1}p}{\sqrt{\langle G^{-1}p, p \rangle}}, \quad H(x, p) = \langle p, v^* \rangle = \frac{\langle G^{-1}p, p \rangle}{\sqrt{\langle G^{-1}p, p \rangle}} = \sqrt{\langle G^{-1}p, p \rangle}.$$

Observe this relation is in agreement with the results of Section 6.

Summarizing the above considerations one can see that the Lagrangian and the Hamiltonian formalisms for the construction of rays are dual when the following important condition holds:

$$\det G(x, \dot{x}) = \det \left[\frac{1}{2}(L^2)_{\dot{x}\dot{x}} \right] = \det \left[\frac{1}{2}(H^2)_{pp} \right] = \det Q^{-1}(x, p) \neq 0.$$

This holds generally whenever \mathcal{V} are \mathcal{W} are convex, which also provides the convexity of the Hamiltonian and Lagrangian. In the case where \mathcal{V} is convex, the Fermat principle is true so that the Pontryagin maximum principle can be applied. For the non-convex case, the construction is more complicated and requires a different technique related to differential games theory that deals as a rule with convex-concave Hamiltonians.

10 Description of waves propagation using differential games approach

Generally, if the slowness surface \mathcal{N} is non-convex, it is impossible to transform the phase velocity surface \mathcal{W} to the wave surface \mathcal{V} to obtain the related variational problem

with a Lagrangian. Therefore, it is necessary to work with the original Hamiltonian constructed on the base of the phase velocity surface \mathcal{W} . Such a Hamiltonian is convex-concave in the impulse variable, which is the reason why this case can not be reduced to an optimal control problem. Note that convex-concave Hamiltonians are typical for differential games [8]. The main goal in the solution of a game is to find the set of optimal trajectories in its complexity, i.e. the optimal phase portrait. Trajectories are not arranged in that regular way as in the calculus of variations or even in the theory of optimal control. There are several specific trajectories, called singular lines, [8], [4], [10], which match regions filled by regular trajectories. The knowledge of types and locations of singularities can deliver important information about the behavior of rays in the associated problem of the propagation of acoustic waves.

The idea of this section is to formulate a differential game whose Hamiltonian coincides with, or is close to, the Hamiltonian of the corresponding wave propagation problem. For the propagation of elastic waves in anisotropic media, it is sufficient to use a game with the so-called “simple dynamics” [8]:

$$\dot{x} = u - v, \quad 0 \leq t \leq \mathcal{T}, \quad u \in P, \quad v \in Q.$$

Here the vectors u and v are controls of two players P_1 and P_2 , respectively. The controls assume values from convex sets P and Q which can depend on the phase-vector x , i.e. $P = P(x)$, $Q = Q(x)$. Such dependence reflects inhomogeneity of the elastic media. The goal of the first player is to bring the vector $x(t)$ to a given terminal surface M as soon as possible, i.e. to minimize the time t of the first event: $x(t) \in M$. The second player strives to maximize the time of reaching M using his controls v . In other words, the payoff of the game is the time of attaining M . For any point x , the value function $T(x)$ gives the optimal guaranteed time of attaining M .

It is well-known [8] that $T(x)$ satisfies the following Hamilton-Jacobi equation

$$\min_{u \in P} \max_{v \in Q} \langle T_x, u - v \rangle = -1$$

or, equivalently,

$$\max_{u \in P} \min_{v \in Q} \langle T_x, -u + v \rangle = 1$$

for all points x where T is differentiable. In the points of non-differentiability it satisfies the above equations in a viscosity sense [15, 7]. Therefore, if we construct the sets P and Q such that the left hand side of the last equation coincides with the Hamiltonian of the wave propagation problem, then the value function $T(x)$ delivers the propagation time of the wave front, i.e the time of the first arrival of the excitation at x .

Thus, assuming that P and Q are symmetric about the origin, we obtain:

$$H(x, p) = \max_u \min_v \langle p, -u + v \rangle = \max_u \langle p, u \rangle - \max_v \langle p, v \rangle, \quad u \in P(x), \quad v \in Q(x).$$

Obviously, the resulting Hamiltonian $H(x, p)$, generally, is not purely convex or concave with respect to p .

