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IDENTIFICATION OF UNKNOWN PARAMETERS FOR HEAT CONDUCTIVITY EQUATIONS

N.D.Botkin[1]

Institute of Mathematics & Statistics of the
Technical University of München.
Dachauer Str. 9a
D-80335 München, Germany.

e-mail: botkin@appl-math.tu-muenchen.de
Phone: (089)55224825
Fax: (089)55224819

Abstract. An algorithm for identification of unknown parameters of nonlinear heat conductivity equations is proposed. Solutions of equations observed with an error are input data of the algorithm. Finite dimensional approximations of input signals and their derivatives are used. The algorithm utilizes the idea of the direct minimization of the residual of equations written in an appropriate variational form. The convergence of the algorithm output to the set of all parameters compatible with the exact solution is proved. The paper is illustrated by computer simulations related to the identification of heat conductivity coefficients depending upon spatial variables and the temperature.

Introduction

This paper proposes an algorithm for identification of unknown parameters of nonlinear heat conductivity equations on the basis of solutions measured with an error. The main idea of the algorithm is related to the method of the direct minimization of the defect of equations. Such methods were considered in [1] for ordinary differential equations. In [2] a method based on the direct solution of the original equation with respect to the unknown heat conductivity coefficient was proposed. An approach related to direct methods and based on stabilization theory was developed in [3].

The algorithm proposed in this paper seems to have extensions to nonlinear problems like those considered in [4, 5, 6]. Such an opportunity is discussed in the conclusion.

[1]Institut für Angewandte Mathematik und Statistik der Technischen Universität München, Dachauer Str. 9a, D-80335 München

Notation

$\mathbb{R}$	is the set of real numbers.
$\mathbb{R}^n$	is the real n-dimensional Euclidean space.
$\langle\cdot,\cdot\rangle$	is the scalar product in $\mathbb{R}^n$.
$\lvert\cdot\rvert$	denotes either the absolute value of a scalar or the Euclidean norm of a vector in $\mathbb{R}^n$.
Ω	is an open and bounded subset of $\mathbb{R}^n$.
$\partial\Omega$	is the boundary of Ω.
Q_T	is the cylinder $\Omega\times(0,T)$.
$(\cdot,\cdot)_X$	is the scalar product of a Hilbert space X.
$\lVert\cdot\rVert_X$	is the norm of a Hilbert space X.
$\lVert\cdot\rVert$	is the canonical norm of a linear operator.

Formulation of the problem

Consider the following nonlinear heat conductivity equation:

$$u_t - \operatorname{div}\,(k(x,u,\alpha(t))\nabla u) = f,\ x\in\Omega,\ t\in(0,T). \tag{1}$$

Here $k(x,u,\alpha)$ is the heat conductivity matrix defined on $\bar{\Omega}\times\mathbb{R}\times A$, where A is a compact subset of a normed space (for example, of $C(\bar{\Omega})$). We assume that k is positive definite, Lipschitzian in x,u, and continuous with respect to α. The function $\alpha(\cdot):[0,T]\to A$ is considered as the unknown parameter, which is to be identified.

Assume the homogeneous initial and boundary conditions

$$\begin{array}{ll} u(x,0)=0, & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t\in(0,T). \end{array}$$

Define solutions of equation (1) as functions $u\in L_2((0,T);H_0^1(\Omega))\cap H^1((0,T);L_2(\Omega))$ satisfying the following variational equation:

$$\int_0^T\int_\Omega (u_t\theta + k(x,u,\alpha(t))\nabla u\nabla\theta - f\theta)dx\,dt = 0 \tag{2}$$

for all $\theta\in L_2((0,T);H_0^1(\Omega))$. It should be noted (see [7]) that such solutions have the property $u\in C([0,T];L_2(\Omega))$.

Denote $B_0^1=\{\psi\in H_0^1(\Omega):\|\psi\|_{H_0^1(\Omega)}\le 1\}$. The following proposition is a simple consequence of the last definition (we omit the proof).

PROPOSITION 1. A function u is solution of (1) iff

$$\int_0^T \max_{\psi\in B_0^1}\left[\int_\Omega (u_t\psi + k(x,u,\alpha(t))\nabla u\nabla\psi - f\psi)dx\right]^2 dt = 0. \tag{3}$$

Input data and approximation

Assume that values of u are measured with an error η, that is, we have a function u^η such that

$$\|u^\eta - u\|_{L_\infty(Q_T)} \le \eta. \tag{4}$$

Let τ be a time step and $\{0 = t_0 < t_1 < ... < t_{N+1} = T\}$ the equidistant partition of the interval $[0, T]$ with the step τ. For functions $v \in L_2(Q_T)$, we introduce an operator I_τ defined as follows:

$$(I_\tau v)(x,t) = \frac{1}{\tau}\int_{t_i}^{t_{i+1}} v(x,\zeta)d\zeta, \quad \text{if} \quad t \in [t_i, t_{i+1}),\ i \in \overline{1,N}.$$

Note that the image of v is a piece-wise constant with respect to t function.

For $\delta > 0$, let $\omega_\delta(x-y)$ be a smoothing function. For example, such a function can be defined as follows:

$$\omega_\delta(x-y) = \frac{1}{\delta^n}\omega_1(|x-y|/\delta),$$

$$\omega_1(r) = \begin{cases} \frac{1}{c}e^{-\frac{1}{1-r^2}} & ; \quad |r| < 1 \\ 0 & ; \quad |r| \ge 1 \end{cases},$$

where the constant c is such that $\int_{\mathbb{R}^n} \omega_\delta(z)dz = 1$. The smoothing operator J_δ is defined by the formula

$$(J_\delta v)(x,t) = \int_\Omega \omega_\delta(x-y)v(y,t)dy.$$

Let $\mathcal{F}_h$ be a regular triangulation of $\bar\Omega$. Consider the standard finite element space with linear elements

$$\Phi_h = \{\theta \in C(\bar\Omega) : \theta|_\mathcal{K} \text{ is linear } \forall \mathcal{K} \in \mathcal{F}_h\} \cap H_0^1(\Omega).$$

For functions $w \in L_2(\Omega)$, we denote by F_h an operator of a local $L_2(\Omega)$ projection defined in [8] as follows: For all finite elements $\mathcal{K} \in \mathcal{F}_h$,

a) $F_h w|_\mathcal{K}$ is a linear function,
b) $\int_\mathcal{K}(F_h w - w)l dx = 0$ for any linear function l.

