A fully coupled model of a nonlinear thin plate excited by piezoelectric actuators

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Abstract. A model describing oscillations of nonlinear thin plates excited by patches made of a piezoelectric ceramic is considered. The specific of the model is that the mutual coupling between elastic deformations and electric fields is assumed. Partial differential equations describing the model are stated and their solvability is proved. The question of homogenization when the number of the piezoelectric patches goes to infinity whereas their dimension goes to zero is discussed.

1 Introduction

This paper is a continuation of the work [1] where the interface between electric fields and elastic deformations was assumed to be rather simple: electric fields generate elastic deformations but not vice versa. In this paper, a full coupling between elastic deformations and electric fields is assumed. This means that the generation of electric fields trough elastic deformations is taken into consideration. Therefore, the model contains additional variables, the potential functions, that describe electric fields arising both through the voltage applied to the piezoelectric patches and due to elastic deformations. The aim of this paper consists in obtaining partial differential equations describing the phenomena and in the statement of their solvability. The question of homogenization is discussed.

It should be mentioned that the elasticity part of our model is the Kármán system (see [2]) with discontinuous coefficients and additional terms related to piezoelectric properties of the patches. Thus, the model is applicable to the case of "great" deformations: the bending must be much less then the longitudinal dimension of the plate (see [3]). Taking into account geometrical nonlinearities is motivated by the fact that linear models are applicable to the case of extremely small bending that must be much less then the thickness of the plate.

2 Model of piezoelectric media

We consider a macroscopic and quasi-static model of piezoelectric media. The first characteristic means that the model contains only mean values of physical magnitudes. The second characteristic assumes that the frequency of electric fields is sufficiently small so that magnetic fields do not appear and electromagnetic waves are absent (see [4] for general models).

Assume that the deformations of the medium are sufficiently small and use the following conventional form of the strain tensor:

$$d_{lm} := 1/2(u_{l,m} + u_{m,l} + u_{k,l} \cdot u_{k,m}),$$

where $u_l = y_l(x_1, x_2, x_3, t) - x_l$ is the displacement. The summation over the repeating indices is assumed. Latin symbols run from 1 to 3 whenever Greek symbols run from 1 to 2. Commas before indices denote differentiation with respect to the corresponding components of the vector x.
Let σ_{ij} be the stress tensor. We consider linear material laws (see [5]):

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} d_{kl} - e_{kij} E_k, \\ D_i &= \varepsilon_{ij} E_j + e_{ikl} d_{kl}. \end{aligned} \tag{1}$$

The coefficients are such that

C_{ijkl}	is the stiffness tensor,
e_{ikl}	is the piezoelectric tensor,
ε_{ij}	is the permittivity tensor.

To cover all possible cases of piezoelectric ceramics, assume that all coefficients may be nonzero and different. This is really the case for the triclinic crystal systems (see [5]). In conclusion of this section, we give the formula for the density of the energy of a piezoelectric medium (see e.g. [4]):

$$\chi = \frac{1}{2}(\sigma_{ij} d_{ij} - E_i D_i). \tag{2}$$

3 Oscillations of thin plates

3.1 A plate with piezoelectric actuators

Without any loss of generality, we consider a thin plate supplied with two symmetric patches made of a piezoelectric ceramic. The plate itself consists of a metal, and the horizontal external surfaces of the patches are covered by a metal. Therefore, the voltage can be applied to the patches as shown in Figure 1.

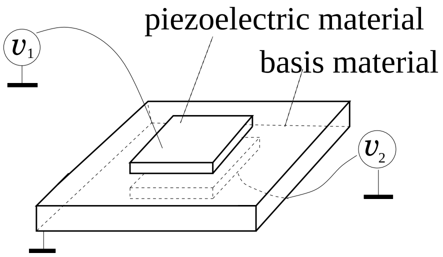

Figure 1.

As usually, we assume the existence of a neutral plane on which all deformations caused by the “pure bending” are equal to zero (see Figure 2). Note that the deformations caused by the stretching of the plate do not vanish on the neutral plane.

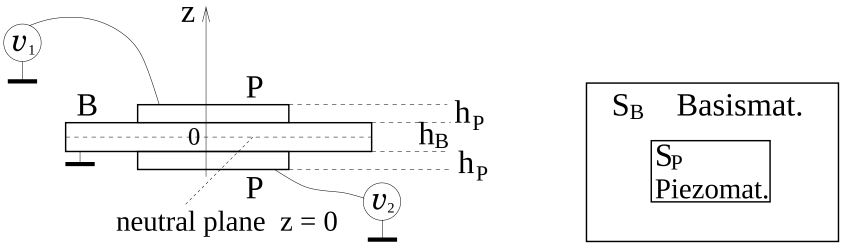

Figure 2.

Let S be the projection of the plate onto the plane $x_3 = 0$, S_P the projection of the piezopatches onto the plane $x_3 = 0$, and S_B the complement of S_P, i.e. $S = S_P \cup S_B$ (see Figure 2). Let I_P and I_B denote indicator functions of S_P and S_B respectively.