The theory of viscosity solutions provides the existence of piece-wise smooth solutions to the Hamilton-Jacobi equation

$$H \left(x, \frac{\partial S}{\partial x} \right) = 1.$$

Generally, the singular surfaces (lines in two dimensions) are of the following types: dispersal, equivocal, and focal, see Fig. 4(a). The arrows in Fig. 4(a) show the motion of the

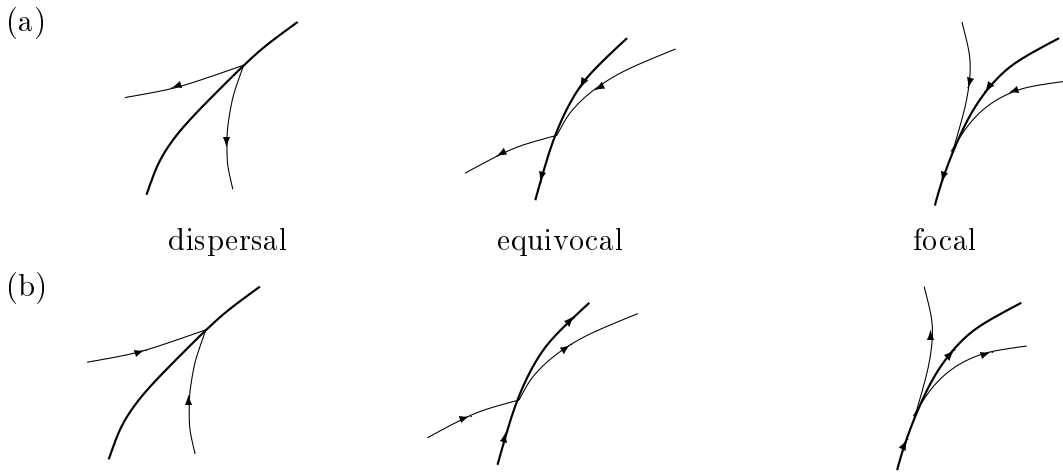


Figure 4: Singular lines: (a) direct time; (b) backward time.

phase point of the game in direct time. The dispersal surface does not contain trajectories, while the equivocal and focal surfaces consist of singular trajectories.

In Fig. 4(b) the directions of all arrows are reversed, which means the inversion of time in order to go on to the associated wave propagation problem. Recall that the terminal surface “absorbs” trajectories in a differential game, whereas it irradiates rays in the related wave propagation problem.

The gradient $\partial S/\partial x$ has a jump on the singular lines. We denote the gradient on different sides of a singular line by p and q , $p \neq q$. One important property of viscosity solutions is the following [10]: characteristics emanating from the singular line at a point x must be tangential to the line (as shown in Fig. 4) whenever the Hamiltonian is smooth in the vicinity of the singular point (x, p) or (x, q) .

In [10], using the definition of viscosity solutions, the following necessary conditions for possible values of p and q are derived in the form of inequalities depending on the character of the gradient jump:

$$H(x, \lambda q + (1 - \lambda)p) \geq 1, \quad 0 \leq \lambda \leq 1, \quad S = \max\{S^+, S^-\},$$

$$H(x, \lambda q + (1 - \lambda)p) \leq 1, \quad 0 \leq \lambda \leq 1, \quad S = \min\{S^+, S^-\}.$$

Here S^+ and S^- are smooth branches of the solution S ; the character of non-smoothness is illustrated in Fig. 5.

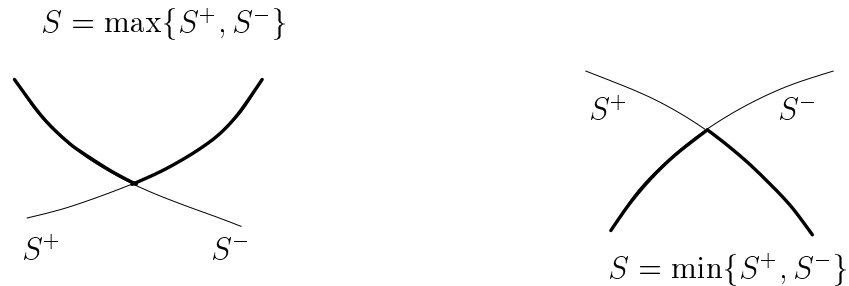


Figure 5: Types of nonsmoothness.

