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THE VARIATIONAL CONTACT PROBLEM FOR ELASTIC OBJECTS OF DIFFERENT DIMENSIONS

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Abstract: We consider the variational free boundary problem describing the contact of an elastic plate with a thin elastic obstacle. The contact domain is unknown a priori and should be determined. The problem is described by a variational inequality for a fourth-order operator. The constraint on the displacement is given on a set of dimension less than that of the solution domain. We find the boundary conditions on the set of the possible contact and their exact statement. We justify the mixed statement of the problem and analyze the limit cases corresponding to the unbounded increase of the elasticity coefficients of the contacting bodies.

Keywords: variational inequality, thin obstacle, contact problem

§ 1. Statement of the Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ such that $\gamma = (0,1) \times \{0\} \subset \Omega$. Introduce the Sobolev spaces $H_0^2(\Omega)$ and $H_0^2(\gamma)$ and the energy functional on $H_0^2(\Omega) \times H_0^2(\gamma)$:

$$\Pi(w,u) = \frac{1}{2}\int\limits_\Omega w_{,ij}w_{,ij} - \int\limits_\Omega fw + \frac{1}{2}\int\limits_\gamma au_{xx}^2 - \int\limits_\gamma gu, \tag{1}$$

where $w_{,i} = \frac{\partial w}{\partial x_i}$, $i = 1, 2$, $(x_1, x_2) \in \Omega$; $u_x = \frac{du}{dx}$, $x = x_1$; and $f \in L^2(\Omega)$, $g \in L^2(\gamma)$, and $a \in L^\infty(\gamma)$ are given functions such that $a \geq c_0 > 0$ and $c_0 = \text{const}$. Repeated indices imply summation. We identify the functions given only on γ with functions of one variable x.

Consider the set of admissible displacements

$$K = \{(w,u) \in H_0^2(\Omega) \times H_0^2(\gamma) \mid w - u \geq 0 \text{ on } \gamma\}$$

and the minimization problem

$$\inf_{(w,u)\in K} \Pi(w,u).$$

Obviously, this problem has a solution satisfying the variational inequality

$$(w,u) \in K, \tag{2}$$

$$\int\limits_\Omega w_{,ij}(\overline{w}_{,ij} - w_{,ij}) - \int\limits_\Omega f(\overline{w} - w)$$
$$+ \int\limits_\gamma au_{xx}(\bar{u}_{xx} - u_{xx}) - \int\limits_\gamma g(\bar{u} - u) \geq 0 \quad \forall (\overline{w}, \bar{u}) \in K. \tag{3}$$

The functions $w(x_1, x_2)$ and $u(x)$ describe the displacements of the points of the plate and the thin elastic obstacle. The domain Ω corresponds to the middle surface of the plate and γ is the thin elastic obstacle.

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Below we find the differential statement of (2), (3). First of all, recall the Green's formula. We suppose that the curve γ can be prolonged to a closed curve Σ of class $C^{1,1}$ in Ω. Moreover, Ω splits into two subdomains Ω_1 and Ω_2 with the respective boundaries Σ and $\Sigma \cup \Gamma$. Denote the outward normal to Γ by $n = (n_1, n_2)$ and the normal to Σ (pointing into Ω_2) by $\nu = (\nu_1, \nu_2)$. Let $\Omega_\gamma = \Omega \setminus \bar{\gamma}$ and $\varphi_\nu = \frac{\partial \varphi}{\partial \nu}$. Consider the function space

$$V = \{v \in H^2(\Omega_1) \mid \Delta^2 v \in L^2(\Omega_1)\}$$

and introduce the notation

$$m(v) = v_{,ij}\nu_j\nu_i, \quad t^\nu(v) = v_{,ijk}s_k s_j \nu_i + v_{,ijj}\nu_i, \quad s = (s_1, s_2) = (-\nu_2, \nu_1).$$

Then, for $u \in V$, we can define $m(u) \in H^{-1/2}(\Sigma)$ and $t^\nu(u) \in H^{-3/2}(\Sigma)$ such that the Green's formula holds [1, 2]:

$$\int\limits_{\Omega_1} \varphi \Delta^2 u = \int\limits_{\Omega_1} \varphi_{,ij} u_{,ij} + \langle t^\nu(u), \varphi \rangle_{3/2,\Sigma} - \langle m(u), \varphi_\nu \rangle_{1/2,\Sigma} \quad \forall \varphi \in H^2(\Omega_1). \tag{4}$$

Here the parentheses $\langle \cdot, \cdot \rangle_{i/2,\Sigma}$ denote the duality between $H^{i/2}(\Sigma)$ and $H^{-i/2}(\Sigma)$, where $H^{-i/2}(\Sigma)$ is the dual of $H^{i/2}(\Sigma)$, $i = 1, 3$.

