

**An Algorithm for Finding the Chebyshev Center
of a Convex Polyhedron¹**

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Abstract. An algorithm for finding the Chebyshev center of a finite point set in the Euclidean space R^n is proposed. The algorithm terminates after a finite number of iterations. In each iteration of the algorithm the current point is projected orthogonally onto the convex hull of a subset of the given point set.

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Abbreviated version of the article title: An Algorithm for Finding the Chebyshev Center

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Introduction

Many problems of control theory and computational geometry require an effective procedure for finding the Chebyshev center of a convex polyhedron. Particularly, it is natural to consider the Chebyshev center as the center of an information set in control problems with uncertain disturbances and state information errors. Chebyshev centers are also useful as an auxiliary tool for some problems of computational geometry. These are the reasons to propose a new algorithm for finding the Chebyshev center (see also [1,4]).

1. Formulation of the Problem. Optimality Conditions

Let $Z = \{z_1, z_2, \dots, z_m\}$ be a finite set of points in the Euclidean space R^n . A point $x_* \in R^n$ is called Chebyshev center of the set Z , if

$$\max_{z \in Z} \|x_* - z\| = \min_{x \in R^n} \max_{z \in Z} \|x - z\| \quad (1)$$

It is easily seen that the point x_* satisfying relation (1) is unique and $x_* \in \text{co}Z$, where symbol "co" means the convex hull of a set.

For any point $x \in R^n$ we denote

$$d_{max}(x) = \max_{z \in Z} \|x - z\|.$$

Let symbol $E(x)$ denote the subset of points of Z which have the largest distance from x , i.e.

$$E(x) = \{z \in Z : \|x - z\| = d_{max}(x)\}.$$

The optimality conditions are given by the following theorem [2].

Theorem 1. *A point $x_* \in R^n$ is the Chebyshev center of Z iff $x_* \in \text{co}E(x_*)$.*

This fact will be used as a criterion of the algorithm termination.

2. The Idea of the Algorithm

Choose an initial point $x_0 \in \text{co}Z$ and find the set $E(x_0)$. Assume that $x_0 \notin \text{co}E(x_0)$. Otherwise $x_0 = x_*$.

Let

$$J = \{j \in \overline{1, m} : z_j \in E(x_0)\}, \quad I = \{i \in \overline{1, m} : z_i \in Z \setminus E(x_0)\}.$$

Find a point $y_0 \in \text{co}E(x_0)$ nearest to the point x_0 . For any $\alpha \in [0, 1]$ consider the point $x_\alpha = x_0 + \alpha(y_0 - x_0)$. Obviously, for $\alpha = 0$ the following inequality holds

$$\max_{j \in J} \|x_\alpha - z_j\| - \max_{i \in I} \|x_\alpha - z_i\| > 0 \quad (2)$$

When we increase α , the point x_α moves from the point x_0 towards y_0 . We will increase α until the left-hand side of the inequality (2) becomes zero. If we reach the point y_0 before the left-hand side of (2) is zero, then the point y_0 is the desired solution x_* . If the left-hand side of (2) is equal zero for some $\alpha = \alpha_0 < 1$,

then we take the point x_{α_0} as new initial point and repeat the above process. Note that finding the value α_0 does not require solving of any equation. The algorithm provides explicit formula for α_0 .

Thus, the algorithm resembles the simplex method in linear programming. The set $E(x_0)$ is an analog of the support basis in the simplex method. After one iteration we obtain a new set $E(x_{\alpha_0})$ by adding to the set $E(x_0)$ some new points and removing some old points. Besides the “objective” function $d_{max}(\cdot)$ decreases, i.e. $d_{max}(x_{\alpha_0}) < d_{max}(x_0)$.

3. The Algorithm

Algorithm 1

Step 0: Choose an initial point $x_0 \in coZ$ and set $k := 0$.

Step 1: Find $E(x_k)$.

If $E(x_k) = Z$, then $x_* := x_k$ and stop.

Step 2: Find $y_k \in coE(x_k)$ as the nearest point to x_k .

If $y_k = x_k$, then $x_* := x_k$ and stop.