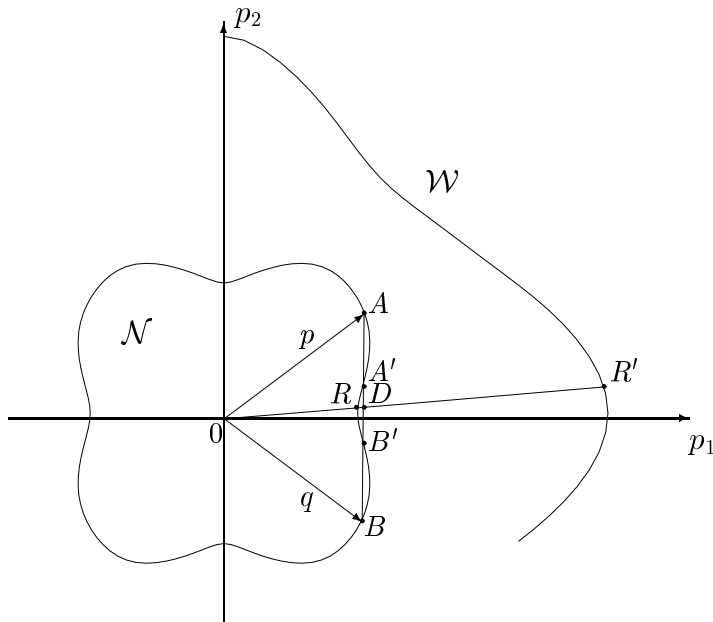


Figure 6: Geometric construction showing that singular lines are possible.

Perform a geometrical analysis (see Fig. 6) of the above inequalities using slowness and velocity surfaces similar to the ones shown in Fig. 2. Bear in mind a differential game with the Hamiltonian $H(x, p) = |p|W(x, p/|p|)$. Fix some vectors p and $q \in \mathcal{N}$ and consider the segment AB given by $\lambda q + (1 - \lambda)p$, $0 \leq \lambda \leq 1$.

Consider a point D of that segment and the ray passing through O and D . Let R and R' be points of the intersection of the ray with the surfaces \mathcal{N} and \mathcal{W} . Let d , r and r' be the lengths of the vectors OD , OR and OR' , respectively. Since $H(x, p) = |p|W(x, p/|p|)$, we have $H = dr'$. By definition of the surfaces \mathcal{V} and \mathcal{W} it holds $rr' = 1$. Since $d > r$ for the point considered, the relation $dr' > 1$ holds, and, consequently, $H > 1$. This consideration shows that the part $A'B'$ of the segment AB satisfies the inequality $H(x, p) \geq 1$, the parts AA' and $B'B$ satisfy the inequality $H(x, p) \leq 1$. This property allows us to specify candidates for vectors p, q for a potential singular line. Figures 7–9 demonstrate five possible configurations of p, q – two per a line of the dispersal or equivocal type (Fig. 7 and Fig. 8, respectively), and one for the focal type (Fig. 9). One can see that unique values of p, q for focal surface can be found through convexification of the surface \mathcal{N} . The tangency of trajectories in Fig. 4 leads to the tangency of the segments in Fig. 8 and Fig. 9 – at a single point for the equivocal case, and at both points for the focal case provided that the Hamiltonian is smooth in neighborhoods of (x, p) and (x, q) , see [10]. Note, that the singular lines specified in Figs. 7–9 are possible but they do not have necessarily to appear in the solution. These surfaces are only potentially allowed by the necessary conditions. Which surfaces will be present in the solution depends on boundary conditions. The exact analysis can be done on the base of the complete (numerical) solution of the problem.

A remark on a BVP for nonlinear Hamilton-Jacobi equation. For a classical solution of nonlinear first-order PDE it is known that locally a disturbance in boundary conditions propagates along a characteristic which is a single line. This means that the value of the solution at a point outside the boundary is defined by the value at one single boundary point. Generally speaking, this is not the case in the presence of singular

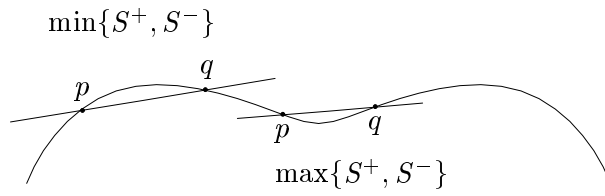


Figure 7: Dispersal situation.

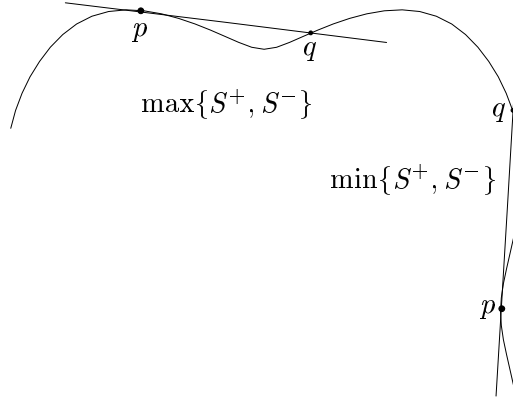


Figure 8: Equivocal situation.