3.2 Kirchhoff-Love-Koiter hypothesis

Let Q be a point of the plate and M its orthogonal projection onto the neutral plane. Let $u_i,\ i = 1, 2, 3$ be components of the displacement of M. According to the Kirchhoff-Love-Koiter hypothesis (see e.g. [6]), the components of the strain tensor at the point Q are defined as follows:

$$\begin{aligned} &d_{\alpha\beta} = 1/2(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}u_{3,\beta}) - x_3 \cdot u_{3,\alpha,\beta}, \\ &d_{3\alpha} = d_{\alpha 3} = 0, \quad d_{33} = -\frac{1}{C_{3333}} C_{33\alpha\beta} d_{\alpha\beta} + \frac{1}{C_{3333}} e_{k33} E_k. \end{aligned} \tag{3}$$

Note that the assumption $d_{3\alpha} = d_{\alpha 3} = 0$, $\alpha = 1, 2$, means the absence of transversal shear strains, which implies the conservation of the normal. The component d_{33} is computed from the relation $\sigma_{33} = 0$, which expresses the in-plain stress condition. Therefore, d_{33} is material and field dependent. Using (1), (2), (3) and taking into account that $\sigma_{33} = 0$ yield the energy density for the piezopatches:

$$\chi^P = \frac{1}{2} c^P_{\alpha\beta\gamma\mu} d_{\alpha\beta} d_{\gamma\mu} - e_{i\alpha\beta} d_{\alpha\beta} E_i - \frac{1}{2} \epsilon_{ij} E_i E_j \tag{4}$$

where

$$c^P_{\alpha\beta\gamma\mu} = C_{\alpha\beta\gamma\mu} - \frac{C_{33\alpha\beta} C_{33\gamma\mu}}{C_{3333}}, \quad e_{i\alpha\beta} = e_{i\alpha\beta} - \frac{C_{33\alpha\beta} e_{i33}}{C_{3333}}, \quad \epsilon_{ij} = \varepsilon_{ij} + \frac{e_{i33} e_{j33}}{C_{3333}}$$

One can verify that the tensors $c^P_{\alpha\beta\gamma\mu}$ and ϵ_{ij} are positive definite for all realistic materials. This means that

$$c^P_{\alpha\beta\gamma\mu} r_{\alpha\beta} r_{\gamma\mu} \geq \nu r_{\alpha\beta} r_{\alpha\beta} \quad \text{and} \quad \epsilon_{ij} E_i E_j \geq \nu E_i E_i$$

for all symmetric $r_{\alpha\beta} \in R^{2\times 2}$ and all $E_i \in R^3$, respectively. Substituting relations (3) into (4), we obtain:

$$\chi^P = \frac{1}{2}c^P_{\alpha\beta\gamma\mu}s_{\alpha\beta}s_{\gamma\mu}+\frac{1}{2}x_3^2c^P_{\alpha\beta\gamma\mu}u_{3,\alpha,\beta}u_{3,\gamma,\mu} - e_{i\alpha\beta}s_{\alpha\beta}E_i - \frac{1}{2}\epsilon_{ij}E_iE_j + \\ + x_3c^P_{\alpha\beta\gamma\mu}s_{\alpha\beta}u_{3,\gamma,\mu} - x_3e_{i\alpha\beta}u_{3,\alpha,\beta}E_i, \tag{5}$$

where

$$s_{\alpha\beta} = 1/2(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}\cdot u_{3,\beta}).$$

The energy density of the base material is given by the formula

$$\chi^B = \frac{1}{2}c^B_{\alpha\beta\gamma\mu}s_{\alpha\beta}s_{\gamma\mu}+\frac{1}{2}x_3^2c^B_{\alpha\beta\gamma\mu}u_{3,\alpha,\beta}u_{3,\gamma,\mu} + x_3c^B_{\alpha\beta\gamma\mu}s_{\alpha\beta}u_{3,\gamma,\mu}\,, \tag{6}$$

where

$$c^B_{\alpha\beta\gamma\mu} = C_{\alpha\beta\gamma\mu} - \frac{C_{33\alpha\beta}C_{33\gamma\mu}}{C_{3333}}$$

with $C_{\alpha\beta\gamma\mu}$ being the stiffness tensor of the base material.

3.3 Electric fields in piezoelectric patches

Note that electric fields in the upper and lower patches are independent each from other because of the metal layer between them. The interface between the electric fields in the piezopatches occurs due to deformation of the whole structure. Thus, we need two potential functions to describe electric fields inside the patches.

Let us consider the upper piezopatch. Let $\bar{E}_j$ be the electric field in the upper piezopatch. It is convenient to introduce a new variable $\bar{z} = x_3 - (h_B + h_P)/2$ such that $\bar{z} = -h_P/2$ if $x_3 = h_B/2$, and $\bar{z} = h_P/2$ if $x_3 = h_B/2 + h_P$.
Since the patch is supposed to be thin, we assume that $\bar{E}_3$ depends linear on $\bar{z}$:

$$\bar{E}_3(x_1, x_2, \bar{z}) = \bar{E}^0(x_1, x_2) + \bar{z}\bar{E}^1(x_1, x_2) \tag{7}$$