We shall consider F_h as an operator from $L_2(Q_T)$ to $L_2(Q_T)$ by applying F_h to functions $v(\cdot, t)$ for almost all $t \in [0, T]$.

The operators I_τ, J_δ, and F_h have the following properties:

$$I_\tau v,\ J_\delta v,\ F_h v \to v \quad \text{in } L_2(Q_T) \quad \text{as } \tau, \delta, h \to 0,$$

$$\|I_\tau\|, \|J_\delta\|, \|F_h\| \le 1, \quad \text{for all } \tau, \delta, h.$$

Now we define some piece-wise constant with respect to t functions which approximate $u_t, u, \nabla u$, and f. We put

$$u_1^{\eta,\tau,h} = F_h\left(\frac{u^\eta(\cdot, t_{i+1}) - u^\eta(\cdot, t_i)}{\tau}\right), \quad \text{if} \quad t \in [t_i, t_{i+1}),\ i \in \overline{1,N},$$

$$u_2^{\eta,\tau,h} = F_h u^\eta(\cdot, t_i), \quad \text{if} \quad t \in [t_i, t_{i+1}),\ i \in \overline{1,N},$$
$$u_3^{\eta,\delta,\tau,h} = F_h I_\tau \nabla J_\delta u^\eta,$$
$$f^\tau = I_\tau f.$$

PROPOSITION 2. Let $\eta, \delta, \tau, h \to 0$ and $\eta/\delta, \eta/\tau \to 0$. Then

a) $u_1^{\eta,\tau,h} \to u_t$ in $L_2(Q_T)$.

b) $u_2^{\eta,\tau,h} \to u$ in $L_2(Q_T)$.

c) $u_3^{\eta,\delta,\tau,h} \to \nabla u$ in $L_2^n(Q_T)$.

d) $f^\tau \to f$ in $L_2(Q_T)$.

The proof is given in the Appendix.

REMARK 1. If the exact solution u is sufficiently smooth with respect to x and t ($u \in W^{1,p}(Q_T)$ with $p > (n+1)$), one can use a simplified approximation of u, u_t and ∇u (see [8]). Namely, one may set

$$u_1^{\eta,\tau,h} = \sum_k \frac{u^\eta(x_k, t_{i+1}) - u^\eta(x_k, t_i)}{\tau} \omega_k(x),$$
$$u_2^{\eta,\tau,h} = \sum_k u^\eta(x_k, t_i)\omega_k(x),$$
$$u_3^{\eta,\tau,h} = \sum_k u^\eta(x_k, t_i)\nabla\omega_k(x)$$

if $t \in [t_i, t_{i+1})$ for some $i \in \overline{1,N}$. Here x_k are vertices of the triangulation $\mathcal{F}_h$ and $\omega_k(x)$ are global form functions of Φ_h. Proposition 2 remains valid.

Algorithm and convergence

For brevity we denote the collection (η, δ, τ, h) by ε. We shall write $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$, and f^ε instead of $u_1^{\eta,\tau,h}, u_2^{\eta,\tau,h}, u_3^{\eta,\delta,\tau,h}$, and f^τ. We say that $\varepsilon \to 0$ if $\eta, \delta, \tau, h \to 0$ and $\eta/\delta, \eta/\tau \to 0$.

For any $t \in [0,T]$, $\alpha \in A$, and ε, let $\psi_{t,\alpha,\varepsilon} \in \Phi_h$ be a function such that

$$(\psi_{t,\alpha,\varepsilon}, \psi)_{H_0^1(\Omega)} = \int_\Omega (u_1^\varepsilon \psi + k(x, u_2^\varepsilon, \alpha) u_3^\varepsilon \nabla\psi - f^\varepsilon \psi) dx, \quad \forall \psi \in \Phi_h. \tag{5}$$

This relation can be considered as the representation of the continuous functional defined on the Hilbert space Φ_h via the inner product. Recall that the functions $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$, and f^ε are piece-wise constant w.r.t. t and, hence, the function $t \to \psi_{t,\alpha,\varepsilon}$ is constant w.r.t. t on each interval $[t_i, t_{i+1})$, $i \in \overline{1,N}$.

For $i \in \overline{1, N}$ and $t \in [t_i, t_{i+1})$, we set

$$\alpha^\varepsilon(t) \equiv \alpha^i = \arg\min_{\alpha \in A} \int_\Omega (u_1^\varepsilon \psi_{t,\alpha,\varepsilon} + k(x, u_2^\varepsilon, \alpha) u_3^\varepsilon \nabla \psi_{t,\alpha,\varepsilon} - f^\varepsilon \psi_{t,\alpha,\varepsilon}) dx, \qquad (6)$$

where $\psi_{t,\alpha,\varepsilon}$ is defined by (5). Note that $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon$ and $\psi_{t,\alpha,\varepsilon}$ belong to Φ_h. So we can apply the technique of the finite element method to solve the optimization problem (5),(6). The following theorem is true.

THEOREM 1. Let $\mathcal{U}$ be the set of all functions $\alpha(\cdot)$ satisfying (1) (the set of all parameters compatible with the exact solution u). Then for any $p \geq 1$,

$$\mathrm{dist}_{L_p((0,T);A)}(\alpha^\varepsilon(\cdot), \mathcal{U}) \to 0$$

as $\varepsilon \to 0$. ■

Denote for simplicity

$$\ell_\varepsilon(t, \alpha, \psi) = \int_\Omega (u_1^\varepsilon \psi + k(x, u_2^\varepsilon, \alpha) u_3^\varepsilon \nabla \psi - f^\varepsilon \psi) dx,$$

$$\ell(t, \alpha, \psi) = \int_\Omega (u_t \psi + k(x, u, \alpha) \nabla u \nabla \psi - f \psi) dx.$$

The proof of Theorem 1 is based on the following proposition.