Step 3: Calculate α_k by the formula:

$$\alpha_k = \min_{i \in I_k^-} \frac{\|z_i - x_k\|^2 - d_{max}^2(x_k)}{2\langle y_k - x_k, z_i - y_k \rangle}, \quad (3)$$

$$I_k^- = \{i : z_i \in Z \setminus E(x_k), \langle y_k - x_k, z_i - y_k \rangle < 0\}$$

(we set formally $\alpha_k = +\infty$ whenever $I_k^- = \emptyset$).

If $\alpha_k \geq 1$, then $x_* := y_k$ and stop.

Step 4: Let

$$x_{k+1} := x_k + \alpha_k(y_k - x_k), \quad k := k + 1$$

and go to Step 1.

Remark. Note that Algorithm 1 comprises the non-trivial operation of finding the distance to convex hull of the point set (Step 2). In our latest program realizing Algorithm 1 we used the recursive algorithm of [3] for the implementation of this operation. However, we will see below that the point y_k of Step 2 is the Chebyshev center of $E(x_k)$. So we come to the following recursive algorithm.

Algorithm 2

Step 0: Choose an initial point $x_0 \in \text{co}Z$ and $k := 0$.

Step 1: Find $E(x_k)$.

If $E(x_k) = Z$, then $x_* := x_k$ and stop.

Step 2: Call Algorithm 2 with $Z = E(x_k)$ and let y_k be an output of this call (the Chebyshev center of $E(x_k)$).

If $y_k = x_k$, then $x_* := x_k$ and stop.

Step 3: Calculate α_k by the formula:

$$\alpha_k = \min_{i \in I_k^-} \frac{\|z_i - x_k\|^2 - d_{\max}^2(x_k)}{2\langle y_k - x_k, z_i - y_k \rangle},$$

$$I_k^- = \{i : z_i \in Z \setminus E(x_k), \langle y_k - x_k, z_i - y_k \rangle < 0\}$$

(we set formally $\alpha_k = +\infty$ whenever $I_k^- = \emptyset$).

If $\alpha_k \geq 1$, then $x_* := y_k$ and stop.

Step 4: Let

$$x_{k+1} := x_k + \alpha_k(y_k - x_k), \quad k := k + 1$$

and go to Step 1.

First we give the proof of Algorithm 1. The proof of Algorithm 2 will simply follow from the fact that y_k obtained at Step 2 of Algorithm 2 is the nearest point to x_k .

4. Auxiliary Propositions

Consider k -th iteration of Algorithm 1. We have the current approximation x_k and the point $y_k \in \text{co}E(x_k)$ nearest to x_k . Let us assume further that $x_k \neq y_k$. Otherwise $x_* = y_k$ by Theorem 1.

Introduce some notations. For any subset $S \subset Z$ we put

$$\text{num}(S) = \{l \in \overline{1, m} : z_l \in S\}.$$

Define the following sets of indices

$$J_k = \text{num}(E(x_k)),$$

$$J_k^0 = \{l \in J_k : \langle y_k - x_k, z_l \rangle = \min_{j \in J_k} \langle y_k - x_k, z_j \rangle\},$$

$$I_k = \text{num}(Z) \setminus J_k,$$

$$I_k^- = \{i \in I_k : \langle y_k - x_k, z_i - y_k \rangle < 0\},$$

$$I_k^+ = I_k \setminus I_k^-.$$

Note that the set I_k^- is the same as in formulas (3).

We now prove two propositions characterizing the point y_k .

Proposition 1. *The following equality holds*

$$\langle y_k - x_k, y_k \rangle = \min_{j \in J_k} \langle y_k - x_k, z_j \rangle.$$

Proof. Choose an arbitrary point $z \in \text{co}E(x_k)$ and consider the function $\delta(\lambda) = \|(1 - \lambda)y_k + \lambda z - x_k\|^2$, $\lambda \in [0, 1]$. Because of the definition of y_k the function $\delta(\cdot)$ takes its minimum at $\lambda = 0$. Hence $\delta'(0) \geq 0$, which implies

$$\langle y_k - x_k, z - y_k \rangle \geq 0.$$

Therefore,

$$\langle y_k - x_k, y_k \rangle = \min_{z \in \text{co}E(x_k)} \langle y_k - x_k, z \rangle.$$

Taking into account the obvious equality

$$\min_{z \in \text{co}E(x_k)} \langle y_k - x_k, z \rangle = \min_{j \in J_k} \langle y_k - x_k, z_j \rangle$$

we obtain the desired result.