Let $\bar{\varphi}$ be the potential function for the upper piezopatch. Then

$$\frac{\partial\bar{\varphi}}{\partial\bar{z}} = \bar{E}_3(x_1, x_2, \bar{z}) = \bar{E}^0(x_1, x_2) + \bar{z}\bar{E}^1(x_1, x_2),$$

and hence

$$\bar{\varphi}(x_1, x_2, \bar{z}) = \bar{z}\bar{E}^0(x_1, x_2) + \frac{1}{2}\bar{z}^2\bar{E}^1(x_1, x_2) + \bar{\phi}(x_1, x_2).$$

From the boundary conditions:

$$\begin{aligned} \bar{\varphi}(x_1, x_2, \bar{z}) &= v_1, \quad \text{if} \quad \bar{z} = h_P/2, \\ \bar{\varphi}(x_1, x_2, \bar{z}) &= 0, \quad\; \text{if} \quad \bar{z} = -h_P/2, \end{aligned} \tag{8}$$

we obtain that

$$\bar{E}^0(x_1,x_2) = \frac{v_1}{h_P}, \quad \bar{E}^1(x_1,x_2) = -\frac{4v_2}{h_P^2} + \frac{8\bar{\phi}}{h_P^2}.$$

Therefore,

$$\bar{\varphi} = \left(\frac{1}{h_P}\bar{z} + \frac{2}{h_P^2}\bar{z}^2\right) v_1 + \left(1 - \frac{4}{h_P^2}\bar{z}^2\right)\bar{\phi}.$$

Thus,

$$\bar{E}_\alpha = \frac{\partial\bar{\varphi}}{\partial x_\alpha} = \left(1 - \frac{4}{h_P^2}\bar{z}^2\right)\frac{\partial\bar{\phi}}{\partial x_\alpha}, \qquad \bar{E}_3 = \frac{\partial\bar{\varphi}}{\partial\bar{z}} = \left(\frac{1}{h_P} + \frac{4}{h_P^2}\bar{z}\right) v_1 - \frac{8}{h_P^2}\bar{z}\bar{\phi}. \tag{9}$$

Let us consider the lower piezopatch. We introduce a variable $\underline{z} = x_3 + (h_B + h_P)/2$ such that $\underline{z} = -h_P/2$ if $x_3 = -h_B/2 - h_P$, and $\underline{z} = h_P/2$ if $x_3 = -h_B/2$. The linear ansatz is used again:

$$\underline{E}_3(x_1,x_2,\underline{z}) = \underline{E}^0(x_1,x_2) + \underline{z}\,\underline{E}^1(x_1,x_2). \tag{10}$$

Let $\underline{\varphi}$ be the potential function. Then

$$\underline{\varphi}(x_1,x_2,\underline{z}) = \underline{z}\,\underline{E}^0(x_1,x_2) + \frac{1}{2}\underline{z}^2\underline{E}^1(x_1,x_2) + \underline{\phi}(x_1,x_2).$$

Taking into account the obvious boundary conditions:

$$\begin{aligned} \underline{\varphi}(x_1,x_2,\underline{z}) &= v_1, \quad \text{if} \quad \underline{z} = -h_P/2, \\ \underline{\varphi}(x_1,x_2,\underline{z}) &= 0, \quad \text{if} \quad \underline{z} = h_P/2, \end{aligned} \tag{11}$$

we obtain

$$\underline{E}^0(x_1,x_2) = -\frac{v_2}{h_P}, \quad \underline{E}^1(x_1,x_2) = \frac{4v_2}{h_P^2} - \frac{8\underline{\phi}}{h_P^2},$$

and hence

$$\underline{\varphi} = \left(\frac{2}{h_P^2}\underline{z}^2 - \frac{1}{h_P}\underline{z}\right) v_2 + \left(1 - \frac{4}{h_P^2}\underline{z}^2\right)\underline{\phi}.$$

Thus,

$$\underline{E}_\alpha = \frac{\partial\underline{\varphi}}{\partial x_\alpha} = \left(1 - \frac{4}{h_P^2}\underline{z}^2\right)\frac{\partial\underline{\phi}}{\partial x_\alpha}, \quad \underline{E}_3 = \frac{\partial\underline{\varphi}}{\partial\underline{z}} = \left(\frac{4}{h_P^2}\underline{z} - \frac{1}{h_P}\right) v_2 - \frac{8}{h_P^2}\underline{z}\underline{\phi}. \tag{12}$$

3.4 Governing equations

Using (5), we obtain the energy densities of the upper and lower piezopatches:

$$\begin{aligned}\bar{\chi}^P = {} & \frac{1}{2} c^P_{\alpha\beta\gamma\mu} s_{\alpha\beta} s_{\gamma\mu} + \frac{1}{2} x_3^2 c^P_{\alpha\beta\gamma\mu} u_{3,\alpha,\beta} u_{3,\gamma,\mu} - e_{i\alpha\beta} s_{\alpha\beta} \bar{E}_i - \frac{1}{2} \epsilon_{ij} \bar{E}_i \bar{E}_j + \\ & + x_3 c^P_{\alpha\beta\gamma\mu} s_{\alpha\beta} u_{3,\gamma,\mu} - x_3 e_{i\alpha\beta} u_{3,\alpha,\beta} \bar{E}_i,\end{aligned} \tag{13}$$