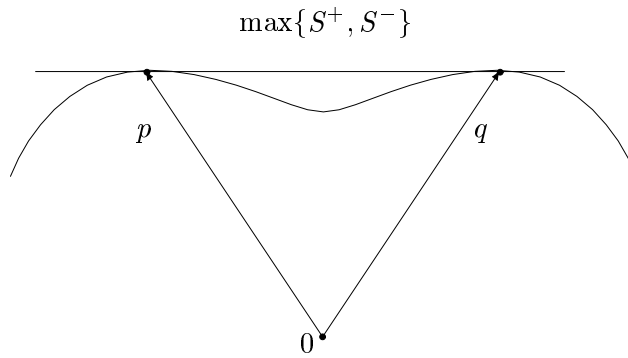


Figure 9: Focal situation.

characteristics typical for nonlinear and non-convex Hamiltonians, for example, in the case of a differential game problem. Due to singularities, the influence region for a point outside the boundary may be a proper part of the boundary but not a single point. In other words, a characteristic (singular) line is formed by a part of the boundary. Going back to a physical problem, one can see that a ray passing through a point of an anisotropic medium is influenced by a set of points, and, generally, can not be described using local properties

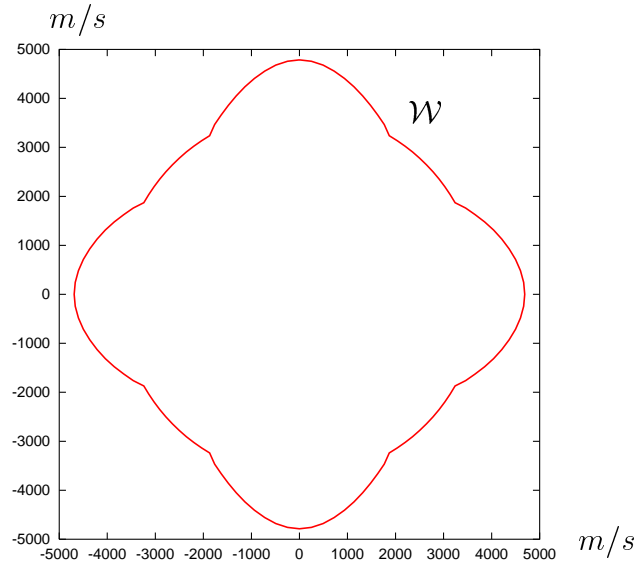


Figure 10: Velocity contour for shear acoustic waves in ST-quartz covered by a $5 \mu\text{m}$ SiO_2 film.

of the medium. Thus, the propagation of excitations in anisotropic media is a global process described by Hamilton-Jacobi equations whose solutions have to be constructed globally. There are many publications devoted to ray field and wave front construction using viscosity solutions and classical techniques (see, for example [3, 5, 14]). However, these works consider convex Hamiltonians. This paper utilizes methods of differential games for the analysis of convex-concave Hamiltonians.

Construction of equivalent differential games. We demonstrate application of differential games to the problem of propagation of surface acoustic waves in a quartz wafer covered by a thin film made of the isotropic silicon dioxide. Such a structure is typical for acoustic sensors whose operation principle is based on the piezoelectric excitation of surface acoustic waves and the detection of the phase shift in the waves that arises because of deposition of an additional mass on the sensor surface. The wave propagation velocity in such multi-layered structures is obtained using numerical treatment of dispersion relations derived via substituting plane waves into material equations and matching the interface conditions between the layers.

In Figure 10, the phase velocity contour for shear surface acoustic waves propagating in an ST-quartz wafer covered by a $5 \mu\text{m}$ SiO_2 film is shown. The contour is symmetric with respect to the origin. In fact, surface shear waves exist for directions from the set $\Omega = \{(\cos \phi, \sin \phi) : \phi \in [-\pi/6 + k\pi/2, \pi/6 + k\pi/2], k = \overline{0, 3}\}$. Such directions will be called feasible. For all non-feasible directions, the velocity value is the same and equals to 3739.79 m/s , which corresponds to the velocity of shear bulk waves in the structure considered.

