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A fitting algorithm for solving inverse problems of heat conduction

Andrzej Frąckowiak [a], Nikolai D. Botkin [b], Michał Ciałkowski [a,*], Karl-Heinz Hoffmann [b]

[a] Poznan University of Technology, Chair of Thermal Engineering, Piotrowo 3, 60-965 Poznań, Poland

[b] Technische Universität München, Zentrum Mathematik, Boltzmannstr.3, D-85747 Garching bei München, Germany

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ABSTRACT

The paper presents an algorithm for solving inverse problems of heat transfer. The method is based on iterative solving of direct and adjoint model equations with the aim to minimize a fitting functional. An optimal choice of the step length along the descent direction is proposed. The algorithm has been used for the treatment of a steady-state problem of heat transfer in a region with holes. The temperature and the heat flux density were known on the outer boundary of the region, whereas these values on the boundaries of the holes are to be determined. The idea of the algorithm consist in solving of Neumann problems where the heat flux on the outer boundary is prescribed, whereas the heat flux on the inner boundary is guessed. The guess is being improved iteratively to minimize the mean quadratic deviation of the solution on the outer boundary from the given distribution.

The results obtained show that the algorithm provides smooth, non-oscillating, and stable solutions to inverse problems of heat transfer, that is, it avoids disadvantages inherent in other computational methods for such problems. The ill-conditioning of inverse problems in the Hadamard sense is exhibited in that a very quick convergence of the fitting functional to its minimum does not imply a comparable rate of convergence of the recovered temperature on the inner boundary to the true distribution.

The considered method can easily be extended to nonlinear problems.

Numerical calculation has been carried out with the FE program Felics developed at the Chair of Mathematical Modelling of the Technical University of Munich.

1. Introduction

Inverse problems, in contrast to stationary and non-stationary direct boundary value problems, are characterized by unknown boundary conditions on a part of the boundary. Such a situation is typical when studying the heat transfer in engineering objects that have complicated geometries with holes. As an example, the problem of cooling of gas turbine blades can be mentioned. Measurements give the temperature and the heat flux density on the outer boundary of a blade, whereas the temperature, the heat flux, and the surface film conductance on the channel walls of the blade should be found (see [5]). Missing information about heat conditions in the unreachable part of the boundary is compensated by a redundant condition (the temperature or the heat flux) on the outer blade boundary. Another statement of inverse problems consists in determining the temperature and the heat flux on the whole boundary or on its part, whereas the temperature and the heat flux are prescribed in selected points located inside the domain where the problem is considered (see [1,2,7,12,13]). The work [9] proposes a method for recovering of unknown data in the case where the temperature is measured in several points that are located close each to other.

Inverse problems belong to ill-conditioned ones in the Hadamard sense [4], which means that small disturbances of the boundary data (the temperature and/or the heat flux) cause large solution errors and oscillations in the heat flux density. In problems of design of appropriate heat loads for turbine blades cooled with a gas, a constant temperature as well as a constant distribution of the surface film conductance α are assumed (knowing of α is equivalent to the knowledge of the heat flux density if the surface temperature is known). Problems stated in such a way can be unsolvable in the classical sense but solvable in a mean-square setting only. Instabilities inherent to inverse problems can destroy numerical procedures since the temperature and heat flux values are always polluted with a measurement error noise. Such a noise is usually eliminated by solving the problem in the mean-square sense (see [2,5,12,13]) or with statistical methods (comp. [7]).

Inverse problems are solved with various methods. Usually, regularization techniques are being applied, which converts ill-conditioned systems of equations into well-conditioned ones (see [1,7,12,13]). Some inverse problems can be solved in the same way as direct ones (i.e. as boundary value problems). The unknown temperature and the heat flux on a part of the boundary are

* Corresponding author.
E-mail addresses: andrzej.frackowiak@put.poznan.pl (A. Frąckowiak), botkin@ma.tum.de (N.D. Botkin), michal.cialkowski@put.poznan.pl (M. Ciałkowski), Hoffmann@ma.tum.de (K.-H. Hoffmann).