$$\begin{aligned}\underline{\chi}^P = {} & \frac{1}{2} c^P_{\alpha\beta\gamma\mu} s_{\alpha\beta} s_{\gamma\mu} + \frac{1}{2} x_3^2 c^P_{\alpha\beta\gamma\mu} u_{3,\alpha,\beta} u_{3,\gamma,\mu} - e_{i\alpha\beta} s_{\alpha\beta} \underline{E}_i - \frac{1}{2} \epsilon_{ij} \underline{E}_i \, \underline{E}_j + \\ & + x_3 c^P_{\alpha\beta\gamma\mu} s_{\alpha\beta} u_{3,\gamma,\mu} - x_3 e_{i\alpha\beta} u_{3,\alpha,\beta} \underline{E}_i,\end{aligned} \tag{14}$$

where $\bar{E}_j$ and $\underline{E}_j$ are defined by (9) and (12) respectively. The total energy is obtained via integration over the volumes of the piezoelectric and base materials:

$$F = \int\limits_{h_B/2}^{h_B/2+h_P} dx_3 \iint\limits_{S_P} \bar{\chi}^P dx_1 dx_2 + \int\limits_{-h_B/2-h_P}^{-h_B/2} dx_3 \iint\limits_{S_P} \underline{\chi}^P dx_1 dx_2 + \int\limits_{-h_B/2}^{-h_B/2} dx_3 \iint\limits_{S} \chi^B dx_1 dx_2.$$

Substituting (13), (14), and (6) into this formula and taking into account (9) and (12), we obtain after the integration over x_3:

$$\begin{aligned}F = \iint\limits_{S_P} \Bigg\{ & h_P c^P_{\alpha\beta\gamma\mu} s_{\alpha\beta} s_{\gamma\mu} + \left(\frac{h_P^3}{12} + \frac{(h_B + h_P)^2 h_P}{4} \right) c^P_{\alpha\beta\gamma\mu} u_{3,\alpha,\beta} u_{3,\gamma,\mu} - \\ & e_{3\alpha\beta} s_{\alpha\beta} v_1 - \left(\tfrac{h_B}{2} + \tfrac{5h_P}{6}\right) e_{3\alpha\beta} u_{3,\alpha,\beta} v_1 + e_{3\alpha\beta} s_{\alpha\beta} v_2 - \left(\tfrac{h_B}{2} + \tfrac{5h_P}{6}\right) e_{3\alpha\beta} u_{3,\alpha,\beta} v_2 \\ & - \frac{4}{15} h_P \epsilon_{\alpha\beta} \frac{\partial \bar{\phi}}{\partial x_\alpha} \frac{\partial \bar{\phi}}{\partial x_\beta} - \frac{8}{3h_P} \epsilon_{33} \bar{\phi}^2 - \frac{2}{3} \epsilon_{3\alpha} \frac{\partial \bar{\phi}}{\partial x_\alpha} v_1 + \frac{8}{3h_P} \epsilon_{33} \bar{\phi} v_1 - \frac{7}{3h_P} \epsilon_{33} v_1^2 \\ & - \frac{4}{15} h_P \epsilon_{\alpha\beta} \frac{\partial \underline{\phi}}{\partial x_\alpha} \frac{\partial \underline{\phi}}{\partial x_\beta} - \frac{8}{3h_P} \epsilon_{33} \underline{\phi}^2 + \frac{2}{3} \epsilon_{3\alpha} \frac{\partial \underline{\phi}}{\partial x_\alpha} v_2 + \frac{8}{3h_P} \epsilon_{33} \underline{\phi} v_2 - \frac{7}{3h_P} \epsilon_{33} v_2^2 \\ & - \frac{2}{3} h_P e_{\gamma\alpha\beta} s_{\alpha\beta} \frac{\partial \bar{\phi}}{\partial x_\gamma} - \frac{(h_B + h_P) h_P}{3} e_{\gamma\alpha\beta} u_{3,\alpha,\beta} \frac{\partial \bar{\phi}}{\partial x_\gamma} + \frac{2}{3} h_P e_{3\alpha\beta} u_{3,\alpha,\beta} \bar{\phi} \\ & - \frac{2}{3} h_P e_{\gamma\alpha\beta} s_{\alpha\beta} \frac{\partial \underline{\phi}}{\partial x_\gamma} + \frac{(h_B + h_P) h_P}{3} e_{\gamma\alpha\beta} u_{3,\alpha,\beta} \frac{\partial \underline{\phi}}{\partial x_\gamma} + \frac{2}{3} h_P e_{3\alpha\beta} u_{3,\alpha,\beta} \underline{\phi} \Bigg\} dx_1 dx_2 \\ & + \iint\limits_{S} \left\{ \frac{h_B}{2} c^B_{\alpha\beta\gamma\mu} s_{\alpha\beta} s_{\gamma\mu} + \frac{h_B^3}{24} c^B_{\alpha\beta\gamma\mu} u_{3,\alpha,\beta} u_{3,\gamma,\mu} \right\} dx_1 dx_2.\end{aligned}$$