Our next goal is to find constraint sets P and Q for the differential game

$$\dot{x} = u - v, \quad u \in P, \quad v \in Q,$$

so that, for every $\ell \in R^2$, $|\ell| = 1$, the value of the Hamiltonian

$$H(\ell) := \max_{u \in P} \min_{v \in Q} \langle \ell, -u + v \rangle$$

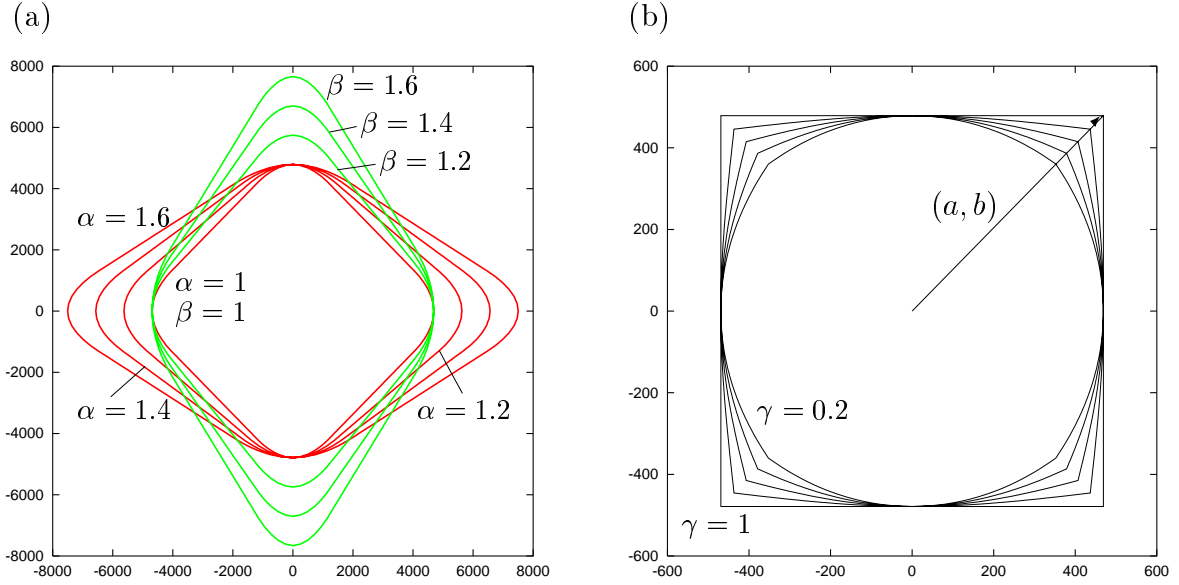


Figure 11: Families (a): $P^{1,\beta}$, $P^{\alpha,1}$; (b): Q^γ .

coincides with or is close to the value $W(\ell)$ taken from the velocity surface \mathcal{W} shown in Fig. 10.

Assuming that P and Q are symmetric with respect to the origin, we obtain

$$H(\ell) = \max_{u \in P} \langle \ell, -u \rangle - \max_{v \in Q} \langle \ell, -v \rangle = \max_{u \in P} \langle \ell, u \rangle - \max_{v \in Q} \langle \ell, v \rangle .$$

Evidently, there exist many sets P and Q satisfying the relation

$$\max_{u \in P} \langle \ell, u \rangle - \max_{v \in Q} \langle \ell, v \rangle = W(\ell), \quad \ell \in \mathbb{R}^2.$$

To approximate the set W along feasible directions, we use the families

$$P^{\alpha,\beta} = \{(\alpha x, \beta y) : (x, y) \in \text{co } W\}, \quad \alpha \geq 1, \beta \geq 1$$

and

$$Q^\gamma = \begin{cases} \frac{|y|^\gamma}{b} = \left(\frac{b^2}{x^2+y^2}\right)^{1-\gamma}, & |y| > \frac{b}{a}|x| \\ \frac{|y|^\xi}{a} = \left(\frac{a^2}{x^2+y^2}\right)^{1-\xi}, & |y| \leq \frac{b}{a}|x| \end{cases}$$

where $\gamma > 0$, $a = W((1,0)')$, $b = W((0,1)')$, and ξ is being calculated from the following formula

$$(b/\sqrt{a^2+b^2})^\xi = \frac{b}{a}(a/\sqrt{a^2+b^2})^\gamma.$$

The families $P^{1,\beta}$ and $P^{\alpha,1}$ are depicted in Fig. 11(a), the family Q^γ is shown in Fig. 11(b).