Choose the test functions in (3) in the form $(\overline{w}, \bar{u}) = (w + \varphi, u)$, where $\varphi \in H_0^2(\Omega)$ and $\varphi \geq 0$ on γ. We obtain

$$\int\limits_\Omega w_{,ij}\varphi_{,ij} - \int\limits_\Omega f\varphi \geq 0. \tag{5}$$

Replacing the integration domain Ω with $\Omega_1 \cup \Omega_2$ and applying the Green's formula in the form (4), from (5) we find that

$$-\langle [m(w)], \varphi_\nu \rangle_{1/2,\Sigma} + \langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} \geq 0, \tag{6}$$

where $[v] = v^+ - v^-$ and $v^\pm$ correspond to the values of v on $\Sigma^\pm$ according to the chosen direction of the normal ν. In the derivation of (6) we also use the equality

$$\Delta^2 w = f \quad \text{in } \Omega_\gamma,$$

which results from (3) as we insert the test functions of the form $(\overline{w}, \bar{u}) = (w \pm \xi, u)$, $\xi \in C_0^\infty(\Omega_\gamma)$. It follows from (6) that

$$[m(w)] = 0 \quad \text{in the sense of } H^{-1/2}(\Sigma), \tag{7}$$

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} \geq 0 \quad \forall \varphi \in H_0^2(\Omega),\ \varphi \geq 0 \text{ on } \gamma. \tag{8}$$

Taking the test functions in (3) in the form $(\overline{w}, \bar{u}) = (w, u + \psi)$ with $\psi \in H_0^2(\gamma)$ and $\psi \leq 0$ on γ, we find that

$$\int\limits_\gamma a u_{xx}\psi_{xx} - \int\limits_\gamma g\psi \geq 0.$$

Consequently,

$$(a u_{xx})_{xx} - g \leq 0 \quad \text{in the sense of } H^{-2}(\gamma). \tag{9}$$

Here $H^{-2}(\gamma)$ is the dual of $H_0^2(\gamma)$. Henceforth we use the parentheses $\langle \cdot, \cdot \rangle_{2,\gamma}$ to denote the duality between $H^{-2}(\gamma)$ and $H_0^2(\gamma)$. Choose the test functions in (3) in the form $(\overline{w}, \bar{u}) = (w + \varphi, u + \psi)$, $(\varphi, \psi) \in K$. We have

$$\int\limits_\Omega w_{,ij}\varphi_{,ij} - \int\limits_\Omega f\varphi + \int\limits_\gamma a u_{xx}\psi_{xx} - \int\limits_\gamma g\psi \geq 0.$$

Splitting Ω into Ω_1 and Ω_2 and applying again the Green's formula in the form (4), we obtain

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} + \langle (a u_{xx})_{xx} - g, \psi \rangle_{2,\gamma} \geq 0 \quad \forall (\varphi, \psi) \in K. \tag{10}$$

Choosing successively the test functions in (3) to be $(\overline{w}, \bar{u}) = (0,0)$ and $(\overline{w}, \bar{u}) = 2(w,u)$, we obtain

$$\int\limits_{\Omega} w_{,ij} w_{,ij} - \int\limits_{\Omega} fw + \int\limits_{\gamma} a u_{xx}^2 - \int\limits_{\gamma} gu = 0,$$

which implies the equality

$$\langle [t^\nu(w)], w \rangle_{3/2,\Sigma} + \langle (a u_{xx})_{xx} - g, u \rangle_{2,\gamma} = 0. \tag{11}$$

Note that conditions (8) and (9) follow from (10).

The above formulas enable us to write down the differential statement of (2), (3) as follows: Find the functions w and u on Ω_γ and γ satisfying

$$\Delta^2 w = f \quad \text{in } \Omega_\gamma, \tag{12}$$

$$w = w_n = 0 \quad \text{on } \Gamma, \tag{13}$$

$$w - u \geq 0, \quad [w] = [w_\nu] = 0, \quad [m(w)] = 0 \quad \text{on } \gamma, \tag{14}$$

$$[t^\nu(w)] \geq 0, \quad [t^\nu(w)](w-u) = 0 \quad \text{on } \gamma, \tag{15}$$

$$[t^\nu(w)] = -(a u_{xx})_{xx} + g \quad \text{on } \gamma, \tag{16}$$

$$u = u_x = 0 \quad \text{on } \partial\gamma. \tag{17}$$

The last condition in (14) holds in the sense of (7); the first condition in (15), together with condition (16), holds in the sense of (10); and the second condition in (15) holds in the sense of (11). Note that Σ in the above formulas is an arbitrary closed curve of class $C^{1,1}$ containing γ.