The variational principle reads:

$$\delta_{u_1}F + \iint\limits_{S_P} h_P\,\rho_P u_{1tt}\delta u_1 dx_1 dx_2 + \iint\limits_{S} h_B\,\rho_B u_{1tt}\delta u_1 dx_1 dx_2 +$$

$$\delta_{u_2}F + \iint\limits_{S_P} h_P\,\rho_P u_{2tt}\delta u_2 dx_1 dx_2 + \iint\limits_{S} h_B\,\rho_B u_{2tt}\delta u_2 dx_1 dx_2 +$$

$$\delta_{u_3}F + \iint\limits_{S_P} h_P\,\rho_P u_{3tt}\delta u_3 dx_1 dx_2 + \iint\limits_{S} h_B\,\rho_B u_{3tt}\delta u_3 dx_1 dx_2 = 0,$$

$$\delta_{\bar\phi}F + \delta_{\underline{\phi}}F = 0.$$

Putting together the integrals over S_P yields a weak formulation of the problem:

$$(\rho\vec{u}_{tt},\vec{\psi}) + \omega^1(\vec{u},\vec{\psi}) - b^1(\vec{u},\vec{\phi},\vec{\psi}) - e^1(\vec{u},\vec{\psi},\vec{v}) = 0, \tag{15}$$

$$a(\vec{\phi},\vec{\eta}) + b(\vec{u},\vec{\eta},\vec{u}) - g(\vec{\eta},\vec{v}) = 0, \tag{16}$$

$$\vec{u}(0) = \vec{u}_0,\ \vec{u}_t(0) = \vec{u}_0'. \tag{17}$$

Here the following notation are used:

$$\vec{u} = (u_1,u_2,u_3), \quad \vec{\psi} = (\psi_1,\psi_2,\psi_3), \quad \vec{\phi} = (\bar\phi,\underline{\phi}), \quad \vec{\eta} = (\bar\eta,\underline{\eta}), \quad \vec{v} = (v_1,v_2),$$

$$(\vec{u},\vec{\psi}) = \iint\limits_{S} \Big\{u_1\psi_1 + u_2\psi_2 + u_3\psi_3\Big\} dx_1 dx_2,$$

$$\omega^1(\vec{u},\vec{\psi}) = \iint\limits_{S} \Big\{\tau_{\alpha\beta}(\vec{u})(\psi_{\alpha,\beta} + u_{3,\alpha}\psi_{3,\beta}) + \gamma_{\alpha\beta\gamma\mu}u_{3,\alpha,\beta}\psi_{3,\gamma,\mu}\Big\} dx_1 dx_2,$$

$$\omega(\vec{u},\vec{\psi}) = \iint\limits_{S} \Big\{\tau_{\alpha\beta}(\vec{u})(\psi_{\alpha,\beta} + \tfrac{1}{2}u_{3,\alpha}\psi_{3,\beta}) + \gamma_{\alpha\beta\gamma\mu}u_{3,\alpha,\beta}\psi_{3,\gamma,\mu}\Big\} dx_1 dx_2,$$

$$b^1(\vec{u},\vec{\phi},\vec{\psi}) = \iint\limits_{S_P} \Big\{\theta e_{\gamma\alpha\beta}(\bar\phi_{,\gamma} + \underline{\phi}_{,\gamma})(\psi_{\alpha,\beta} + u_{3,\alpha}\psi_{3,\beta}) - \theta e_{3\alpha\beta}(\bar\phi + \underline{\phi})\psi_{3,\alpha,\beta} + \sigma e_{\gamma\alpha\beta}(\bar\phi_{,\gamma} - \underline{\phi}_{,\gamma})\psi_{3,\alpha,\beta}\Big\} dx_1 dx_2,$$

$$b(\vec{u},\vec{\phi},\vec{\psi}) = \iint\limits_{S_P} \Big\{\theta e_{\gamma\alpha\beta}(\bar\phi_{,\gamma} + \underline{\phi}_{,\gamma})(\psi_{\alpha,\beta} + \tfrac{1}{2}u_{3,\alpha}\psi_{3,\beta}) - \theta e_{3\alpha\beta}(\bar\phi + \underline{\phi})\psi_{3,\alpha,\beta} + \sigma e_{\gamma\alpha\beta}(\bar\phi_{,\gamma} - \underline{\phi}_{,\gamma})\psi_{3,\alpha,\beta}\Big\} dx_1 dx_2,$$

$$e^1(\vec{u},\vec{\psi},\vec{v}) = \int\!\!\int_{S_P} \Big\{ h_B^{-1} e_{3\alpha\beta}(\psi_{\alpha,\beta} + u_{3,\alpha}\psi_{3,\beta})(v_1 - v_2) + \ell e_{3\alpha\beta}\psi_{3,\alpha,\beta}(v_1 + v_2) \Big\},$$

$$e(\vec{u},\vec{\psi},\vec{v}) = \int\!\!\int_{S_P} \Big\{ h_B^{-1} e_{3\alpha\beta}(\psi_{\alpha,\beta} + \tfrac{1}{2}u_{3,\alpha}\psi_{3,\beta})(v_1 - v_2) + \ell e_{3\alpha\beta}\psi_{3,\alpha,\beta}(v_1 + v_2) \Big\},$$