Denote

$$R^{\alpha,\beta,\gamma} = \max_{u \in P^{\alpha,\beta}} \langle \ell, u \rangle - \max_{v \in Q^\gamma} \langle \ell, v \rangle .$$

We compute α_* , β_* , γ_* that provide minimum in

$$\min_{\alpha,\beta,\gamma} \{W(\ell) - R^{\alpha,\beta,\gamma}(\ell), \ell \in \Omega\}.$$

The sets P^{α_*,β_*} , Q^{γ_*} , and $R^{\alpha_*,\beta_*,\gamma_*}$ are depicted in Fig. 12. The sets W and $R^{\alpha_*,\beta_*,\gamma_*}$ practically coincide on Ω . To approximate W along the non-feasible directions, the set

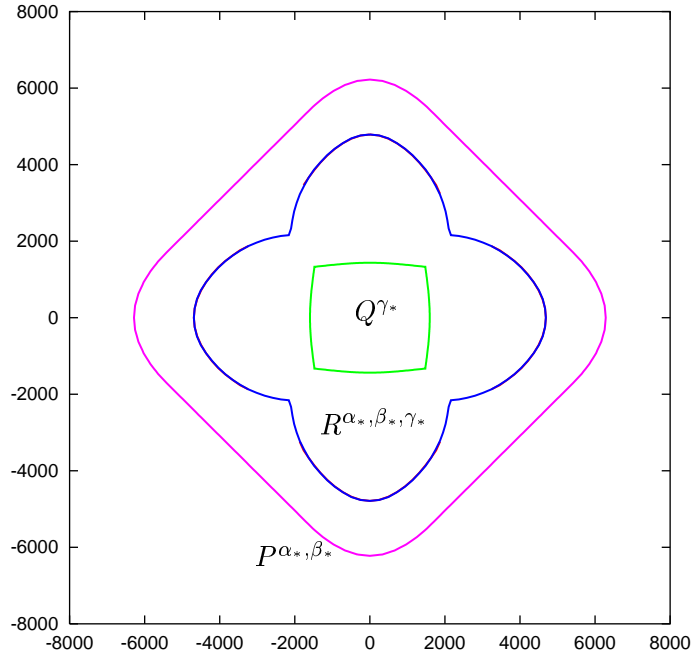


Figure 12: Sets P^{α_*, β_*} , Q^{γ_*} , and $R^{\alpha_*, \beta_*, \gamma_*}$.

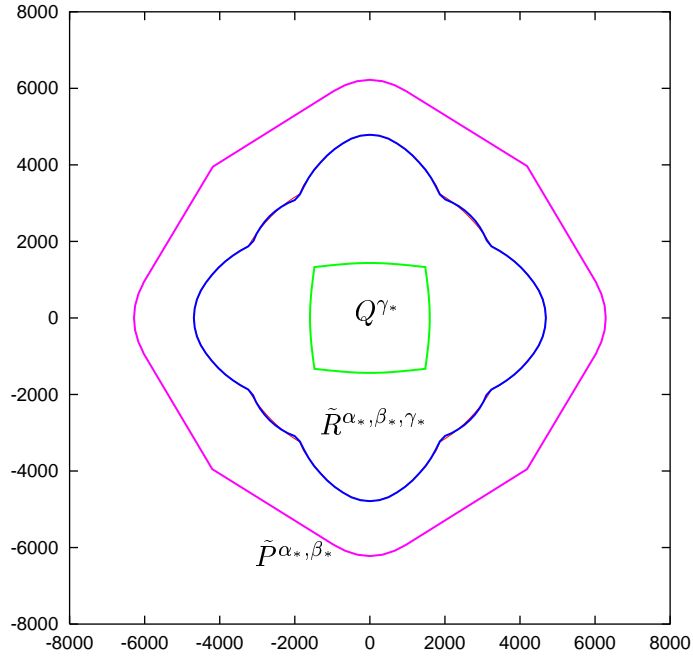


Figure 13: Sets $\tilde{P}^{\alpha_*, \beta_*}$, Q^{γ_*} , $\tilde{R}^{\alpha_*, \beta_*, \gamma_*}$, and W .

P^{α_*, β_*} is transformed into the set $\tilde{P}^{\alpha_*, \beta_*}$ (see Fig. 13) being a convex hull of a set that differs from P^{α_*, β_*} along four directions that are close to $(\cos(\pi/4+k\pi/2), \sin(\pi/4+k\pi/2))$, $k = \overline{0, 3}$.