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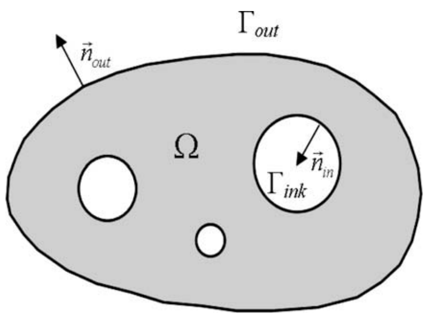

Fig. 1. Problem setting for a multiply connected region; a redundant condition is given on the outer boundary, whereas the heat conditions on the boundaries of the holes are to be found.

guessed, which yields a well-conditioned system of equations and provides stability of solutions (see [2]). The unknown (guessed) temperature and the heat flux are then determined from additional conditions such as entropy or energy dissipation minimum principle (see [3]).

The paper presents an algorithm for solving inverse problems outlined in Fig. 1. Here, the temperature and the heat flux are given on the outer boundary of the region. The heat conditions on the inner boundary are to be found. The idea of the algorithm consist in solving of Neumann problems where the heat flux on the outer boundary is prescribed, whereas the heat flux on the inner boundary is guessed and designated as control. The quality of the control is estimated by the mean quadratic deviation of the solution from the given temperature on the outer boundary. In each iteration of the algorithm, a unique solution of the direct Neumann boundary value problem is computed and substituted into the right-hand-side of an adjoint equation whose solution defines the derivative of the error functional with respect to the control. Such techniques have been developed in fundamental works [12,13]. An advanced feature of our approach is that the optimal step length of the line minimum search along the vector representing the derivative of the error functional can be computed exactly in the case of linear equations. Another new feature consists in the utilization of the $H^{1/2}$norm in the boundary error functional.

The present method is expected to be free of disadvantages related to oscillations of the heat flux density or the loss of stability of solutions because well posed boundary value problems are solved in each iteration step. Nevertheless, it is impossible to trick out the physical nature of the ill-posedness of inverse problems and, therefore, each method is more or less limited. First experience show that the method of the present paper is less limited then other approaches. Results of numerical calculations given in the present paper show the scale of the problem.

Note that some nonlinear steady-state problems can be converted to linear ones with the help of the Helmholtz substitution.

2. Formulation of inverse problems for a multiply connected region

Consider stationary heat flow in a multiply connected region Ω with the boundary Γ (see Fig. 1), where

$$\Gamma = \Gamma_{out} \bigcup \Gamma_{in}, \quad \Gamma_{in} = \bigcup_{k=1}^{n} \Gamma_{ink}, \tag{1}$$

and the temperature T is governed by the equation:

$$\Delta T = 0. \tag{2}$$

With a knowledge of the temperature and the heat flux on the boundary Γ_{out},

$$T|_{\Gamma_{out}} = T_\alpha, \quad \left.\frac{\partial T}{\partial n}\right|_{\Gamma_{out}} = q_\alpha, \tag{3}$$

find the distribution of the heat flux and the temperature on the inner boundaries Γ_{in}.

The problem is solved iteratively by setting direct Neumann problems with the following boundary data:

$$\left.\frac{\partial T}{\partial n}\right|_{\Gamma_{out}} = q_\alpha, \quad \left.\frac{\partial T}{\partial n}\right|_{\Gamma_{in}} = g. \tag{4}$$

The control (guess) g on Γ_{in} is iteratively being modified to achieve the minimal value of the functional

$$J[g] = \frac{1}{2}\int_{\Gamma_{out}} (T(g) - T_a)^2 ds. \tag{5}$$

The next section describes application of techniques of adjoint equations (see [9,11]) to the minimization of the functional (5).

3. Solution of the inverse problem

Because of arbitrary shape of Ω, Eq. (2) and other equations are solved with Finite Element Method. Therefore, it is convenient to utilize weak formulation techniques. Multiplication of the Laplace equation by a test function φ, integration over Ω, and application of the Green–Ostrogradski–Gauss theorem yield:

$$\int_\Omega \Delta T\,\varphi\,dx = -\int_\Omega \nabla T\,\nabla\varphi\,dx + \int_{\Gamma_{out}} \frac{\partial T}{\partial n}\,\varphi\,ds + \int_{\Gamma_{in}} \frac{\partial T}{\partial n}\,\varphi\,ds = 0. \tag{6}$$

Accounting for (4) gives

$$\int_\Omega \nabla T\,\nabla\varphi\,dx = \int_{\Gamma_{out}} q_a\,\varphi\,ds + \int_{\Gamma_{in}} g\,\varphi\,ds. \tag{7}$$