The system of boundary conditions (13)–(17) is complete; in particular, from (12)–(17) we can derive the variational inequality (2), (3).

In spite of the fact that a solution w to the variational problem (2), (3) is sought in the smooth domain Ω, the differential statement of the problem is formulated in a nonsmooth domain Ω_γ with the cut γ.

Nonsmooth boundaries are also typical for the problems of mathematical theory of cracks; in particular, in [2] the theory of cracks with an unknown contact domain is exposed. The general theory of boundary value problems with nonsmooth boundaries is given in [3]. For the contact problems with unknown boundaries the reader can refer to [4, 5].

§ 2. The Mixed Statement of the Problem

In this section we study the so-called mixed statement of (2), (3) (or (12)–(17)). All values with two subscripts are supposed symmetric in these subscripts. Let $m = \{m_{ij}\}$, $i,j = 1,2$. Put $\nabla\nabla m = m_{ij,ij}$ and define the boundary operators on Σ as follows:

$$m_\nu = m_{ij}\nu_j\nu_i, \quad T^\nu(m) = m_{ij,k}s_k s_j \nu_i + m_{ij,j}\nu_i.$$

Moreover, if φ is a scalar function on Ω_γ then we put

$$\nabla\nabla\varphi = \{\varphi_{,ij}\}, \quad i,j = 1,2.$$

Introduce the additional functions $m = \{m_{ij}\}$, $m_{ij} = w_{,ij}$, $i,j = 1,2$, and $M = a u_{xx} - G$, where G is a solution to the boundary value problem

$$G_{xx} = g \quad \text{on } \gamma, \quad G = 0 \quad \text{on } \partial\gamma,$$

and rewrite (12)–(17) as

$$\nabla\nabla m = f \quad \text{in } \Omega_\gamma, \tag{18}$$

$$m = \nabla\nabla w \quad \text{in } \Omega_\gamma, \tag{19}$$

$$w = w_n = 0 \quad \text{on } \Gamma, \tag{20}$$

$$w - u \geq 0, \quad [w] = [w_\nu] = 0, \quad [m_\nu] = 0 \quad \text{on } \gamma, \tag{21}$$

$$[T^\nu(m)] \geq 0, \quad [T^\nu(m)](w-u) = 0 \quad \text{on } \gamma, \tag{22}$$

$$[T^\nu(m)] = -M_{xx} \quad \text{on } \gamma, \tag{23}$$

$$(M+G)a^{-1} = u_{xx} \quad \text{on } \gamma, \tag{24}$$

$$u = u_x = 0 \quad \text{on } \partial\gamma. \tag{25}$$

Here the last condition in (21) is valid in the sense

$$\langle [m_\nu], \varphi\rangle_{1/2,\Sigma} = 0 \quad \forall \varphi \in H^{1/2}(\Sigma). \tag{26}$$

The first condition in (22) together with (23) is valid in the sense of the inequality

$$\langle [T^\nu(m)], \varphi\rangle_{3/2,\Sigma} + \langle M_{xx}, \psi\rangle_{2,\gamma} \geq 0 \quad \forall (\varphi,\psi) \in K, \tag{27}$$

and the second condition in (22) is understood in the sense of the equality

$$\langle [T^\nu(m)], w\rangle_{3/2,\Sigma} + \langle M_{xx}, u\rangle_{2,\gamma} = 0. \tag{28}$$

Observe one important consequence of (27). If $\varphi = 0$ on $\Sigma \setminus \gamma$ and $\varphi = \psi$ on γ then (27) implies

$$\langle [T^\nu(m)], \varphi\rangle_{3/2,00,\gamma} + \langle M_{xx}, \varphi\rangle_{2,\gamma} = 0, \tag{29}$$

where $\langle\,\cdot,\cdot\,\rangle_{3/2,00,\gamma}$ denotes the duality between $H_{00}^{-3/2}(\gamma)$ and $H_{00}^{3/2}(\gamma)$. Here the space $H_{00}^{3/2}(\gamma)$ is defined as follows:

$$H_{00}^{3/2}(\gamma) = \left\{ v \in H_0^{3/2}(\gamma) \mid \int\limits_\gamma \frac{|\nabla v|^2}{\rho} < \infty \right\},$$

where $\rho(x)$ is the distance from x to $\partial\gamma$. Note that $H_0^2(\gamma) = H_{00}^2(\gamma)$, and the embedding $H_{00}^2(\gamma) \subset H_{00}^{3/2}(\gamma)$ is dense and continuous; therefore, (29) yields the equality