PROPOSITION 3.

$$\int_0^T \max_{\alpha \in A} |\ell_\varepsilon(t, \alpha, \psi_{t,\alpha,\varepsilon}) - \max_{\psi \in B_0^1} \ell^2(t, \alpha, \psi)| dt \to 0$$

as $\varepsilon \to 0$. The proof is given in the appendix. ■

Consider the space $\mathcal{M}$ of weakly measurable functions $t \to \mu(t|\cdot)$ mapping the interval $[0, T]$ into the set of all regular Borel probability measures defined on A. Measurability of $t \to \mu(t|\cdot)$ is understood in the following sense: For any continuous function $g : A \to \mathbb{R}$, the function $t \to \int_A g(\alpha) \mu(t|d\alpha)$ is Lebesque - measurable. It is said that a sequence $\mu^k \in \mathcal{M}$ converges to $\nu \in \mathcal{M}$ if for any function $g \in L_1((0,T); C(A))$,

$$\int_0^T \int_A g(t, \alpha) \mu^k(t|d\alpha) dt \to \int_0^T \int_A g(t, \alpha) \nu(t|d\alpha) dt.$$

The set $\mathcal{M}$ endowed with the topology defined by such a convergence is compact [9]. One can consider measurable functions $t \to \alpha(t)$ as elements of $\mathcal{M}$, setting the correspondence as follows: $\alpha(t) \leftrightarrow \delta_{\alpha(t)}$, where $\delta_{\alpha(t)}$ is the Dirac measure concentrated at the point $\alpha(t)$.

The following proposition is a simple consequence of Proposition 3 (we omit the proof).

PROPOSITION 4. Let $\mu^k \to \nu$. Then

$$\int_0^T \int_A \ell_\varepsilon(t,\alpha,\psi_{t,\alpha,\varepsilon})\mu^k(t|d\alpha)dt \to \int_0^T \int_A \max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi)\nu(t|d\alpha)dt$$

as $\varepsilon \to 0, k \to \infty$.

The next proposition immediately follows from Proposition 4. It is sufficient to replace minima by integrals in measures concentrated for each time instant at minimizing elements (we omit the proof).

PROPOSITION 5.

$$\int_0^T \min_{\alpha\in A} \ell_\varepsilon(t,\alpha,\psi_{t,\alpha,\varepsilon})dt \to \int_0^T \min_{\alpha\in A}\max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi)dt$$

as $\varepsilon \to 0$. ■

PROOF OF THEOREM 1. Assume that

$$\mathrm{dist}_{L_p((0,T);A)}(\alpha^\varepsilon(\cdot),\mathcal{U}) \not\to 0$$

as $\varepsilon \to 0$. Then we can find a sequence $\alpha^{\varepsilon_k}(\cdot)$ such that

$$\mathrm{dist}_{L_p((0,T);A)}(\alpha^{\varepsilon_k}(\cdot),\mathcal{U}) \geq r > 0. \tag{7}$$

We identify functions $\alpha^{\varepsilon_k}(\cdot)$ with elements $\mu^k \in \mathcal{M}$. Without any loss of generality we may assume that $\alpha^{\varepsilon_k}(\cdot)$ (i.e., the correspondent μ^k) converge in $\mathcal{M}$ to some element ν. Then, using Propositions 4,5 and the definition of $\alpha^{\varepsilon_k}(\cdot)$ (see (6)), we conclude that

$$\int_0^T \int_A \ell_{\varepsilon_k}(t,\alpha,\psi_{t,\alpha,\varepsilon_k})\mu^k(t|d\alpha)dt \to \int_0^T \int_A \max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi)\nu(t|d\alpha)dt$$

and

$$\int_0^T \int_A \ell_{\varepsilon_k}(t,\alpha,\psi_{t,\alpha,\varepsilon_k})\mu^k(t|d\alpha)dt = \int_0^T \min_{\alpha\in A} \ell_{\varepsilon_k}(t,\alpha,\psi_{t,\alpha,\varepsilon_k})dt \to$$
$$\to \int_0^T \min_{\alpha\in A}\max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi)dt = 0.$$

The equality to zero follows from (3). Therefore,

$$\int_0^T \int_A \max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi)\nu(t|d\alpha)dt = 0. \tag{8}$$

Consider a multivalued function defined by the relation

$$Q(t) = \left\{\alpha \in A : \max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi) = 0\right\}.$$

Comparison with (3) shows that $\mathcal{U}$ is the set of all measurable selectors of $Q(\cdot)$, i.e.

$$\mathcal{U} = \operatorname{sel} Q := \{q(\cdot) : q(t) \in Q(t),\ \text{a.e. } t \in [0,T],\ q(\cdot) \text{ is measurable }\}.$$

Let us show that a.e. $t \in [0,T]$ the measure $\nu(t|\cdot)$ is concentrated on $Q(t)$, that is

$$\nu(t|A \setminus Q(t)) = 0,\ \text{a.e. } t \in [0,T]. \tag{9}$$

Denote $P(t) = A \setminus Q(t)$ and assume that $\nu(t|P(t)) > 0$ for some t. By the definition of $P(\cdot)$, we have

$$P(t) = \left\{\alpha \in A : \max_{\psi \in B_0^1} \ell^2(t,\alpha,\psi) > 0\right\} = \bigcup_{m=1}^{\infty} P_m(t),$$

where

$$P_m(t) = \left\{\alpha \in A : \max_{\psi \in B_0^1} \ell^2(t,\alpha,\psi) > 1/m\right\}.$$

Since $\nu(t|P(t)) = \lim_{m\to\infty} \nu(t|P_m(t))$, we conclude that $\nu(t|P_m(t)) > 0$ for some m. Hence,

$$\int_A \max_{\psi \in B_0^1} \ell^2(t,\alpha,\psi)\nu(t|d\alpha) \geq \int_{P_m(t)} \max_{\psi \in B_0^1} \ell^2(t,\alpha,\psi)\nu(t|d\alpha) \geq 1/m\nu(t|P_m(t)) > 0.$$

Therefore, the set of such t can not have any positive measure because of (8). So, we have proved that ν is concentrated on Q. At this point we apply the following result obtained in [1].