Proposition 2. *The following inclusion holds*

$$y_k \in \text{co}\{z_j : j \in J_k^0\}.$$

Proof. Since $y_k \in \text{co}E(x_k)$, we have

$$y_k = \sum_{j \in J_k} \lambda_j z_j, \quad \lambda_j \geq 0, \quad \sum_{j \in J_k} \lambda_j = 1.$$

Proposition 1 gives

$$\langle y_k - x_k, \sum_{j \in J_k} \lambda_j z_j \rangle = \min_{j \in J_k} \langle y_k - x_k, z_j \rangle$$

or

$$\sum_{j \in J_k} \lambda_j [\langle y_k - x_k, z_j \rangle - \min_{l \in J_k} \langle y_k - x_k, z_l \rangle] = 0.$$

This implies

$$\lambda_j [\langle y_k - x_k, z_j \rangle - \min_{l \in J_k} \langle y_k - x_k, z_l \rangle] = 0$$

for any $j \in J_k$. Hence $\lambda_j = 0$ for any $j \notin J_k^0$ (the term in the square brackets is positive for $j \notin J_k^0$). Thus, Proposition 2 is proved.

Suppose that $I_k^- \neq \emptyset$ and consider for $\alpha \geq 0$ the following function

$$g(\alpha) = \max_{j \in J_k} \|x_k + \alpha(y_k - x_k) - z_j\|^2 - \max_{i \in I_k^-} \|x_k + \alpha(y_k - x_k) - z_i\|^2. \quad (4)$$

Lemma 1. *The function $g(\cdot)$ is monotone decreasing and has an unique zero $\alpha_k > 0$ given by formula (3).*

Proof. First we note that $g(0) > 0$ by definition of the sets J_k, I_k^- . Rewrite $g(\cdot)$ as follows

$$\begin{aligned}
g(\alpha) &= \min_{i \in I_k^-} \max_{j \in J_k} [\|x_k + \alpha(y_k - x_k) - z_j\|^2 - \|x_k + \alpha(y_k - x_k) - z_i\|^2] = \\
&\quad \min_{i \in I_k^-} \max_{j \in J_k} [\langle 2x_k + 2\alpha(y_k - x_k) - (z_j + z_i), z_i - z_j \rangle] = \\
&\quad \min_{i \in I_k^-} \max_{j \in J_k} [2\alpha \langle y_k - x_k, z_i - z_j \rangle + \langle 2x_k - (z_j + z_i), z_i - z_j \rangle] = \\
\min_{i \in I_k^-} \max_{j \in J_k} [2\alpha \langle y_k - x_k, (z_i - y_k) + (y_k - z_j) \rangle + \langle (x_k - z_j) + (x_k - z_i), (z_i - x_k) - (z_j - x_k) \rangle] &= \\
\min_{i \in I_k^-} \max_{j \in J_k} [2\alpha \langle y_k - x_k, z_i - y_k \rangle + 2\alpha \langle y_k - x_k, y_k - z_j \rangle + \|x_k - z_j\|^2 - \|x_k - z_i\|^2] &= \\
\min_{i \in I_k^-} [2\alpha \langle y_k - x_k, z_i - y_k \rangle - \|x_k - z_i\|^2] + \max_{j \in J_k} [2\alpha \langle y_k - x_k, y_k - z_j \rangle + \|x_k - z_j\|^2]. &
\end{aligned}$$

Since $\|x_k - z_j\| = d_{max}(x_k)$ for any $j \in J_k$, and

$$\max_{j \in J_k} \langle y_k - x_k, y_k - z_j \rangle = \langle y_k - x_k, y_k \rangle - \min_{j \in J_k} \langle y_k - x_k, z_j \rangle = 0$$

due to Proposition 1, we obtain the following representation of $g(\cdot)$

$$g(\alpha) = \min_{i \in I_k^-} [2\alpha \langle y_k - x_k, z_i - y_k \rangle - \|x_k - z_i\|^2] + d_{max}^2(x_k). \quad (5)$$