$$[T^\nu(m)] = -M_{xx} \quad \text{in the sense of } H^{-2}(\gamma). \tag{30}$$

Introduce the so-called set of admissible moments

$$L = \{(\overline{m}, \overline{M}) \mid \overline{m} = \{\overline{m}_{ij}\},\ i,j = 1,2;\ \overline{m},\ \nabla\nabla\overline{m} \in L^2(\Omega_\gamma),\ \overline{M} \in L^2(\gamma);$$
$$[\overline{m}_\nu] = 0,\ [T^\nu(\overline{m})] \geq 0,\ [T^\nu(\overline{m})] = -\overline{M}_{xx} \text{ on } \gamma\},$$

where $\overline{m}$ and $\overline{M}$ satisfy the boundary conditions on γ in the sense of (26)–(27). Here we should note that if $\overline{m}$ and $\nabla\nabla\overline{m} \in L^2(\Omega_\gamma)$ then the jumps $[\overline{m}_\nu]$ and $[T^\nu(\overline{m})]$ are defined on Σ as elements of $H^{-1/2}(\Sigma)$ and $H^{-3/2}(\Sigma)$ (see [1, 4]).

We can now give the mixed statement of (18)–(25). Multiply (19) by $\overline{m} - m$ and (24) by $\overline{M} - M$, where $(\overline{m}, \overline{M}) \in L$, integrate over Ω_γ and γ respectively, and sum the results to obtain the relation

$$\int\limits_{\Omega_\gamma} m(\overline{m} - m) - \int\limits_{\Omega_\gamma} \nabla\nabla w(\overline{m} - m)$$
$$+ \int\limits_\gamma a^{-1}(M+G)(\overline{M} - M) - \int\limits_\gamma u_{xx}(\overline{M} - M) = 0. \tag{31}$$

From (28) and the definition of L we obtain the inequality

$$-\langle [T^\nu(\overline{m})] - [T^\nu(m)], w\rangle_{3/2,\Sigma} - \int\limits_\gamma u_{xx}(\overline{M} - M) \le 0.$$

Therefore, it follows from (31) that

$$\int\limits_{\Omega_\gamma} m(\overline{m} - m) - \int\limits_{\Omega_\gamma} w(\nabla\nabla\overline{m} - \nabla\nabla m) + \int\limits_\gamma a^{-1}(M + G)(\overline{M} - M) \ge 0.$$

Thus, we arrive at the following mixed statement of the variational problem (2), (3): *Find functions* w *and* $m = \{m_{ij}\}$, $i, j = 1, 2, M$, *satisfying*

$$w \in L^2(\Omega_\gamma), \quad (m, M) \in L, \tag{32}$$

$$\nabla\nabla m = f \quad \text{in } \Omega_\gamma, \tag{33}$$

$$\int\limits_{\Omega_\gamma} m(\overline{m} - m) - \int\limits_{\Omega_\gamma} w(\nabla\nabla\overline{m} - \nabla\nabla m)$$
$$+ \int\limits_\gamma a^{-1}(M + G)(\overline{M} - M) \ge 0 \quad \forall (\overline{m}, \overline{M}) \in L. \tag{34}$$

Once problem (32)–(34) is solved, the function u is determined uniquely from (24), (25), and (30); moreover, $u \in H_0^2(\gamma)$. Also, by the arguments before (30), we easily see that $u \in H^{\frac{5}{2}-\delta}$ for every $\delta > 0$.

We can prove existence of a solution to (32)–(34) without referring to (2), (3), using the regularization of the form

$$\varepsilon w^\varepsilon + \nabla\nabla m^\varepsilon = f \quad \text{in } \Omega_\gamma,\ \varepsilon > 0, \tag{35}$$

$$\int\limits_{\Omega_\gamma} m^\varepsilon(\overline{m} - m^\varepsilon) - \int\limits_{\Omega_\gamma} w^\varepsilon(\nabla\nabla\overline{m} - \nabla\nabla m^\varepsilon)$$
$$+ \int\limits_\gamma a^{-1}(M^\varepsilon + G)(\overline{M} - M^\varepsilon) \ge 0 \quad \forall (\overline{m}, \overline{M}) \in L. \tag{36}$$

We can express the function w^ε from (35) and insert it into (36), which leads to the variational inequality in $(m^\varepsilon, M^\varepsilon)$ having a unique solution in L. We can now obtain a priori estimates uniform in ε, pass to the limit in (35) and (36) as $\varepsilon \to 0$, and prove existence of a solution to (32)–(34). Since a close situation arises in the boundary value problems for plates with cracks, we drop down the details (they can be found in [6]).