If the control g varies on Γ_{in}, then the solution $T(g)$ of (7) changes accordingly. The directional derivative of this solution is defined as:

$$T' = \lim_{\lambda\to 0}\frac{T(g+\lambda\delta g) - T(g)}{\lambda}. \tag{8}$$

Consideration of Eq. (7) with g (as it is) and $g + \lambda\delta g$ yields:

$$\int_\Omega \nabla T(g)\,\nabla\varphi\,dx = \int_{\Gamma_{out}} q_a\,\varphi\,ds + \int_{\Gamma_{in}} g\,\varphi\,ds. \tag{9}$$

$$\int_\Omega \nabla T(g+\lambda\delta g)\,\nabla\varphi\,dx = \int_{\Gamma_{out}} q_a\,\varphi\,ds + \int_{\Gamma_{in}} (g+\lambda\delta g)\,\varphi\,ds, \tag{10}$$

Subtraction of (9) from (10), division over λ, and passing to the limit as $\lambda \to 0$ give:

$$\int_\Omega \nabla T'\,\nabla\varphi\,dx = \int_{\Gamma_{in}} \delta g\,\varphi\,ds. \tag{11}$$

The same procedure applied to the functional $J(g)$ gives:

$$\begin{aligned}\frac{\partial J}{\partial\lambda} &= \lim_{\lambda\to 0}\frac{J(g+\lambda\delta g) - J(g)}{\lambda}\\ &= \lim_{\lambda\to 0}\frac{1}{2\lambda}\left[\int_{\Gamma_{out}} (T(g+\lambda\delta g) - T_a)^2 ds - \int_{\Gamma_{out}} (T(g) - T_a)^2 ds\right]\\ &= \int_{\Gamma_{out}} (T(g) - T_a)\,T'\,ds.\end{aligned} \tag{12}$$

Since we consider the variation δg of the heat flux on the inner boundary Γ_{in}, the derivative of the error functional should be also expressed through a function defined on Γ_{in} (compare with (12)). This can be achieved using the following adjoint equation defining an adjoint variable p:

$$\int_{\Omega} \nabla\psi\,\nabla p dx = \int_{\Gamma_{out}} (T(g) - T_a)\,\psi\, ds. \tag{13}$$

Substituting p instead of φ in Eq. (11), and u instead of ψ in (13) (this can be done because φ and ψ have the same properties as p and T', respectively), gives the following equation:

$$\int_{\Gamma_{in}} p\,\delta g ds = \int_{\Gamma_{out}} (T(g) - T_a)\,T'\, ds = J'(g)\delta g. \tag{14}$$

Therefore, $p|_{\Gamma_{in}}$ represents the derivative of the functional with respect to g. Thus, the algorithm looks as follows. Having the flux distribution g^n on Γ_{in} in the $n\,th$ iteration step allows us to find solution T^n of the equation

$$\int_{\Omega} \nabla T^n\,\nabla\varphi\, dx = \int_{\Gamma_{out}} q_a\,\varphi\, ds + \int_{\Gamma_{in}} g^n\,\varphi\, ds. \tag{15}$$

The equation

$$\int_{\Omega} \nabla\psi\,\nabla p^n\, dx = \int_{\Gamma_{out}} (T^n - T_a)\,\psi\, ds \tag{16}$$

serves for finding the adjoint function p^n. A new approximation g^{n+1} of the control on Γ_{in} is given by the formula

$$g^{n+1} = g^n - \eta p^n, \tag{17}$$

where $\eta \in (0, \infty)$ is normally being found using the line minimum search. Iterations are terminated when the condition

$$\|T^{n+1} - T^n\|_{L^2(\Gamma_{out})} < \varepsilon \tag{18}$$

is fulfilled. The value of the parameter η can be optimally chosen using linearity of the problem under consideration. Really, the solution in the $n+1\,th$ iteration step depends on η and satisfies the equation

$$\int_{\Omega} \nabla T^{n+1}_{\eta}\,\nabla\varphi\, dx = \int_{\Gamma_{out}} q_a\,\varphi\, ds + \int_{\Gamma_{in}} (g^n - \eta p^n)\,\varphi\, ds. \tag{19}$$

Therefore,

$$T^{n+1}_{\eta} = T^n - \eta Z, \tag{20}$$

where Z solves the equation

$$\int_{\Omega} \nabla Z\,\nabla\varphi\, dx = \int_{\Gamma_{in}} p^n\,\varphi\, ds. \tag{21}$$