Since $\langle y_k - x_k, z_i - y_k \rangle < 0$ for any $i \in I_k^-$, the function $g(\cdot)$ is the minimum of a finite set of strictly decreasing affine functions of α . Hence $g(\cdot)$ is strictly decreasing. Using inequality $g(0) > 0$ we obtain that there exists a unique zero $\alpha' > 0$ of $g(\cdot)$. Due to the monotonicity of the function $g(\cdot)$ the value α' can be found by

$$\alpha' = \max\{\alpha \geq 0 : g(\alpha) \geq 0\}.$$

Let us prove now that α' coincides with α_k defined by formula (3). Choose an arbitrary $\bar{\alpha} > \alpha_k$. Then from definition of α_k there exists $i_* \in I_k^-$ such that

$$\frac{\|z_{i_*} - x_k\|^2 - d_{max}^2(x_k)}{2\langle y_k - x_k, z_{i_*} - y_k \rangle} < \bar{\alpha}.$$

Since $\langle y_k - x_k, z_{i_*} - y_k \rangle < 0$ we obtain

$$2\bar{\alpha}\langle y_k - x_k, z_{i_*} - y_k \rangle - \|z_{i_*} - x_k\|^2 + d_{max}^2(x_k) < 0.$$

A comparison with (5) gives $g(\bar{\alpha}) < 0$. Using the monotonicity of $g(\cdot)$ we get $\bar{\alpha} > \alpha'$. Thus, for any $\bar{\alpha} > \alpha_k$ we have $\bar{\alpha} > \alpha'$. Hence $\alpha_k \geq \alpha'$.

Let us prove the opposite inequality. For any $i \in I_k^-$ we have

$$\frac{\|z_i - x_k\|^2 - d_{max}^2(x_k)}{2\langle y_k - x_k, z_i - y_k \rangle} \geq \alpha_k.$$

This implies

$$2\alpha_k \langle y_k - x_k, z_i - y_k \rangle - \|z_i - x_k\|^2 + d_{max}^2(x_k) \geq 0 \quad \text{for any } i \in I_k^-.$$

A comparison with (5) gives $g(\alpha_k) \geq 0$. Hence $\alpha_k \leq \alpha'$ and Lemma 1 is proved.

Lemma 2. For any $\alpha > 0$ the maximum in the expression

$$\max_{j \in J_k} \|x_k + \alpha(y_k - x_k) - z_j\|^2$$

is attained on the subset J_k^0 , and all $j \in J_k^0$ are maximizing.

Proof. Let $\alpha > 0$, $s, q \in J_k$. Consider the function

$$\varphi(\alpha) = \|x_k + \alpha(y_k - x_k) - z_s\|^2 - \|x_k + \alpha(y_k - x_k) - z_q\|^2,$$

with $\varphi(0) = 0$. Since $\varphi'(\alpha) = \langle y_k - x_k, z_q - z_s \rangle > 0$ for $s \in J_k^0$, $q \notin J_k^0$, $\varphi(\alpha)$ increases strictly with $\alpha > 0$. On the other hand $\varphi'(\alpha) \equiv 0$ for $s, q \in J_k^0$, which proves the lemma.

Lemma 3. For any $i \in I_k^+$ and any $\alpha \geq 0$ the following inequality holds

$$\max_{j \in J_k} \|x_k + \alpha(y_k - x_k) - z_j\|^2 > \|x_k + \alpha(y_k - x_k) - z_i\|^2.$$

Proof. Let $i \in I_k^+$ and consider a fixed arbitrary index $s \in J_k^0$. One can easily see from the definition of I_k^+ and Proposition 1 that the function

$$\varphi(\alpha) = \|x_k + \alpha(y_k - x_k) - z_s\|^2 - \|x_k + \alpha(y_k - x_k) - z_i\|^2$$

satisfies $\varphi(0) > 0$ and $\varphi'(\alpha) = \langle y_k - x_k, z_i - z_s \rangle = \langle y_k - x_k, z_i - y_k \rangle + \langle y_k - x_k, y_k - z_s \rangle \geq 0$, hence $\varphi(\alpha)$ is increasing with $\alpha \geq 0$. This gives the proof.