The following assertion holds:

Theorem 1. *The mixed statement* (32)–(34), (24), (25), (30) *and the variational statement* (2), (3) *of* (12)–(17) *are equivalent.*

PROOF. Since the mixed statement is obtained from (12)–(17), it suffices to verify that the solution to the mixed problem possesses the necessary smoothness and satisfies all necessary boundary conditions. It follows from (34) that $m_{ij} = w_{,ij}$, $i, j = 1, 2$; therefore, $w \in H^2(\Omega_\gamma)$. Moreover, we can prove that the boundary conditions (13) are valid.

For example, we show how to validate the boundary conditions

$$[w] = [w_\nu] = 0 \quad \text{on } \gamma. \tag{37}$$

To this end, find a solution to the boundary value problem

$$\Delta^2 \widetilde{w} = f \quad \text{in } \Omega_\gamma, \tag{38}$$

$$\widetilde{w} = \widetilde{w}_n = 0 \quad \text{on } \Gamma, \tag{39}$$

$$m(\widetilde{w}) = \varphi, \quad t^\nu(\widetilde{w}) = \psi \quad \text{on } \gamma^\pm, \tag{40}$$

where φ and ψ are arbitrary fixed functions in $L^2(\gamma)$. This problem admits a variational statement. Namely, find a function $\widetilde{w}$ satisfying

$$\widetilde{w} \in H^2_\Gamma(\Omega_\gamma), \tag{41}$$

$$\int\limits_{\Omega_\gamma} \widetilde{w}_{,ij} v_{,ij} - \int\limits_{\Omega_\gamma} f v - \int\limits_\gamma \psi [v] + \int\limits_\gamma \varphi \left[\frac{\partial v}{\partial \nu}\right] = 0 \quad \forall v \in H^2_\Gamma(\Omega_\gamma), \tag{42}$$

where

$$H^2_\Gamma(\Omega_\gamma) = \{v \in H^2(\Omega_\gamma) \mid v = v_n = 0 \text{ on } \Gamma\}.$$

Observe that a solution to (41), (42) possesses the property

$$[m(\widetilde{w})] = 0 \quad \text{in the sense of } H^{-1/2}(\Sigma),$$

$$[t^\nu(\widetilde{w})] = 0 \quad \text{in the sense of } H^{-3/2}(\Sigma).$$

Take the test functions in (34) in the form $(\overline{m}, \overline{M}) = (m \pm \widetilde{m}, M)$, where $\widetilde{m} = \{\widetilde{m}_{ij}\}$, $\widetilde{m}_{ij} = \widetilde{w}_{,ij}$, $i, j = 1, 2$. We obtain

$$\int\limits_{\Omega_\gamma} m\widetilde{m} - \int\limits_{\Omega_\gamma} w \nabla\nabla \widetilde{m} = 0.$$

Hence, we find that

$$\langle T^\nu(\widetilde{m}), [w] \rangle_{3/2,\Sigma} - \left\langle \widetilde{m}_\nu, \left[\frac{\partial w}{\partial \nu}\right] \right\rangle_{1/2,\Sigma} = 0.$$

However, from (41) and (42) we derive the equality

$$\langle T^\nu(\widetilde{m}), [v] \rangle_{3/2,\Sigma} - \left\langle \widetilde{m}_\nu, \left[\frac{\partial v}{\partial \nu}\right] \right\rangle_{1/2,\Sigma} = \int\limits_\gamma \psi[v] - \int\limits_\gamma \varphi \left[\frac{\partial v}{\partial \nu}\right] \quad \forall v \in H^2_\Gamma(\Omega_\gamma).$$

Thus,

$$\int\limits_\gamma \psi[w] - \int\limits_\gamma \varphi \left[\frac{\partial w}{\partial \nu}\right] = 0,$$

and, in view of the arbitrariness of φ and ψ, the boundary conditions (37) are justified.

Thus, we are left with validating the boundary conditions

$$[T^\nu(m)](w - u) = 0 \quad \text{on } \gamma, \tag{43}$$

$$w - u \geq 0 \text{ on } \gamma. \tag{44}$$

Checking (43) is obvious. Firstly, insert $(\overline{m}, \overline{M}) = 0$ in (34) and secondly, put $(\overline{m}, \overline{M}) = 2(m, M)$. Using (24), we obtain

$$\int\limits_{\Omega_\gamma} mm - \int\limits_{\Omega_\gamma} w \nabla\nabla m + \int\limits_\gamma u_{xx} M = 0.$$

Hence (28) follows, as required.