RESULT([1]). Let $Q : [0,T] \to A$ be a multivalued measurable function. Suppose that a measure ν is concentrated on Q (see 9). Let $\alpha^k(\cdot)$ be an arbitrary sequence of functions converging to ν in $\mathcal{M}$. Then, for any $p \geq 1$,

$$\operatorname{dist}_{L_p((0,T);A)}(\alpha^k(\cdot), \operatorname{sel} Q) \to 0$$

as $k \to \infty$.

So, we conclude that

$$\operatorname{dist}_{L_p((0,T);A)}(\alpha^{\varepsilon_k}(\cdot), \mathcal{U}) \to 0.$$

This contradiction with (7) proves Theorem 1. ■

Numerical examples

Consider the following one-dimensional equation:

$$u_t - (k(x,u)\ u_x)_x = f, \quad x \in (0,1), \quad t \in (0,T).$$

Initial and boundary conditions are homogeneous:

$$\begin{aligned} u(x,0) &= 0, && x \in (0,1), \\ u(0,t) = u(1,t) &= 0, && t \in [0,T]. \end{aligned}$$

The function f is fixed throughout the examples:

$$f(x,t) = 2 + \cos 5x.$$

We use the following parameterization:

$$k(x,u) = \sum_{l=1}^{m} \alpha_l \chi_l(x,u),$$

where $\chi_l,\ l = 1, ..., m$, is a fixed set of functions (we shall define these function more exactly in each concrete case).

Let $\omega_i(x),\ i = 1, ...q$, be a system of global form-functions of the spaces of linear finite elements corresponding to the equidistant partition $0 < x_1 < x_2 ... < x_q < 1$ of the interval $(0,1)$. We assume that the exact solution u has the appropriate smoothness and use the approximations indicated in Remark 1. It is easily to verify that (6) has the following form in this case:

$$\alpha^{\varepsilon}(t) \equiv \alpha^i = \arg\min_{\alpha \in A} (Q\alpha + Mr - g)^T R^{-1} (Q\alpha + Mr - g), \tag{10}$$

if $t \in [t_i, t_{i+1})$ for some $i \in \overline{1,N}$. Here

$$R = \left\{ \rho_{jk} = \int_0^1 \frac{\partial \omega_j}{\partial x}(x) \frac{\partial \omega_k}{\partial x}(x) dx \right\}, \qquad M = \left\{ m_{jk} = \int_0^1 \omega_j(x)\omega_k(x) dx \right\},$$

$$Q = \left\{ q_{jl} = \sum_{s=1}^{q} u^{\eta}(x_s, t_i) \int_0^1 \chi_l(x, u^{\eta}(x,t_i)) \frac{\partial \omega_j}{\partial x}(x) \frac{\partial \omega_s}{\partial x}(x) dx \right\},$$

$$r = \left\{ r_j = \frac{u^{\eta}(x_j, t_{i+1}) - u^{\eta}(x_j, t_i)}{\tau} \right\},$$

$$g = \left\{ g_j = \int_0^1 f(x, t_i) \omega_j(x) dx \right\},$$

where $j, k \in \overline{1,q}$ and $l \in \overline{1,m}$.

It is also meaningful to add to the right-hand-side of (10) the term $c(\alpha - \alpha^{i-1})^2$ to provide uniqueness of the minimizing element. Here c is a sufficiently small coefficient, whose value should be defined more precisely from numerical experiments. This term will play a part of the regularizing component in case the set $\mathcal{U}$ is non-one-element. We obtain the following equation for finding α^i:

$$(Q^T R^{-1} Q + cE)\alpha = Q^T R^{-1}(g - Mr) + c\alpha^{i-1}.$$

Let us consider concrete cases.

a) $k(x,u) = k(u) = 1 + 20u^2$. We set $q = 20$, $m = 11$, and

$$\{\chi_l(u)\} = \left\{ 1, \sin 2\pi u, \cos 2\pi u, ..., \sin 2\pi \frac{m-1}{2} u, \cos 2\pi \frac{m-1}{2} u \right\}.$$