$$a(\vec{\phi},\vec{\eta}) = \int\!\!\int_{S_P} \Big\{ a\epsilon_{\alpha\beta}(\bar{\phi}_{,\alpha}\bar{\eta}_{,\beta} + \underline{\phi}_{,\alpha}\underline{\eta}_{,\beta}) + b\epsilon_{33}(\bar{\phi}\bar{\eta} + \underline{\phi}\,\underline{\eta}) \Big\} dx_1 dx_2,$$

$$g(\vec{\eta},\vec{v}) = \int\!\!\int_{S_P} \Big\{ - k\epsilon_{3\gamma}\bar{\eta}_{,\gamma}v_1 + g\epsilon_{33}\bar{\eta}v_1 + k\epsilon_{3\gamma}\underline{\eta}_{,\gamma}v_2 + g\epsilon_{33}\underline{\eta}v_2 \Big\} dx_1 dx_2,$$

$$\rho = \left(\rho^B + \frac{2h_P}{h_B}\rho^P\right) I_P(x_1,x_2) + \rho_B I_B(x_1,x_2),$$

$$c_{\alpha\beta\gamma\mu} = \left(c^B_{\alpha\beta\gamma\mu} + \frac{2h_P}{h_B}c^P_{\alpha\beta\gamma\mu}\right) I_P(x_1,x_2) + c^B_{\alpha\beta\gamma\mu} I_B(x_1,x_2),$$

$$\tau_{\alpha\beta} = c_{\alpha\beta\gamma\mu}s_{\alpha\beta}, \quad s_{\alpha\beta} = 1/2(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}u_{3,\beta}),$$

$$\gamma_{\alpha\beta\gamma\mu} = \left[\frac{h_B^2}{12}c^B_{\alpha\beta\gamma\mu} + \left(\frac{h_P^3}{6h_B} + \frac{(h_B + h_P)^2 h_P}{2h_B} \right) c^P_{\alpha\beta\gamma\mu} \right] I_P(x_1,x_2) + \frac{h_B^2}{12}c^B_{\alpha\beta\gamma\mu} I_B(x_1,x_2),$$

$$\theta = \frac{2h_P}{h_B}, \quad \sigma = \frac{(h_B + h_P)h_P}{3h_B}, \quad \ell = \frac{1}{2} + \frac{5h_P}{6h_B},$$

$$a = \frac{8h_P}{15h_B}, \quad b = \frac{16}{3h_P h_B}, \quad g = \frac{8}{3h_P h_B}, \quad k = \frac{2}{3h_B}.$$

The variations $\delta\vec{u} = (\delta u_1, \delta u_2, \delta u_3)$ and $\delta\vec{\phi} = (\delta\bar{\phi}, \delta\underline{\phi})$ were replaced by test functions $\vec{\psi} = (\psi_1, \psi_2, \psi_3)$ and $\vec{\eta} = (\bar{\eta}, \underline{\eta})$ from $V = \big(H_0^1(S)\big)^2 \times H_0^2(S)$ and $\Phi = H^1(S_P) \times H^1(S_P)$ respectively. Let us denote $H = \big(L_2(S_P)\big)^3$.

We end up this section with the classical formulation of equations (15),(16) and (17).

$$\rho u_{3\,tt} + \tfrac{\partial}{\partial x_\beta}(\tau_{\alpha\beta}u_{3,\alpha}) + \tfrac{\partial^2}{\partial x_\gamma \partial x_\mu}(\gamma_{\alpha\beta\gamma\mu}u_{3,\alpha,\beta}) - \theta e_{\gamma\alpha\beta}\tfrac{\partial}{\partial x_\beta}(I_P u_{3,\alpha}(\bar{\phi}_{,\gamma} + \underline{\phi}_{,\gamma})) +$$
$$\theta e_{3\alpha\beta}\tfrac{\partial^2}{\partial x_\alpha \partial x_\beta}(I_P(\bar{\phi} + \underline{\phi})) - \sigma e_{\gamma\alpha\beta}\tfrac{\partial^2}{\partial x_\alpha \partial x_\beta}(I_P(\bar{\phi}_{,\gamma} - \underline{\phi}_{,\gamma})) -$$
$$h_B^{-1}e_{3\alpha\beta}\tfrac{\partial}{\partial x_\beta}(I_P u_{3,\alpha})(v_1 - v_2) - \ell e_{3\alpha\beta}\tfrac{\partial^2}{\partial x_\alpha \partial x_\beta}I_P(v_1 + v_2) = 0,$$

$$\rho u_{1tt} - \tfrac{\partial}{\partial x_\beta}\tau_{1\beta} + \theta e_{\gamma 1\beta}\tfrac{\partial}{\partial x_\beta}(I_P(\bar{\phi}_{,\gamma} + \underline{\phi}_{,\gamma})) + h_B^{-1}e_{31\beta}\tfrac{\partial}{\partial x_\beta}I_P(v_1 - v_2) = 0,$$
$$\rho u_{2tt} - \tfrac{\partial}{\partial x_\beta}\tau_{2\beta} + \theta e_{\gamma 2\beta}\tfrac{\partial}{\partial x_\beta}(I_P(\bar{\phi}_{,\gamma} + \underline{\phi}_{,\gamma})) + h_B^{-1}e_{32\beta}\tfrac{\partial}{\partial x_\beta}I_P(v_1 - v_2) = 0,$$