Finally, validate (44). To this end, solve the boundary value problem of finding a function $\widetilde{w}$ such that

$$\begin{gathered}\Delta^2\widetilde{w} = f \quad \text{in } \Omega_\gamma,\\ \widetilde{w} = \widetilde{w}_n = 0 \quad \text{on } \Gamma,\\ m(\widetilde{w}) = 0 \quad \text{on } \gamma^\pm,\\ t^\nu(\widetilde{w}) = h \quad \text{on } \gamma^+, \quad t^\nu(\widetilde{w}) = 0 \quad \text{on } \gamma^-,\end{gathered}$$

where $h \geq 0$ is an arbitrary fixed function in $L^2(\gamma)$. This problem can be written in variational form. Namely, find a function $\widetilde{w}$ satisfying

$$\widetilde{w} \in H^2_\Gamma(\Omega_\gamma), \tag{45}$$

$$\int\limits_{\Omega_\gamma} \widetilde{w}_{,ij}\varphi_{,ij} - \int\limits_{\Omega_\gamma} f\varphi - \int\limits_{\gamma^+} h\varphi = 0 \quad \forall\varphi \in H^2_\Gamma(\Omega_\gamma). \tag{46}$$

Solve one more problem. Find a function $\tilde{u}$ satisfying

$$(a\tilde{u}_{xx})_{xx} = -h \quad \text{on } \gamma, \tag{47}$$

$$\tilde{u} = \tilde{u}_x = 0 \quad \text{on } \partial\gamma. \tag{48}$$

Obviously, such a function $\tilde{u}$ exists and $\tilde{u} \in H^2_0(\gamma)$.

Now, put $\widetilde{m}_{ij} = \widetilde{w}_{,ij}$, $i, j = 1, 2$. Then it follows from (45) and (46) that

$$\langle [T^\nu(\widetilde{m})], \varphi\rangle_{3/2,\Sigma} = \int\limits_\gamma h\varphi \quad \forall\varphi \in H^2_0(\Omega). \tag{49}$$

Putting $\widetilde{M} = a\tilde{u}_{xx}$, from (47) and (48) we find that

$$\langle \widetilde{M}_{xx}, \psi\rangle_{2,\gamma} = -\int\limits_\gamma h\psi \quad \forall\psi \in H^2_0(\gamma). \tag{50}$$

Summing (49) and (50), we obtain the relation

$$\langle [T^\nu(\widetilde{m})], \varphi\rangle_{3/2,\Sigma} + \langle \widetilde{M}_{xx}, \psi\rangle_{2,\gamma} = \int\limits_\gamma h(\varphi - \psi). \tag{51}$$

If $\varphi - \psi \geq 0$ on γ then the right-hand side of (51) is nonnegative and so $(\widetilde{m}, \widetilde{M}) \in L$. Consequently, the substitution $(\overline{m}, \overline{M}) = (m, M) + (\widetilde{m}, \widetilde{M})$ in (34) is admissible, which leads to the inequality

$$\int\limits_{\Omega_\gamma} m\widetilde{m} - \int\limits_{\Omega_\gamma} w\nabla\nabla\widetilde{m} + \int\limits_\gamma u_{xx}\widetilde{M} \geq 0.$$

Hence, we obtain the relation

$$\langle [T^\nu(\widetilde{m})], w\rangle_{3/2,\Sigma} + \langle \widetilde{M}_{xx}, u\rangle_{2,\gamma} \geq 0.$$

By the definition of $\widetilde{m}$ and $\widetilde{M}$, this means that

$$\int\limits_\gamma h(w - u) \geq 0.$$

Since $h \geq 0$ is an arbitrary function on γ, we obtain $w - u \geq 0$ on γ, as required. Thus, the theorem is proven.

§ 3. The Limit Cases

In applications, the problem of the form (2), (3) contains some parameters that characterize the physical and geometric properties of elastic materials. In this section we consider two passages to the limit which correspond to passage of the plate and the thin obstacle from an elastic state to a rigid state.

First consider the case when the bending rigidity of the plate increases to infinity. Let $\varepsilon > 0$ be a parameter to vanish. If, instead of (1), we consider the functional

$$\Pi_\varepsilon(w,u) = \frac{1}{2\varepsilon}\int\limits_\Omega w_{,ij}w_{,ij} - \int\limits_\Omega fw + \frac{1}{2}\int\limits_\gamma au_{xx}^2 - \int\limits_\gamma gu,$$

then the variational inequality (2), (3) for each ε takes the form

$$(w^\varepsilon, u^\varepsilon) \in K, \tag{52}$$

$$\frac{1}{\varepsilon}\int\limits_\Omega w^\varepsilon_{,ij}(\overline{w}_{,ij} - w^\varepsilon_{,ij}) - \int\limits_\Omega f(\overline{w} - w^\varepsilon)$$
$$+\int\limits_\gamma au^\varepsilon_{xx}(\bar{u}_{xx} - u^\varepsilon_{xx}) - \int\limits_\gamma g(\bar{u} - u^\varepsilon) \geq 0 \quad \forall(\overline{w},\bar{u}) \in K. \tag{53}$$