Thus,

$$\begin{aligned} J(g^n - \eta p^n) &= \frac{1}{2}\int_{\Gamma_{out}} (T^n - \eta Z - T_a)^2 ds \\ &= \frac{1}{2}\int_{\Gamma_{out}} (T^n - T_a)^2 ds - \eta\int_{\Gamma_{out}} (T^n - T_a) Z\, ds \\ &\quad + \frac{1}{2}\eta^2 \int_{\Gamma_{out}} Z^2\, ds = a\eta^2 + b\eta + c, \end{aligned} \tag{22}$$

where

$$\begin{aligned} a &= \frac{1}{2}\int_{\Gamma_{out}} Z^2\, ds, \quad b = -\int_{\Gamma_{out}} (T^n - T_a) Z\, ds, \\ c &= \frac{1}{2}\int_{\Gamma_{out}} (T^n - T_a)^2 ds. \end{aligned} \tag{23}$$

Finally, the optimal value of the parameter η is given by the formula:

$$\eta_{opt} = -\frac{b}{2a} = \frac{\int_{\Gamma_{out}} (T^n - T_a) Z(p^n) ds}{\int_{\Gamma_{out}} Z^2(p^n) ds}. \tag{24}$$

4. Usage of the $H^{1/2}(\Gamma_{out})$ norm in the error functional

In the previous section, the $L^2(\Gamma_{out})$ norm is used as the error functional J. Now, the $H^{1/2}(\Gamma_{out})$ norm is tested for that. It is known (see e.g. [10]) that the functional of the form

$$J = \alpha/2 \int_{\Omega} (\nabla v)^2\, dx + \beta/2 \int_{\Gamma_{out}} (T - T_a)^2 ds, \tag{25}$$

where the auxiliary function v is a solution of the problem

$$-\Delta v = 0, \quad v|_{\Gamma_{out}} = T - T_a, \quad \frac{\partial v}{\partial n}|_{\Gamma_{in}} = 0,$$

defines an $H^{1/2}(\Gamma_{out})$ norm. The derivative of the functional is:

$$\begin{aligned} J' &= \alpha \int_{\Omega} \nabla v\,\nabla v'\, dx + \beta \int_{\Gamma_{out}} (T - T_a) T'\, ds \\ &= \alpha \int_{\Omega} -\cancelto{0}{\Delta v}\, v'\, dx + \alpha \int_{\Gamma_{out}} \frac{\partial v}{\partial n} v' + \alpha \int_{\Gamma_{in}} \cancelto{0}{\frac{\partial v}{\partial n}} v' \\ &\quad + \beta \int_{\Gamma_{out}} (T - T_a) T'\, ds = \alpha \int_{\Gamma_{out}} \frac{\partial v}{\partial n} v' \\ &\quad + \beta \int_{\Gamma_{out}} (T - T_a) T'\, ds = \alpha \int_{\Gamma_{out}} \frac{\partial v}{\partial n} T' \\ &\quad + \beta \int_{\Gamma_{out}} (T - T_a) T'\, ds. \end{aligned} \tag{26}$$

Finally

$$J' = \alpha \int_{\Omega} \nabla v \nabla T'\, dx + \beta \int_{\Gamma_{out}} (T - T_a) T'\, ds.$$

The adjoint equation reads:

$$\int_{\Omega} \nabla\psi\,\nabla p\, dx + \epsilon \int_{\Omega} \psi\, p\, dx = \alpha \int_{\Omega} \nabla v\,\nabla\psi\, dx + \beta \int_{\Gamma_{out}} (T - T_0)\psi\, ds,$$

where ϵ is a very small regularizing parameter. The functions $T, T', p, \phi, \psi \in H^1(\Omega)$ are not subjected any boundary conditions. The derivative of the functional is given by the formula:

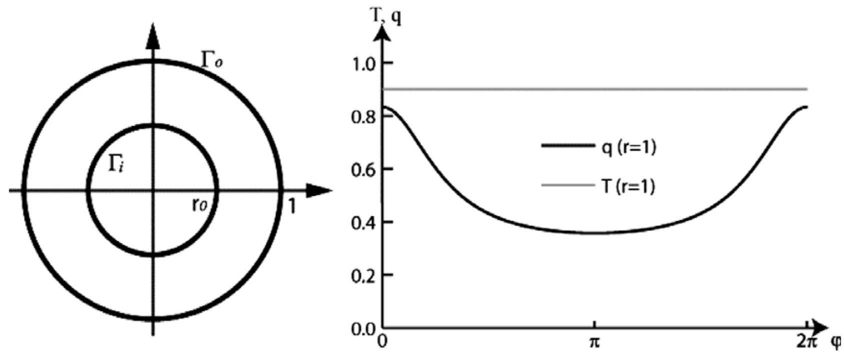

Fig. 2. The temperature and the heat flux density distributions on the outer boundary of the ring.