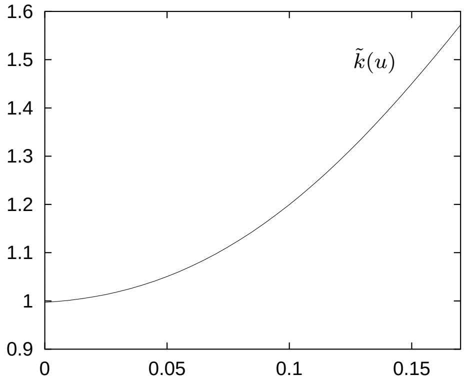

Figure 1

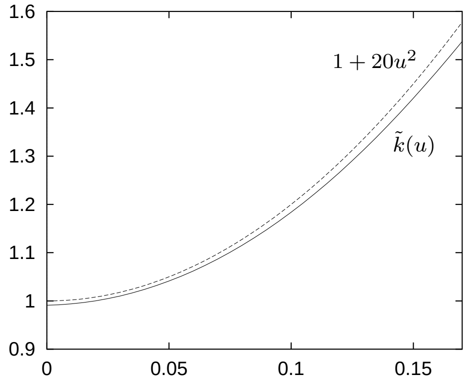

Figure 2

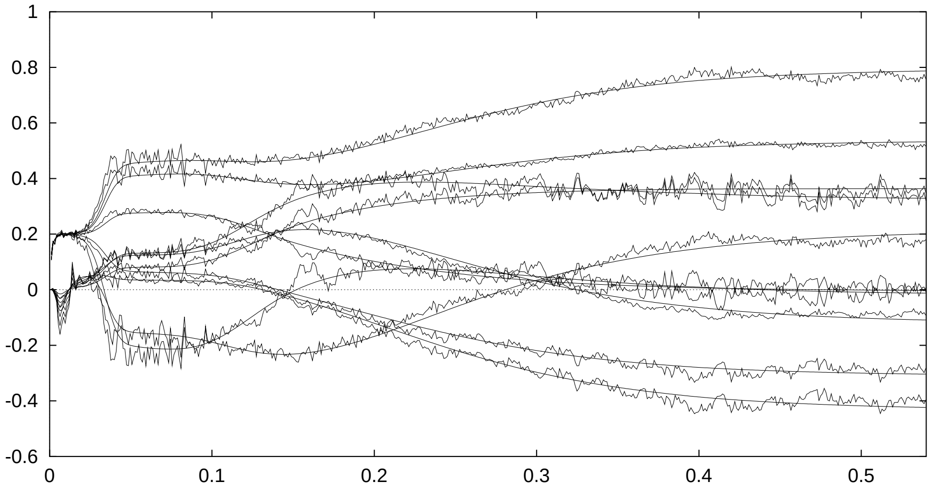

Figure 3

Figure 1 shows the recovered function $\tilde{k}(u)$ in case of the absence of observation errors ($\eta = 0$). The difference between $\tilde{k}(u)$ and $k(u)$ is less then $2 \cdot 10^{-3}$ for all u belonging to the range of the exact solution $u(t, x)$. Figure 2 shows the simulation in case of the presence of the uniformly distributed random error of the amplitude $\eta \approx \tau$. Figure 3 depicts time evolution of the coefficients α_j in the presence of such an error and without errors.

b) $k(x, u) = k(x) = 1 + 0.5 \sin 20x$. We put $q = 20$, $m = 11$, $\eta = 0$. The functions χ_l are defined as in the previous example but with x in place of u. We do not give a picture in this case because the difference between the exact and recovered functions is less then 10^{-5} for all $x \in (0, 1)$.

In the next two examples: $q = 20$, $\eta = 0$, and $\chi_l, l = 1, ..., 15$, are polynomials of x and u.

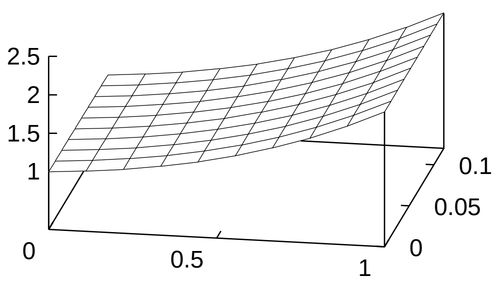

Figure 4

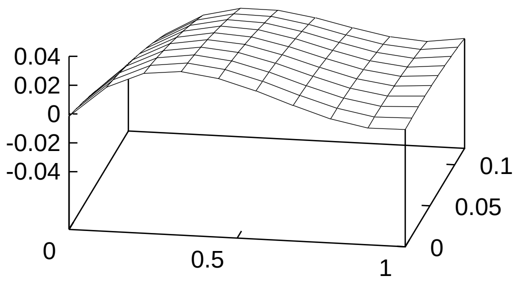

Figure 5

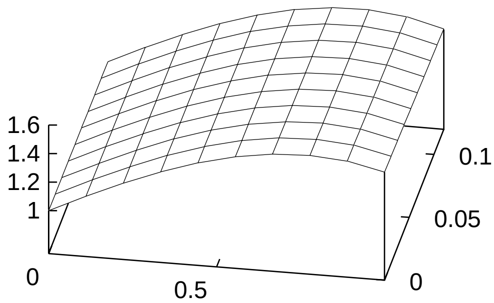

Figure 6

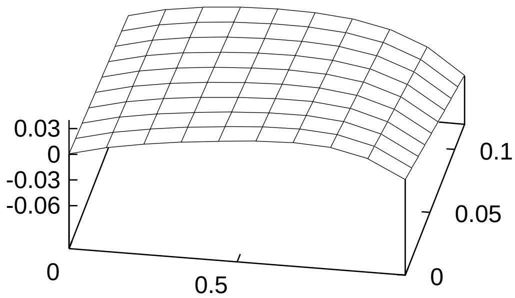

Figure 7

c)$k(x,u) = 1 + x^2 + u^2$. Figure 4 shows the recovered function. Figure 5 depict the difference between the exact and recovered functions.

d)$k(x,u) = 1 + 0.5 \sin 2x + 0.5u^3$. Figures 6 and 7 show the recovered function and the difference between the exact and recovered functions.

REMARK 2. Note that the linear representation of $k(x,u)$ is not the only way to solve the problem. The function $k(x,u)$ can be represented, for example, as follows:

$$k(x,u) = \mathcal{R}(x, u, \alpha_1, \alpha_2, ..., \alpha_m),$$

where $\mathcal{R}$ is an interpolation operator and $\alpha_1, \alpha_2, ..., \alpha_m$ are values of k on a given grid. ■

Figures 8,9 show testruns in the case where $\mathcal{R}$ is the linear interpolation operator. The recovered functions are shown. Figure 8 corresponds to the following data: $k(x,u) = k(u) = 1 + 20u^2$, $q = 20$, $m = 10$ and $\eta = 0$. The difference between $k(u)$ and $\tilde{k}(u)$ is less then $5 \cdot 10^{-3}$. Figure 9 stands for the data: $k(x,u) = k(x) = 1 + 0.5 \sin 20x$, $q = 40$, $m = 40$ and $\eta = 0$. The difference between $k(x)$ and $\tilde{k}(x)$ is less then $2 \cdot 10^{-2}$.

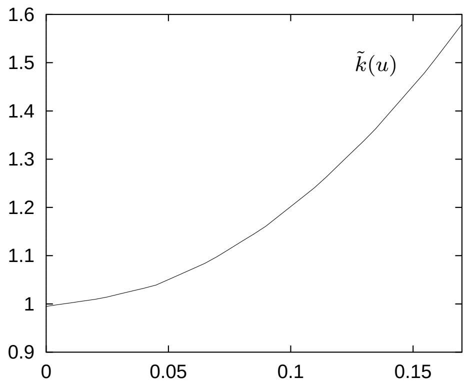

Figure 8

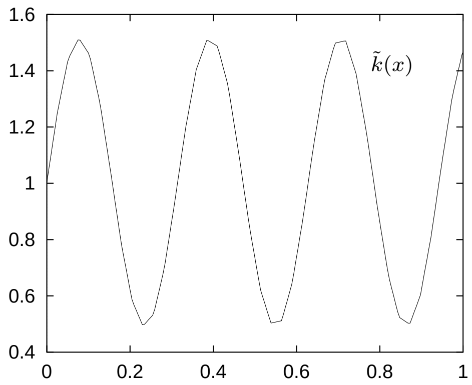

Figure 9

Conclusion: Other types of equations

Now we discuss briefly the possibility to extend the method proposed to more complicated problems. Let us look closer why this algorithm works in the case considered. The crucial point of the proof of Theorem 1 is Proposition 3. Some analysis shows that this proposition is valid due to the following property of the variational equation (2): The solution u, its derivatives, and test functions permit approximations such that the left-hand-side of (2) is continuous with respect to the parameter ε of these approximations. For linear partial differential equations, such a continuity follows from the continuity of the corresponding bilinear forms. For correctly stated nonlinear problems where the existence of solution can be proved, solutions posses some regularity that enables to construct approximations such that the variational functional is continuous with respect to parameters of these approximations.