We can write down the differential statement of the form (12)–(17) for (52), (53). In this case the parameter ε enters in both the equation of equilibrium (12) and the boundary conditions (14)–(16). The goal of the arguments below is to justify the passage to the limit in (52) and (53) as $\varepsilon \to 0$. From (53) we obtain the equality

$$\frac{1}{\varepsilon}\int\limits_\Omega w^\varepsilon_{,ij}w^\varepsilon_{,ij} - \int\limits_\Omega fw^\varepsilon + \int\limits_\gamma au^\varepsilon_{xx}u^\varepsilon_{xx} - \int\limits_\gamma gu^\varepsilon = 0$$

from which we derive the following estimate uniform in $0 < \varepsilon \leq \varepsilon_0$:

$$\frac{1}{\varepsilon}\|w^\varepsilon\|^2_{H^2_0(\Omega)} + \|u^\varepsilon\|^2_{H^2_0(\gamma)} \leq c.$$

Drop to a subsequence possessing the following properties as $\varepsilon \to 0$ and keep the former notation for it:

$$u^\varepsilon \to u \text{ weakly in } H^2_0(\Omega), \quad w^\varepsilon \to 0 \text{ strongly in } H^2_0(\gamma).$$

Pass then to the limit as $\varepsilon \to 0$ in (52) and (53). To this end, insert the test functions of the form $(\overline{w},\bar{u}) = (0,\bar{u})$, with $\bar{u} \leq 0$ on γ and $\bar{u} \in H^2_0(\gamma)$, in inequality (53). We obtain the relation

$$\int\limits_\Omega fw^\varepsilon + \int\limits_\gamma au^\varepsilon_{xx}\bar{u}_{xx} - \int\limits_\gamma g(\bar{u} - u^\varepsilon) \geq \frac{1}{\varepsilon}\int\limits_\Omega w^\varepsilon_{,ij}w^\varepsilon_{,ij} + \int\limits_\gamma au^\varepsilon_{xx}u^\varepsilon_{xx}.$$

Passing to the lower limit in the above relation yields the variational inequality

$$u \in H^2_0(\gamma), \quad u \leq 0 \text{ on } \gamma, \tag{54}$$

$$\int\limits_\gamma au_{xx}(\bar{u}_{xx} - u_{xx}) - \int\limits_\gamma g(\bar{u} - u) \geq 0 \quad \forall\bar{u} \in H^2_0(\gamma), \quad \bar{u} \leq 0 \text{ on } \gamma. \tag{55}$$

We have thus demonstrated the following assertion:

Theorem 2. *The solution to* (52), (53) *converges to the solution to* (54), (55) *as* $\varepsilon \to 0$.

Observe that the limit problem (54), (55) describes the contact of a thin elastic obstacle with a rigid surface.

Now, consider another case in which the bending rigidity of a thin elastic obstacle increases. Namely, for $\varepsilon > 0$ define the function

$$a^\varepsilon = \begin{cases} a & \text{on } \gamma^-, \\ \varepsilon^{-1} a & \text{on } \gamma^+, \end{cases}$$

where $\gamma^- = (0, s_0) \times \{0\}$, $\gamma^+ = (s_0, 1) \times \{0\}$, and $s_0 \in [0,1]$ is a given number. Instead of (1), consider the energy functional of the form

$$\Pi_\varepsilon(w,u) = \frac{1}{2}\int\limits_\Omega w_{,ij}w_{,ij} - \int\limits_\Omega fw + \frac{1}{2}\int\limits_\gamma a^\varepsilon u_{xx}^2 - \int\limits_\gamma gu.$$

In this case the variational inequality (2), (3) takes the form

$$(w^\varepsilon, u^\varepsilon) \in K, \tag{56}$$

$$\int\limits_\Omega w^\varepsilon_{,ij}(\overline{w}_{,ij} - w^\varepsilon_{,ij}) - \int\limits_\Omega f(\overline{w} - w^\varepsilon)$$

$$+ \int\limits_\gamma a^\varepsilon u^\varepsilon_{xx}(\bar{u}_{xx} - u^\varepsilon_{xx}) - \int\limits_\gamma g(\bar{u} - u^\varepsilon) \geq 0 \quad \forall (\overline{w}, \bar{u}) \in K. \tag{57}$$

We can rewrite (56), (57) in differential form (12)–(17). In this event, the parameter ε enters in the boundary condition (16). The arguments below demonstrate that we can pass to the limit in (56) and (57) as $\varepsilon \to 0$.