Consider now a problem where the first player having the control parameter $u \in P = \tilde{P}^{\alpha_*, \beta_*}$ at his disposal minimizes the time of attaining a given terminal set M , whereas the second player whose control parameter is $v \in Q = Q^{\gamma_*}$ maximizes this time. The isochrones or level sets of the value function of this problem are wave fronts in the wave

propagation problem with the velocity contour \mathcal{W} provided that M is an excitation source that generates shear waves in all directions. Using a further development of algorithms (see [11], [12]) for computation of level sets of value functions for time-optimal game problems, one can obtain a portrait of the propagation of wave fronts for sources with very complicated geometries. The propagation time can also be found very precisely which might be helpful when estimating the sensitivity of acoustic sensors.

In Figure 14(a), numerically computed wave fronts propagating from a curved source M in x_1 -direction are shown. Corner points of the fronts, being connected, form singular lines (see Fig. 14(b)). These lines can be classified using some additional analysis when computing the wave fronts with the above mentioned algorithm. Black lines in Fig. 14(b) are equivocal, white lines are focal, and grey lines are dispersal. Fig. 15 shows the behavior of optimal trajectories schematically, the arrows indicate the motion in backward time. We expect that singular lines are closely related to the propagation of physical rays. For example, dispersal lines correspond to the intersection of rays (they can be generally continued beyond the first arrival); equivocal lines absorb rays from one side, guide them and radiate to the other side; focal lines guide rays and radiate them to the both sides. This knowledge can be useful for the study of focusing and beam-steering properties of specially designed transducers.