$$a\epsilon_{\alpha\beta}\bar{\phi}_{,\alpha,\beta} + b\epsilon_{33}\bar{\phi} + \theta e_{\gamma\alpha\beta}\tfrac{\partial}{\partial x_\gamma}(u_{\alpha x_\beta} + u_{3,\alpha}u_{3,\beta}) + \sigma e_{\gamma\alpha\beta}\tfrac{\partial}{\partial x_\gamma}u_{3,\alpha x_\beta}$$
$$-\theta e_{3\alpha\beta}u_{3,\alpha x_\beta} - g\epsilon_{33}v_1 = 0,$$

$$a\epsilon_{\alpha\beta}\underline{\phi}_{,\alpha,\beta} + b\epsilon_{33}\underline{\phi} + \theta e_{\gamma\alpha\beta}\tfrac{\partial}{\partial x_\gamma}(u_{\alpha x_\beta} - u_{3,\alpha}u_{3,\beta}) - \sigma e_{\gamma\alpha\beta}\tfrac{\partial}{\partial x_\gamma}u_{3,\alpha x_\beta}$$
$$-\theta e_{3\alpha\beta}u_{3,\alpha x_\beta} - g\epsilon_{33}v_2 = 0.$$

Remember that:

$\tau_{\alpha\beta} = c_{\alpha\beta\lambda\nu}\,(u_{\lambda,\nu} + u_{\nu,\lambda} + u_{3,\lambda}u_{3,\nu})$ is the in-plane stress tensor,

$c_{\alpha\beta\lambda\nu}, \gamma_{\alpha\beta\lambda\nu}, \rho$	are discontinuous piecewise constant functions, the discontinuity arises due to the patches,
$v_1(t), v_2(t)$	are the applied voltages,
I_P	is the indicator function of the patch region,
$h_B a, b, \theta, \sigma, \ell, e_{i\alpha\beta}, \epsilon_{\alpha\beta}$	are constants.

4 Existence of solutions

We say that functions

$$\vec{u} \in L_2(0,T;V) \cap H^1(0,T;H), \quad \vec{\phi} \in L^2(0,T;\Phi)$$

satisfying the initial conditions $\vec{u}(0) = \vec{u}_0$ form an energy solution to system (15)-(17), if the following equalities hold:

$$\int_0^T \Big\{ -(\rho\vec{u}_t, \vec{\psi}_t) + \omega^1(\vec{u}, \vec{\psi}) - b^1(\vec{u}, \vec{\phi}, \vec{\psi}) - e^1(\vec{u}, \vec{\psi}, \vec{v}) \Big\} dt - (\rho\vec{u}_0', \vec{\psi}(0)) = 0, \tag{18}$$

$$\int_0^T \Big\{ a(\vec{\phi}, \vec{\eta}) + b(\vec{u}, \vec{\eta}, \vec{u}) - g(\vec{\eta}, \vec{v}) \Big\} dt = 0 \tag{19}$$

for all

$$\vec{\psi} \in L_2(0,T;V) \cap H^1_T(0,T;H), \quad \vec{\eta} \in L^2(0,T;\Phi).$$

Here the index T points out to the additional condition: $\vec{\psi}(T) = 0$.

Theorem. Let $\vec{u}_0 \in V$, $\vec{u}_0' \in H$, and $\vec{v} \in H^1(0,T;R^2)$. Then the system (15)-(17) has an energy solution such that

$$\vec{u} \in L_\infty(0,T;V), \quad \vec{u}_t \in L_\infty(0,T;H), \quad \vec{\phi} \in L_\infty(0,T;\Phi). \blacksquare$$

Sketch of the proof. Let $\{\vec{\omega}_i\}_{i=1}^{\infty}$ be a basis of V and $\{\vec{\theta}_i\}_{i=1}^{\infty}$ a basis of Φ. Consider Galerkin approximations of the form:

$$\vec{u}^m = \sum_{i=0}^{m} a_i^m(t)\vec{\omega}_i, \quad \vec{\phi}^m = \sum_{i=0}^{m} c_i^m(t)\vec{\theta}_i, \tag{20}$$

where $a_i^m(t),\ c_i^m(t)$ are unknown functions of t. Using properties of the forms appearing in (18), we obtain the following energy estimate:

$$\begin{aligned}(\rho\vec{u}_t^m, \vec{u}_t^m) + \omega(\vec{u}^m, \vec{u}^m) + a(\vec{\phi}^m, \vec{\phi}^m) \le C + 2|e(\vec{u}^m, \vec{u}^m, \vec{v})| +& \\ 2\int_0^t \Big\{|g(\vec{\phi}^m, \vec{v}_t)| + |e(\vec{u}^m, \vec{u}^m, \vec{v}_t)|\Big\}dt.&\end{aligned} \tag{21}$$