From (57) we obtain the equality

$$\int\limits_\Omega w^\varepsilon_{,ij}w^\varepsilon_{,ij} - \int\limits_\Omega fw^\varepsilon + \int\limits_{\gamma^-} au^\varepsilon_{xx}u^\varepsilon_{xx} + \int\limits_{\gamma^+} \frac{a}{\varepsilon}u^\varepsilon_{xx}u^\varepsilon_{xx} - \int\limits_\gamma gu^\varepsilon = 0,$$

which guarantees the following uniform estimate (in $0 < \varepsilon \leq \varepsilon_0$) for the solution:

$$\|u^\varepsilon\|^2_{H^2_0(\gamma)} + \|w^\varepsilon\|^2_{H^2_0(\Omega)} + \frac{1}{\varepsilon}\|u^\varepsilon\|^2_{H^{2,0}(\gamma^+)} \leq c. \tag{58}$$

Here

$$H^{2,0}(\gamma^+) = \{v \in H^2(\gamma^+) \mid v = v_x = 0 \text{ on } (\partial\gamma) \cap (\partial\gamma^+)\}.$$

By (58), choose a subsequence (keeping the former notation for it) such that

$$\begin{gathered} u^\varepsilon \to u \quad \text{weakly in } H^2_0(\gamma), \\ u^\varepsilon \to 0 \quad \text{strongly in } H^{2,0}(\gamma^+), \\ w^\varepsilon \to w \quad \text{weakly in } H^2_0(\Omega) \end{gathered} \tag{59}$$

as $\varepsilon \to 0$. Now, pass to the limit in (56) and (57) as $\varepsilon \to 0$. Choose the test function $(\overline{w}, \bar{u}) \in K$ so that $\bar{u} = 0$ on γ^+ and insert this function in (57). We obtain

$$\int\limits_\Omega w^\varepsilon_{,ij}(\overline{w}_{,ij} - w^\varepsilon_{,ij}) - \int\limits_\Omega f(\overline{w} - w^\varepsilon) + \int\limits_{\gamma^-} au^\varepsilon_{xx}(\bar{u}_{xx} - u^\varepsilon_{xx})$$

$$-\frac{1}{\varepsilon}\int\limits_{\gamma^+} au^\varepsilon_{xx}u^\varepsilon_{xx} - \int\limits_{\gamma^-} g(\bar{u} - u^\varepsilon) + \int\limits_{\gamma^+} gu^\varepsilon \geq 0.$$

By (59), we can pass to the limit in this relation and obtain the variational inequality

$$w - u \geq 0 \text{ on } \gamma^-, \quad w \geq 0 \text{ on } \gamma^+, \quad w \in H_0^2(\Omega),\ u \in H_0^2(\gamma^-), \tag{60}$$

$$\int\limits_{\Omega} w_{,ij}(\overline{w}_{,ij} - w_{,ij}) - \int\limits_{\Omega} f(\overline{w} - w) + \int\limits_{\gamma^-} a u_{xx}(\bar{u}_{xx} - u_{xx}) - \int\limits_{\gamma^-} g(\bar{u} - u) \geq 0 \tag{61}$$

which is valid for all test functions $(\overline{w}, \bar{u})$ such that

$$\overline{w} - \bar{u} \geq 0 \text{ on } \gamma^-, \quad \overline{w} \geq 0 \text{ on } \gamma^+, \quad \overline{w} \in H_0^2(\Omega), \quad \bar{u} \in H_0^2(\gamma^-). \tag{62}$$

Thus, the above arguments prove the following assertion:

Theorem 3. *The solution to* (56), (57) *converges to the solution to* (60)–(62) *as* $\varepsilon \to 0$.

The limit variational inequality (60)–(62) describes the contact of a plate with a thin obstacle which is elastic on γ^- and rigid on γ^+. Observe that, in particular, we can take $\gamma^+ = \gamma$. In this case problem (60)–(62) describes the contact of an elastic plate with a thin rigid obstacle (see [7]).

References

1. *Temam R.*, Mathematical Problems in Plasticity [Russian translation], Nauka, Moscow (1991).
2. *Khludnev A. M. and Kovtunenko V. A.*, Analysis of Cracks in Solids, WIT Press, Southampton; Boston (2000).
3. *Grisvard P.*, Elliptic Problems in Nonsmooth Domains, Pitman, Boston; London; Melbourne (1985).
4. *Khludnev A. M., Sokolowski J.*, Modelling and Control in Solid Mechanics, Birkhauser, Basel; Boston; Berlin (1987).
5. *Fichera G.*, Existence Theorems in Elasticity [Russian translation], Mir, Moscow (1974).
6. *Khludnev A. M.*, “The method of smooth domains in the equilibrium problem for a plate with a crack,” Sibirsk. Mat. Zh., **43**, No. 6, 1388–1400 (2002).
7. *Schild B.*, “On the coincidence set in biharmonic variational inequalities with thin obstacles,” Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (13), **4**, No. 4, 559–616 (1986).

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