Now define x_{k+1} by

$$x_{k+1} = \begin{cases} x_k + \alpha_k(y_k - x_k) & ; \quad \alpha_k \leq 1 \\ y_k & ; \quad \alpha_k > 1 \end{cases}. \quad (6)$$

Lemma 4. The following equality holds

$$\text{num}(E(x_{k+1})) = \begin{cases} J_k^0 \cup I_k^{-0} & ; \quad \alpha_k \leq 1 \\ J_k^0 & ; \quad \alpha_k > 1, \end{cases} \quad (7)$$

where

$$I_k^{-0} = \{l \in I_k^- : \|x_{k+1} - z_l\|^2 = \max_{i \in I_k^-} \|x_{k+1} - z_i\|^2\}.$$

Proof. Consider first the case $\alpha_k \leq 1$. Since α_k is a zero of the function $g(\cdot)$ defined by (4), we have

$$\max_{j \in J_k} \|x_{k+1} - z_j\|^2 = \max_{i \in I_k^-} \|x_{k+1} - z_i\|^2. \quad (8)$$

Taking into account the definition of the set I_k^{-0} , we obtain

$$\max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in I_k^- \setminus I_k^{-0}} \|x_{k+1} - z_i\|^2.$$

Using Lemma 3, we get

$$\max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in (I_k^- \setminus I_k^{-0}) \cup I_k^+} \|x_{k+1} - z_i\|^2.$$

With Lemma 2 we get for any $s \in J_k^0$

$$\|x_{k+1} - z_s\|^2 = \max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in (I_k^- \setminus I_k^{-0}) \cup I_k^+ \cup (J_k \setminus J_k^0)} \|x_{k+1} - z_i\|^2. \quad (9)$$

Taking into account the definition of the set I_k^{-0} and (8), we obtain that relation (9) is valid for any $s \in J_k^0 \cup I_k^{-0}$.

Since

$$J_k^0 \cup I_k^{-0} \cup (I_k^- \setminus I_k^{-0}) \cup I_k^+ \cup (J_k \setminus J_k^0) = \text{num}(Z),$$

we conclude that

$$\text{num}(E(x_{k+1})) = J_k^0 \cup I_k^{-0}.$$

Let now $\alpha_k > 1$ but $I_k^- \neq \emptyset$. Then $g(1) > 0$ which means that

$$\max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in I_k^-} \|x_{k+1} - z_i\|^2.$$

With Lemma 3 this implies

$$\max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in I_k^- \cup I_k^+} \|x_{k+1} - z_i\|^2.$$

Using Lemma 2, we obtain for any $s \in J_k^0$

$$\|x_{k+1} - z_s\|^2 = \max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in I_k^- \cup I_k^+ \cup (J_k \setminus J_k^0)} \|x_{k+1} - z_i\|^2.$$

Since

$$J_k^0 \cup I_k^- \cup I_k^+ \cup (J_k \setminus J_k^0) = \text{num}(Z),$$

we have

$$\text{num}(E(x_{k+1})) = J_k^0.$$

If $I_k^- = \emptyset$ then $I_k = I_k^+$, and from Lemma 3 we get

$$\max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in I_k} \|x_{k+1} - z_i\|^2.$$

With the help of Lemma 2 this gives for any $s \in J_k^0$

$$\|x_{k+1} - z_s\|^2 = \max_{j \in J_k} \|x_{k+1} - z_j\|^2 > \max_{i \in I_k \cup (J_k \setminus J_k^0)} \|x_{k+1} - z_i\|^2.$$

Since

$$J_k^0 \cup I_k \cup (J_k \setminus J_k^0) = \text{num}(Z),$$

we conclude that

$$\text{num}(E(x_{k+1})) = J_k^0,$$

which completes the proof.

Lemma 2 shows that for any $j \in J_k^0$ the expression $\|y_k - z_j\|$ assumes the same value. By Proposition 2 this value is the Chebyshev radius of the set $\{z_j : j \in J_k^0\}$, which we denote by r_k .