As an illustration of these speculations, we consider a model describing phase transitions in shape memory alloys [5]. The equations of this model look in the variational form as follows:

$$\int_{Q_T} [\theta S_t \xi + \mu u_t u_{xt} \xi_x + k\theta_x \xi_x + k\theta_{xt} \xi_x - \lambda \xi] + k_1 \int_0^T \int_\Gamma [(\theta - \theta_\Gamma) + (\theta_t - \theta_{\Gamma,t})]\xi = 0,$$

$$\int_{Q_T} [u_{tt} - (\sigma_q)_x + (\sigma_q)_x - f]\eta = 0,$$

$\forall \eta \in L_2((0,T); L_2(\Omega))$ and $\forall \xi \in L_2((0,T); H^1(\Omega))$.

These equations determine the temperature θ and displacement u. Here S is the entropy, σ_q and σ_q are quasiconservative and dissipative parts of the stress. Consider, for example, the term $\theta S_t \xi$. According to [5] we have

$$\theta S_t \xi = \psi_0''(\alpha, \theta)\theta\theta_t \xi + \psi_1''(\alpha, \theta)\theta u_x^2 \theta_t \xi + 2\psi_1'(\alpha, \theta)\theta u_x u_{xt} \xi,$$

where ψ_0 and ψ_1 are coefficients determining the Helmholtz free energy. We assume they depend on some fitting parameter α, and we are to find α on the basis of a given solution. It is shown in [5] that $\theta, u_x, \in L_\infty(Q_T)$ and $\theta_t, u_{xt} \in L_2(Q_T)$. Hence, if $u_1 \to \theta, u_2 \to u_x, u_3 \to \theta_t, u_4 \to u_{xt}, u_5 \to \xi$ in $L_2(Q_T)$ and $\|u_1\|_{L_\infty(Q_T)}, \|u_2\|_{L_\infty(Q_T)} \leq C$, then

$$\int_{Q_T} [\psi_0''(\alpha, u_1)u_1u_3u_5 + \psi_1''(\alpha, u_1)u_1u_2^2u_3u_5 + 2\psi_1'(\alpha, u_1)u_1u_2u_4u_5] \to \int_{Q_T} \theta S_t \xi.$$

One can verify that arguments like these are valid for the other nonlinear terms. This enables to prove a proposition like Proposition 3 and, hence, Theorem 1.

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APPENDIX

PROOF OF PROPOSITION 2.

a) Consider the function

$$u_1^{\eta,\tau} = \frac{u^\eta(\cdot, t_{i+1}) - u^\eta(\cdot, t_i)}{\tau},$$

if $t \in [t_i, t_{i+1})$ for some $i \in \overline{1, N}$. One can see that

$$\frac{u^\eta(\cdot, t_{i+1}) - u^\eta(\cdot, t_i)}{\tau} = \frac{u(\cdot, t_{i+1}) - u(\cdot, t_i)}{\tau} + \frac{\gamma(\cdot, t_i)}{\tau},$$

where

$$\|\gamma\|_{L_\infty(Q_T)} \leq 2\eta.$$

Taking into account the relation

$$\frac{u(\cdot, t_{i+1}) - u(\cdot, t_i)}{\tau} = \frac{1}{\tau} \int_{t_i}^{t_{i+1}} u_t(\cdot, \zeta) d\zeta,$$

we have

$$\|u_1^{\eta,\tau} - u_t\|^2_{L_2(Q_T)} \leq 2 \sum_{i=1}^{N} \int_{t_i}^{t_{i+1}} \left\| \frac{1}{\tau} \int_{t_i}^{t_{i+1}} [u_t(\cdot, \zeta) - u_t(\cdot, t)] d\zeta \right\|^2_{L_2(\Omega)} dt + 4\text{mes}(Q_T) \frac{\eta^2}{\tau^2}.$$

Finally, we obtain

$$\|u_1^{\eta,\tau} - u_t\|^2_{L_2(Q_T)} \leq 2 \max_{|\zeta| \leq \tau} \int_0^T \|u_t(\cdot, t + \zeta) - u_t(\cdot, t)]\|^2_{L_2(\Omega)} dt + 4\text{mes}(Q_T) \frac{\eta^2}{\tau^2} \to 0$$

as $\tau \to 0$ and $\eta/\tau \to 0$. Therefore

$$u_1^{\eta,\tau,h} = F_h u_1^{\eta,\tau} \to u_t, \quad \text{in} \quad L_2(Q_T). \blacksquare$$

b) To simplify notations, we denote

$$u^\tau(x,t) = u(x,t_i)$$

if $t \in [t_i, t_{i+1})$ for some $i \in \overline{1,N}$. With the use of (4) we have

$$\|u_2^{\eta,\tau,h} - u\|_{L_2(Q_T)} \le \|F_h u^\tau - u\|_{L_2(Q_T)} + \mathrm{mes}(Q_T)^{1/2}\eta$$

$$\le \|F_h\|\|u_2^\tau - u\|_{L_2(Q_T)} + \|F_h u - u\|_{L_2(Q_T)} + \mathrm{mes}(Q_T)^{1/2}\eta.$$

Taking into account that $\|F_h\| \le 1$ independent from h, we obtain the proof.