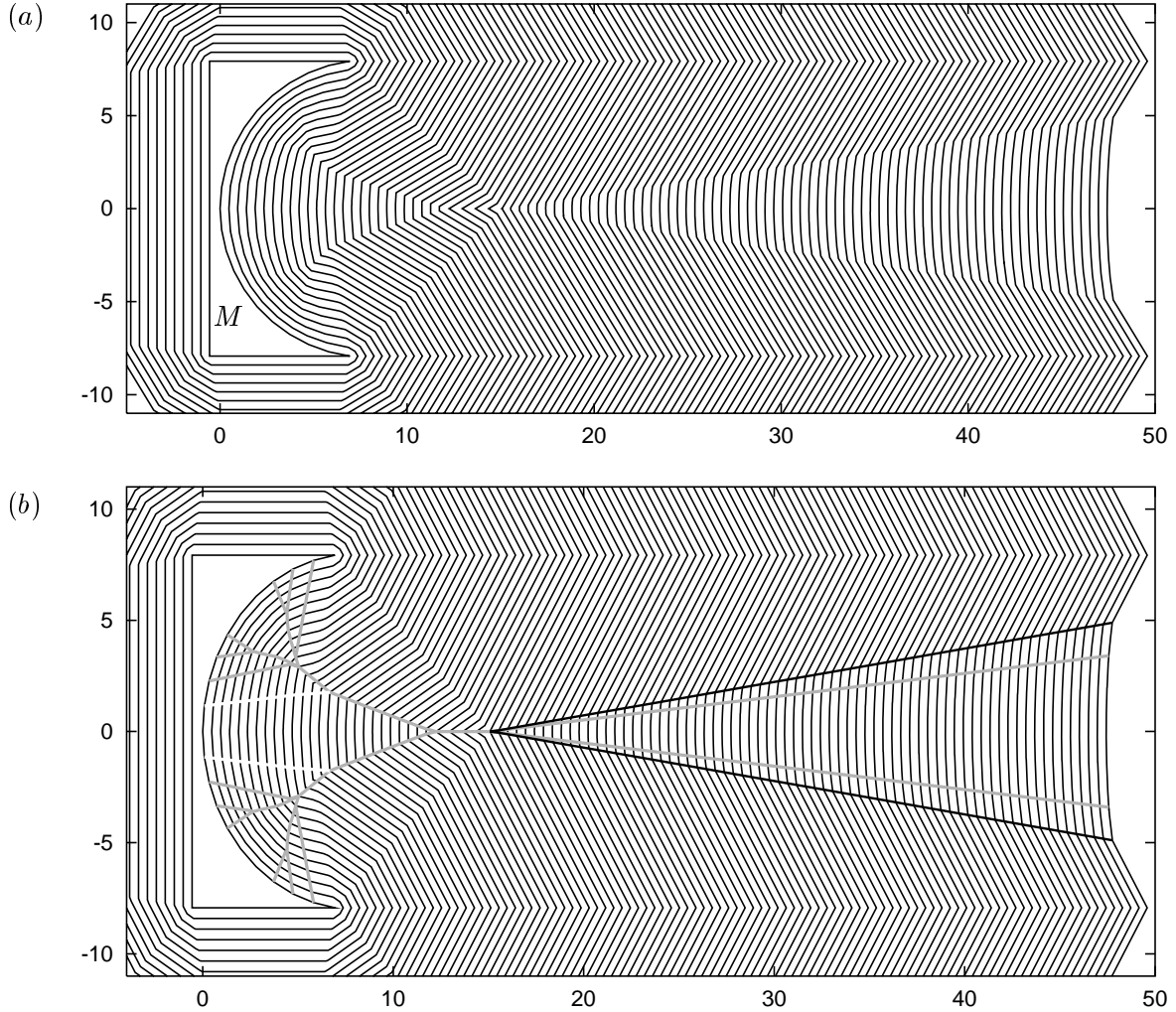


Figure 14: Propagation of waves: (a) Wave fronts; (b): Singular lines.

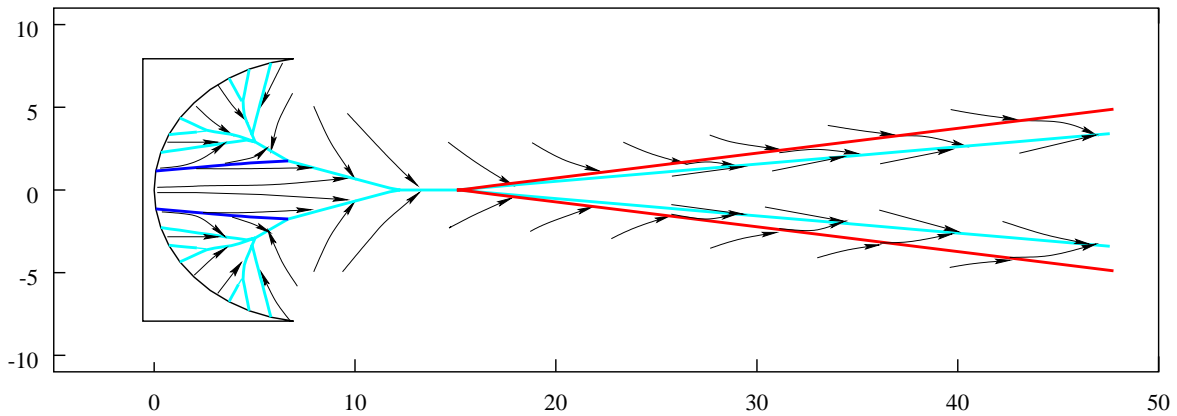


Figure 15: Behavior of optimal trajectories in reverse time.

11 Conclusions

Local properties of a medium in the propagation of excitations are described by the *Wave surface* \mathcal{V} , *Slowness surface* \mathcal{N} , and *Velocity surface* \mathcal{W} . There is a duality between the surfaces \mathcal{V} and \mathcal{N} expressed through the Lagrangian $L(x, \dot{x})$ and the Hamiltonian $H(x, p)$ as follows:

$$\mathcal{V} : L(x, v) = 1; \quad \mathcal{N} : H(x, n) = 1,$$

where v and n are velocity and refractive index vectors. Transformation between L and H is possible analytically whenever these functions are convex in the second variable. This holds for all isotropic media and for some anisotropic materials, if the anisotropy is not too large. The Lagrangian formalism describes rays as extremals of the corresponding Euler equation, which expresses Fermat's principle. The Hamiltonian formalism describes wave fronts and rays as characteristics of the Hamilton-Jacobi (eikonal) equation.

When the Hamiltonian generates a convex-concave contour (given by the equation above), which is typical for anisotropic media, the Lagrangian expressing Fermat's principle is not a single-valued function so that the set of rays can not be computed through the Lagrangian. Nevertheless, the characteristics of the Hamilton-Jacobi (eikonal) equation can be identified with physical rays. The set of these characteristics contains singular paths that are well known in the theory of differential games (conflict-controlled systems). The idea of the paper is to find a differential game whose Hamilton-Jacoby-Bellman equation coincides with the eikonal equation of the medium. Regular characteristics are known to represent physical rays. We conjecture that the same is true for singular characteristics so that the optimal phase portrait of such a game represents the complete set of rays of first arrival in the corresponding physical problem. The whole picture of rays depends, in addition to the Hamiltonian, upon the initial conditions. Numerical results of the paper show the realizability of such an approach.

An interesting question for further research is related to physical interpretation of the objectives of the players (minimization and maximization of the propagation time). Physical experiments confirming the structure of ray fields computed seem to be very useful.

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