Lemma 5. *The following formulas for r_k are valid*

$$r_k = \sqrt{d_{max}^2(x_k) - \|y_k - x_k\|^2}, \quad (10)$$

$$r_k = \sqrt{d_{max}^2(x_{k+1}) - \|y_k - x_{k+1}\|^2}. \quad (11)$$

Proof. Choose an arbitrary $s \in J_k^0$. Then we have

$$z_s - x_k = y_k - x_k + z_s - y_k.$$

By squaring both sides of this expression we obtain

$$\|z_s - x_k\|^2 = \|y_k - x_k\|^2 + r_k^2 + 2\langle y_k - x_k, z_s - y_k \rangle.$$

From Proposition 1 and the definition of the set J_k^0 we have

$$\langle y_k - x_k, z_s - y_k \rangle = 0. \quad (12)$$

Using that $\|z_s - x_k\| = d_{max}(x_k)$ for any $s \in J_k^0 \subset J_k$, we obtain (10). As for (11), we have

$$z_s - x_{k+1} = y_k - x_{k+1} + z_s - y_k.$$

Squaring gives again

$$\|z_s - x_{k+1}\|^2 = \|y_k - x_{k+1}\|^2 + r_k^2 + 2\langle y_k - x_{k+1}, z_s - y_k \rangle.$$

Since $y_k - x_{k+1} = (1 - \alpha_k)(y_k - x_k)$, we get analogously to (12) $\langle y_k - x_{k+1}, z_s - y_k \rangle = 0$. Since $s \in J_k^0 \subset \text{num}(E(x_{k+1}))$ (see (7)), we conclude that $\|z_s - x_{k+1}\|^2 = d_{max}^2(x_{k+1})$. Thus we obtain (11) and lemma is proved.

Lemma 6. *If $\alpha_k < 1$ then $r_{k+1} > r_k$.*

Proof. In fact, from (10) we have

$$r_{k+1} = \sqrt{d_{max}^2(x_{k+1}) - \|y_{k+1} - x_{k+1}\|^2}, \quad (13)$$

where $y_{k+1} \in \text{co}E(x_{k+1})$ is the nearest point to x_{k+1} . Comparing (11) and (13), we see that we have to prove inequality

$$\|y_{k+1} - x_{k+1}\|^2 < \|y_k - x_{k+1}\|^2. \quad (14)$$

To do this we note that from Proposition 2 and (7) the point $y_k \in \text{co}E(x_{k+1})$. Choose an arbitrary point z_q , $q \in I_k^{-0}$. It is seen from (7) that $(1 - \lambda)y_k + \lambda z_q \in \text{co}E(x_{k+1})$ for any $\lambda \in [0, 1]$. Consider the expression

$$\begin{aligned} \|x_{k+1} - (1 - \lambda)y_k - \lambda z_q\|^2 &= \|x_{k+1} - y_k + \lambda(y_k - z_q)\|^2 = \\ &= \|x_{k+1} - y_k\|^2 + 2\lambda(1 - \alpha_k)\langle y_k - x_k, z_q - y_k \rangle + \lambda^2\|y_k - z_q\|^2. \end{aligned}$$

Since $q \in I_k^{-0} \subset I_k^-$ the following inequality holds

$$\langle y_k - x_k, z_q - y_k \rangle < 0.$$

Hence, taking small enough $\lambda' > 0$, we obtain

$$\|x_{k+1} - (1 - \lambda')y_k - \lambda'z_q\|^2 < \|x_{k+1} - y_k\|^2.$$

Since $y_{k+1} \in \text{co}E(x_{k+1})$ is the nearest point to x_{k+1} and $(1 - \lambda')y_k + \lambda'z_q \in \text{co}E(x_{k+1})$, we get

$$\|x_{k+1} - y_{k+1}\|^2 \leq \|x_{k+1} - (1 - \lambda')y_k - \lambda'z_q\|^2 < \|x_{k+1} - y_k\|^2.$$

Thus (14) is true and inequality $r_{k+1} > r_k$ also holds.