c) We have

$$\|u_3^{\eta,\delta,\tau,h} - \nabla u\|_{L_2^n(Q_T)} \le \|F_h I_\tau \nabla J_\delta u - \nabla u\|_{L_2^n(Q_T)} + \|F_h I_\tau \nabla J_\delta \gamma\|_{L_2^n(Q_T)},$$

where $\|\gamma\|_{L_\infty(Q_T)} \le 2\eta$. It is easy to see that

$$\|F_h I_\tau \nabla J_\delta \gamma\|_{L_2^n(Q_T)} \le C_1\eta/\delta,$$

where C_1 depends only on the dimension n and $\mathrm{mes}(Q_T)$. Moreover, using integration by parts and taking into account that $u|_{\partial\Omega} = 0$, we obtain

$$\nabla J_\delta u = J_\delta \nabla u.$$

So,

$$\|u_3^{\eta,\delta,\tau,h} - \nabla u\|_{L_2^n(Q_T)} \le \|F_h I_\tau J_\delta \nabla u - \nabla u\|_{L_2^n(Q_T)} + C_1\eta/\delta.$$

Taking into consideration that $\nabla u \in L_2^n(Q_T)$ and the properties of the operators F_h, I_τ, J_δ, we obtain the proof.

d) The convergence $f^\tau \to f$ in $L_2(Q_T)$ immediately follows from the properties of I_τ. ■

PROOF OF PROPOSITION 3. Let us denote

$$S^h = \{\psi \in \Phi_h : \|\psi\|_{H_0^1(\Omega)} \le 1\}.$$

We establish first that

$$\ell_\varepsilon(t,\alpha,\psi_{t,\alpha,\varepsilon}) = \max_{\psi \in S^h} \ell_\varepsilon^2(t,\alpha,\psi). \tag{11}$$

Indeed, setting $\psi = \psi_{t,\alpha,\varepsilon}$ in (5), we have

$$\ell_\varepsilon(t,\alpha,\psi_{t,\alpha,\varepsilon,}) = (\psi_{t,\alpha,\varepsilon,}, \psi_{t,\alpha,\varepsilon,})_{H_0^1(\Omega)} = \|\psi_{t,\alpha,\varepsilon,}\|^2_{H_0^1(\Omega)}. \tag{12}$$

On the other hand, from (5), we have

$$\max_{\psi \in S^h} \ell_\varepsilon^2(t,\alpha,\psi) = \max_{\psi \in S^h} (\psi_{t,\alpha,\varepsilon,}, \psi)^2_{H_0^1(\Omega)} = \|\psi_{t,\alpha,\varepsilon,}\|^2_{H_0^1(\Omega)}. \tag{13}$$

From (12) and (13), we obtain (11).

Moreover, we have

$$\int_0^T \max_{\alpha\in A} |\max_{\varphi\in S^h} \ell_\varepsilon^2(t,\alpha,\varphi) - \max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi)|dt \le$$

$$\le \int_0^T \max_{\alpha\in A}(\max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi) - \max_{\varphi\in S^h} \ell^2(t,\alpha,\varphi))dt + \tag{14}$$

$$+ \int_0^T \max_{\alpha\in A} |\max_{\varphi\in S^h} \ell_\varepsilon^2(t,\alpha,\varphi) - \max_{\varphi\in S^h} \ell^2(t,\alpha,\varphi)|dt.$$

Consider the first term of the right-hand-side of (14). Let

$$g_h(t) = -\max_{\alpha\in A}(\max_{\psi\in B_0^1} \ell^2(t,\alpha,\psi) - \max_{\varphi\in S^h} \ell^2(t,\alpha,\varphi)),$$

then we have

$$g_h(t) = -\max_{\alpha\in A} \max_{\psi\in B_0^1} \min_{\varphi\in S^h} (\ell^2(t,\alpha,\psi) - \ell^2(t,\alpha,\varphi)) =$$

$$= -\max_{\psi\in B_0^1} \max_{\alpha\in A} \min_{\varphi\in S^h} (\ell^2(t,\alpha,\psi) - \ell^2(t,\alpha,\varphi)).$$

For any fixed $t \in [0,T]$, let $\psi_t^0 \in B_0^1$ be a maximizing element in the last formula. There exists an element $\varphi_t^h \in S^h$ such that $\|\varphi_t^h - \psi_t^0\|_{H_0^1(\Omega)} \to 0$ as $h \to 0$. Therefore, for any fixed $t \in [0,T]$,

$$g_h(t) = -\max_{\alpha\in A} \min_{\varphi\in S^h} (\ell^2(t,\alpha,\psi_t^0) - \ell^2(t,\alpha,\varphi)) \ge$$

$$\ge -\max_{\alpha\in A}(\ell^2(t,\alpha,\psi_t^0) - \ell^2(t,\alpha,\varphi_t^h)).$$

Taking into consideration that

$$|\ell^2(t,\alpha,\psi_t^0) - \ell^2(t,\alpha,\varphi_t^h)| \le$$

$$\le |\ell^2(t,\alpha,\psi_t^0) + \ell^2(t,\alpha,\varphi_t^h)||\ell^2(t,\alpha,\psi_t^0) - \ell^2(t,\alpha,\varphi_t^h)| \le K(t)\|\varphi_t^h - \psi_t^0\|_{H_0^1(\Omega)},$$

where $K(t)$ is an independent from h and α constant, we obtain

$$0 \ge g_h(t) \ge -K(t)\|\varphi_t^h - \psi_t^0\|_{H_0^1(\Omega)} \to 0 \text{ as } h \to 0.$$

Therefore, by the Lebesque theorem,

$$\int_0^T g_h(t)dt \to 0$$

as $h \to 0$.