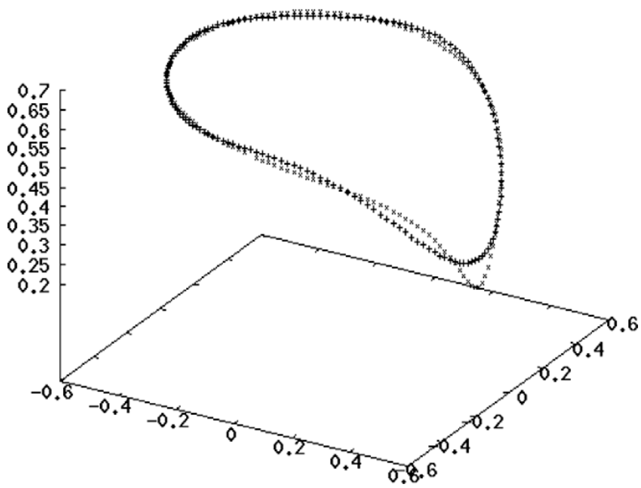

Fig. 3. Comparison of the numerically estimated temperature (large black crosses) on the inner boundary of the ring with the analytically obtained values (small gray crosses). The L^2 error functional is used.

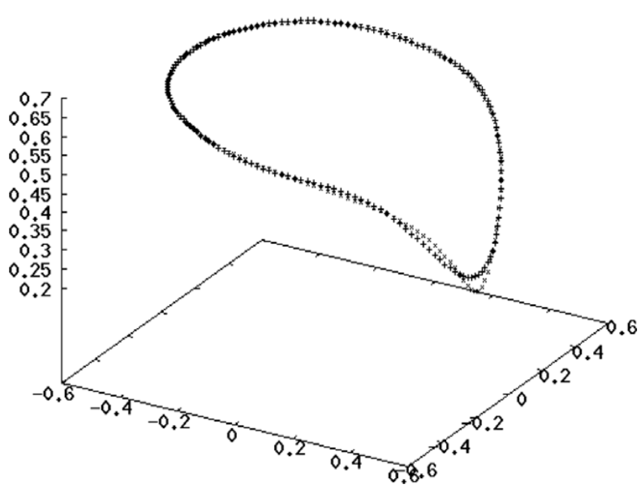

Fig. 4. Comparison of the numerically estimated temperature (large black crosses) on the inner boundary of the ring with the analytically obtained values (small gray crosses). The $H^{1/2}$ error functional is used.

$$J' = \int_{\Gamma_{in}} p\, \delta g\, ds,$$

and, therefore, the gradient of J is associated with $p|_{\Gamma_{in}}$.

5. Numerical examples

The algorithm described above is implemented with the FE program Felics developed at the Chair of Mathematical Modelling of the Technical University of Munich. Two dimensional simulations are performed. A ring of inner and outer radii 0.5 and 1, respectively, (see Fig. 2) is considered as the region Ω. The ring region is divided into 11348 triangle elements. In order to approximate the heat flux density, the outer and inner boundaries of the ring are divided into 312 and 156 segments, respectively.

For comparison, the temperature and the heat flux density on the inner boundary of the ring (Fig. 2) have been determined analytically [8]:

$$T(r,\varphi) = T_c + C \ln r + C \sum_{m=1}^{\infty} \frac{1}{2m} \left[(ar)^m - \left(\frac{a}{r}\right)^m \right] \cos(m\varphi). \tag{27}$$

The estimated distributions of the temperature on the inner boundary of the ring and the analytical values (27) are shown in Figs. 3 and 4.

6. Summary

The results presented in Fig. 3 show that the functional (5) converges more quickly to the minimum (the final value of the functional (5) was of the order of 10^{-4}) than the temperature on the inner boundary of the ring tends to the exact values. This points out to ill-posedness of the inverse problem considered. More important is that solutions obtained using the present method are not subjected to oscillations as in paper [5] where the oscillation had to be damped with the SVD algorithm. Note that the results shown in Fig. 4 (usage of an $H^{1/2}$ error functional) look better then these obtained for the L^2 error functional (see Fig. 3).

The aim of the present paper was to explain the idea of control theory aided algorithms for the treatment of inverse problems. The further work will be concentrated on the improvement of the stability and the convergence speed of such algorithms.

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