5. The Properties of Finiteness and Monotonicity

Theorem 2. *The Algorithm 1 terminates within a finite number of iterations.*

Proof. Notice that the inequality $r_{k+1} > r_k$ of Lemma 6 was obtained under assumptions $x_k \notin \text{co}E(x_k)$ and $\alpha_k < 1$. It is easily seen that after a finite number of iterations one of these conditions will be violated. Indeed, there is only a finite number of distinct sets $\{z_j : j \in J_k^0\}$, $k = 1, 2, \dots$. This number does not exceed the number of all subsets of Z . By the properties of the algorithm each set $\{z_j : j \in J_k^0\}$ can arise only once, because the corresponding Chebyshev radius r_k increases strictly with k . Hence, after a finite number of iterations we obtain either $x_k \in \text{co}E(x_k)$, or $\alpha_k \geq 1$. The first case means termination of the algorithm.

Consider the case $\alpha_k \geq 1$. Let x_{k+1} be defined by formula (6). Then $x_{k+1} = y_k$ and from Proposition 2 and (7) we have

$$x_{k+1} = y_k \in \text{co}\{z_j : j \in J_k^0\} \subset \text{co}E(x_{k+1}).$$

Thus, by Theorem 1 y_k is the Chebyshev center and the algorithm stops. Theorem 2 is proved.

Theorem 3. *The Algorithm 2 terminates within a finite number of iterations.*

Proof. If $y_k \in \text{co}E(x_k)$ is the nearest point to x_k , then Lemma 2 (with $\alpha = 1$) implies that $\{z_j : j \in J_k^0\}$ is the subset of such points of $E(x_k)$ which have the largest

distance from y_k . Then from Proposition 2 and Theorem 1 we obtain that y_k is the Chebyshev center of $E(x_k)$. So, with the uniqueness of the Chebyshev center we deduce that y_k obtained at Step 2 of Algorithm 2 is the nearest point to x_k .

It remains to prove that there is an exit from the recursion. In fact, if the point x_k is not the Chebyshev center of Z , then the set $E(x_k)$ has fewer points than Z has. So, the depth of recursion less or equal m . This proves Theorem 3.

Theorem 4. *The Algorithms 1,2 have the property of monotonicity, that is*

$$d_{max}(x_{k+1}) < d_{max}(x_k).$$

Proof. From (10) and (11) we have

$$d_{max}(x_k) = \sqrt{r_k^2 + \|x_k - y_k\|^2},$$

$$d_{max}(x_{k+1}) = \sqrt{r_k^2 + \|x_{k+1} - y_k\|^2} = \sqrt{r_k^2 + (1 - \alpha_k)^2 \|x_k - y_k\|^2}.$$

This gives the desired result.

6. Numerical Examples

Let us consider some numerical examples characterizing the efficiency of the algorithm. These examples were implemented with a QuickBASIC program on a PC (80486, 33 MHz).

Example 1. $Z = \{z : z = (\delta_1, \delta_2, \dots, \delta_n)^T, \delta_j = \pm 1, j = 1, \dots, n\}$ (vertices of the unit cube in R^n). The number of vertices is $m = 2^n$. The Chebyshev center is $x_* = 0$. The program realizing Algorithm 1 was run with the initial point $x_0 = (\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{n})^T$ for dimensions $n = 2, 3, \dots, 10$. For each n the solution was obtained just in n iterations. The CPU-time for $n = 10$ was 8.62 s.

Example 2. $Z = \{z_i : z_i = (\xi_1, \dots, \xi_m)^T, i = 1, \dots, m\}$, where ξ_j are random values uniformly distributed on the interval $[-1, 1]$. The results obtained for Algorithm 1 are presented in the following table. The initial point was the same as in Example 1. The average values were computed on the basis of 20 testruns.

n=10	Number of iterations			Time (s)		
m	min.	av.	max.	min.	av.	max.
100	8	12.55	18	1.1	3.4	14.66
200	8	13.45	18	1.49	5.7	12.97
500	9	15.65	24	2.85	10.26	25.04
1000	11	17.4	26	8.56	16.77	33.61
5000	13	21.6	29	71.46	115.06	151.21

The average speed of Algorithm 2 is approximately 1.5 times lower than the speed of Algorithm 1, but the corresponding computer program is shorter and looks more elegant. Computer tests show that the efficiency of the algorithms proposed is comparable with the rapidity of the algorithm of [4]. Our test results compare favourable with the test results reported in [1] for various methods to compute the

Chebyshev center proposed by other authors.

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