Consider the last term of (14). We have

$$\int_0^T \max_{\alpha\in A} |\max_{\psi\in S^h} \ell_\varepsilon^2(t,\alpha,\psi) - \max_{\psi\in S^h} \ell^2(t,\alpha,\psi)|dt \le$$

$$\leq \int_0^T \max_{\alpha\in A}\max_{\psi\in S^h} |\ell_\varepsilon^2(t,\alpha,\psi) - \ell^2(t,\alpha,\psi)|dt \leq$$

$$\leq \int_0^T \max_{\alpha\in A}\max_{\psi\in S^h} |\ell_\varepsilon(t,\alpha,\psi) + \ell(t,\alpha,\psi||\ell_\varepsilon(t,\alpha,\psi) - \ell(t,\alpha,\psi)|dt \leq$$

$$\leq \left(\int_0^T \max_{\alpha\in A}\max_{\psi\in S^h}(\ell_\varepsilon(t,\alpha,\psi) + \ell(t,\alpha,\psi))^2 dt\right)^{1/2} \times$$

$$\times \left(\int_0^T \max_{\alpha\in A}\max_{\psi\in S^h}(\ell_\varepsilon(t,\alpha,\psi) - \ell(t,\alpha,\psi))^2 dt\right)^{1/2}.$$

It can be easily proved that

$$\left(\int_0^T \max_{\alpha\in A}\max_{\psi\in S^h}(\ell_\varepsilon(t,\alpha,\psi) + \ell(t,\alpha,\psi))^2 dt\right)^{1/2} \leq C_2,$$

where C_2 is a constant independent from ε.
If we prove that

$$\int_0^T \max_{\alpha\in A}\max_{\psi\in S^h}(\ell_\varepsilon(t,\alpha,\psi) - \ell(t,\alpha,\psi))^2 dt \to 0,$$

then Proposition 3 is proved.
We have

$$(\ell_\varepsilon(t,\alpha,\psi) - \ell(t,\alpha,\psi))^2 =$$

$$= \left(\int_\Omega (u_1^\varepsilon - u_t)\psi + (k(x,u_2^\varepsilon,\alpha)u_3^\varepsilon - k(x,u,\alpha)\nabla u)\nabla\psi + (f^\varepsilon - f)\psi\right)^2 \leq$$

$$\leq 2\int_\Omega (u_1^\varepsilon - u_t)^2 dx \int_\Omega \psi^2 dx +$$

$$+2\int_\Omega (k(x,u_2^\varepsilon,\alpha)u_3^\varepsilon - k(x,u,\alpha)\nabla u)^2 \int_\Omega (\nabla\psi)^2 dx +$$

$$+2\int_\Omega (f^\varepsilon - f)^2 \int_\Omega \psi^2 dx.$$

Taking into account that $\int_\Omega (\nabla\psi)^2 dx \leq 1$ and $\int_\Omega \psi^2 dx \leq C_3$ (Poincore theorem), we obtain

$$\int_0^T \max_{\alpha\in A}\max_{\psi\in S^h}(\ell_\varepsilon(t,\alpha,\psi) - \ell(t,\alpha,\psi))^2 dt \leq$$

$$\leq C_4 \int_{Q_T} (u_1^\varepsilon(x,t) - u_t(x,t))^2 dx\, dt +$$

$$+C_4 \int_{Q_T} \max_{\alpha\in A}(k(x,u_2^\varepsilon(x,t),\alpha)u_3^\varepsilon - k(x,u(x,t),\alpha)\nabla u(x,t))^2 dx\, dt +$$

$$+C_4 \int_{Q_T} (f^\varepsilon - f)^2 dx\, dt.$$

It remains to prove that

$$\int_{Q_T} \max_{\alpha\in A}(k(x,u_2^\varepsilon(x,t),\alpha)u_3^\varepsilon(x,t)-k(x,u(x,t),\alpha)\nabla u(x,t))^2 dx\,dt \to 0$$

as $\varepsilon \to 0$. Assume the opposite. Then we can find a sequence ε_j such that

$$\int_{Q_T} \max_{\alpha\in A}(k(x,u_2^{\varepsilon_j}(x,t),\alpha)u_3^{\varepsilon_j}(x,t)-k(x,u(x,t),\alpha)\nabla u(x,t))^2 dx\,dt \geq r > 0.$$

On the other hand, we may assume that $u_2^{\varepsilon_j} \to u$ and $u_3^{\varepsilon_j} \to \nabla u$ for almost all $(x,t) \in Q_T$. We have

$$\max_{\alpha\in A}(k(x,u_2^{\varepsilon_j},\alpha)u_3^{\varepsilon_j}-k(x,u,\alpha)\nabla u)^2 =$$

$$= \max_{\alpha\in A}(k(x,u_2^{\varepsilon_j},\alpha)u_3^{\varepsilon_j}-k(x,u_2^{\varepsilon_j},\alpha)\nabla u + k(x,u_2^{\varepsilon_j},\alpha)\nabla u - k(x,u,\alpha)\nabla u)^2 \leq$$

$$\leq 2\max_{\alpha\in A}|(k(x,u_2^{\varepsilon_j},\alpha)-k(x,u,\alpha))\nabla u|^2 +$$

$$+2\max_{\alpha\in A}|k(x,u_2^{\varepsilon_j},\alpha)(u_3^{\varepsilon_j}-\nabla u)|^2.$$

We observe that, due to the continuity of k,

$$\max_{\alpha\in A}|(k(x,u_2^{\varepsilon_j},\alpha)-k(x,u,\alpha))\nabla u|^2 \to 0,$$

for almost all $(x,t) \in Q_T$. Because of boundness of k, we have

$$\max_{\alpha\in A}|(k(x,u_2^{\varepsilon_j},\alpha)-k(x,u,\alpha))\nabla u|^2 \leq 4M^2|\nabla u|^2$$

and

$$\max_{\alpha\in A}|k(x,u_2^{\varepsilon_j},\alpha)(u_3^{\varepsilon_j}-\nabla u)|^2 \leq M^2|u_3^{\varepsilon_j}-\nabla u|^2. \tag{15}$$

Using the Lebesque theorem, we obtain

$$\int_{Q_T} \max_{\alpha\in A}|(k(x,u_2^{\varepsilon_j},\alpha)-k(x,u,\alpha))\nabla u|^2 dx\,dt \to 0.$$

From (15), we have

$$\int_{Q_T} \max_{\alpha\in A}|k(x,u_2^{\varepsilon_j},\alpha)(u_3^{\varepsilon_j}-\nabla u)|^2 dx\,dt \leq M^2\int_{Q_T}|u_3^{\varepsilon_j}-\nabla u|^2 dx\,dt \to 0$$

Therefore,

$$\int_{Q_T} \max_{\alpha\in A}(k(x,u_2^\varepsilon(x,t),\alpha)u_3^\varepsilon-k(x,u(x,t),\alpha)\nabla u(x,t))^2 dx\,dt \to 0.$$

This finishes the proof of Proposition 3.

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