Using the Gronwall–Lemma yields:

$$\|\vec{u}_t^m\|_H^2 + \|s_{\alpha\beta}^m\|_{L_2(S)}^2 + \|u_{3,\alpha,\beta}^m\|_{L_2(S)}^2 + \|\vec{\phi}^m\|_{\Phi}^2 \le C,$$

where C depends on $\|\vec{u}_0\|_V^2, \|\vec{u}_0'\|_H^2, \|\vec{v}\|_{H^1(0,T)}^2$ and tends to zero as $\|\vec{u}_0\|_V^2$, $\|\vec{u}_0'\|_H^2$ and $\|\vec{v}\|_{H^1(0,T)}^2$ go to zero. Now, using arguments similar to those in [1], we establish that

$$\|\vec{u}_t^m\|_H^2 + \|\vec{u}^m\|_V^2 + \|\vec{\phi}^m\|_{\Phi}^2 \le C$$

for all $t \in [0, T]$. Therefore, we have established that

$$\begin{aligned}&\{\vec{u}^m\} \text{ is bounded in } L_\infty\big(0, T; V\big),\\ &\{\vec{u}_t^m\} \text{ is bounded in } L_\infty\big(0, T; H\big),\\ &\{\vec{\phi}^m\} \text{ is bounded in } L_\infty\big(0, T; \Phi\big).\end{aligned} \tag{22}$$

From the compactness of the embedding $H_0^2(S) \subset W^{1,q}(S)$, for any $q > 1$, and from (22) we conclude that $\{u_3^m\}$ is relative compact in $C\big([0, T]; W^{1,q}(S)\big)$ for any $q > 1$ (see [7]). The estimate obtained implies the following convergence:

$$\begin{aligned}u_3^m &\to u_3^0 \quad \text{weak}* \text{ in } \quad L_\infty\big(0, T; H_0^2(S)\big),\\ u_{3t}^m &\to u_{3t}^0 \quad \text{weak}* \text{ in } \quad L_\infty\big(0, T; L_2(S)\big),\\ u_3^m &\to u_3^0 \quad \text{in} \quad C\big([0, T]; W^{1,q}(S)\big),\\ u_\alpha^m &\to u_\alpha^0 \quad \text{weak}* \text{ in } \quad L_\infty\big(0, T; H_0^1(S)\big),\\ u_{\alpha t}^m &\to u_{\alpha_t}^0 \quad \text{weak}* \text{ in } \quad L_\infty\big(0, T; L_2(S)\big),\\ \bar{\phi}^m &\to \bar{\phi}^0 \quad \text{weak}* \text{ in } \quad L_\infty\big(0, T; H^1(S_P)\big),\\ \underline{\phi}^m &\to \underline{\phi}^0 \quad \text{weak}* \text{ in } \quad L_\infty\big(0, T; H^1(S_P)\big).\end{aligned} \tag{23}$$

This is sufficient to prove the pass to the limit in the nonlinear terms of the energy formulation, which completes the proof.■

5 Conclusion remark

We have proved the existence of energy solutions under assumption that the time derivatives dv_1/dt and dv_2/dt are quadratic integrable. To prove the uniqueness, it is necessary to establish some additional smoothness of solutions maybe under additional assumptions on the initial data and the controls v_1 and v_2. Some arguments show that the smoothness can not be improved in such a way because of the discontinuity of the coefficients and the presence of terms like $\Delta I_P \in H^{-2}(S)$ on the right-hand-sides of the equations.

Now we want to say a pair words about the homogenization of the equations when the number of the piezoelectric patches goes to infinity whereas their dimension tends to zero. This problem was solved in [8] for the case of electrically uncoupled model (see the introduction).

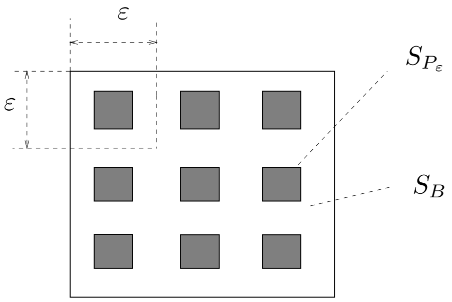

Figure 3. Self similar structure.

The condition $\partial\bar{\phi}/\partial\vec{n}|_{\partial S_{P_\varepsilon}} = \partial\underline{\phi}/\partial\vec{n}|_{\partial S_{P_\varepsilon}} = 0$ allows us to apply a method for homogenization of perforated domains developed by G.Allaire (see [9]). Roughly speaking, the mechanical equations will be homogenized as in [8]. When homogenizing the field equations, we consider the patches as an "electrically solid" part of S. The complement S_B to the piezoelectric patches is considered to be "electrically void". The application of the modified two-scale convergence proposed in [9] yields the result like the one obtained in [8].

Note that the approximation $\partial\bar{\phi}/\partial\vec{n}|_{\partial S_{P_\varepsilon}} = \partial\underline{\phi}/\partial\vec{n}|_{\partial S_{P_\varepsilon}} = 0$ is not realistic for the homogenization. Actually, we have to solve the field equation outside the piezoelectric patches. The jump condition should be taken into account. Some analysis shows that the potential functions decrease very rapidly outside the patches. So, the normal derivative is very small on the boundary of S_P but, if the density of the piezoelectric patches grows, each of them interacts with many others. This interaction can be treated using quadrupole approximation to the exact global solution of the field equation.

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