ESTIMATION OF PARAMETERS OF A LINEAR THIN PLATE EXCITED BY A PIEZOELECTRIC PATCH

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Abstract. A model describing a linear thin plate exited by a piezoelectric patch is considered. The application of the method of least squares to the estimation of unknown parameters of the model is discussed. The main purpose of the paper is to prove the existence of derivatives of solutions with respect to parameters. Such a differentiability is proved for an appropriate space, which enables us to handle pointwise measurements of data. It is shown that the derivatives can be computed via the corresponding variational equations. The paper is illustrated by computer simulations.

Key Words. Linear thin plates, Piezoelectric patches, Estimation of parameters, Variational equations.

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1. Problem setting

We consider a linear system describing oscillations of a thin plate excited by a patch made of a piezoelectric ceramic. Assume that the plate occupies a domain S, where S is a bounded simple connected open set in R^2 with the piecewise infinitely differentiable boundary ∂S. The patch occupies a domain $S_P \subset S$ with the same properties as S. Denote $S_B = S \setminus S_P$. The equation describing the model reads:

$$\rho \xi_{tt} + \Delta\Big(\gamma \Delta \xi\Big) - \mu \Delta \xi_t = v(t) K \Delta\Big(I_{S_P}\Big), \quad x \in S. \tag{1}$$

Here ξ is the vertical displacement of the plate, Δ is the Laplace operator, $v(t)$ is the voltage applied to the piezoelectric patch S_P (see Figure 1), I_{S_P} is the indicator function of the patch.

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The coefficients μ and K are constant; ρ and γ are piecewise constant functions defined as follows :

$$\rho = \begin{cases} \rho_P, \ (x_1, x_2) \in S_P, \\ \rho_B, \ (x_1, x_2) \in S_B, \end{cases} \qquad \gamma = \begin{cases} \gamma_P, \ (x_1, x_2) \in S_P, \\ \gamma_B, \ (x_1, x_2) \in S_B. \end{cases}$$

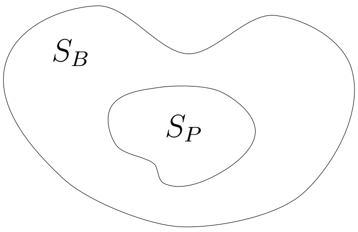

Figure 1. The plate S with the patch S_P, $S_B := S \setminus S_P$.

Indices P and B point out to the piezoelectric–basis–mixed and basis materials, respectively. It is assumed that $(\gamma_P, \gamma_B, \mu, K) \in \mathcal{P}$, where

$$\mathcal{P} := \{(\gamma_P, \gamma_B, \mu, K) \in R^4 : \gamma_P > 0, \gamma_B > 0, \mu \geq 0\}. \tag{2}$$

Boundary and initial conditions are:

$$\xi|_{\partial S} = 0, \quad \partial\xi/\partial\vec{n}|_{\partial S} = 0, \quad \xi|_{t=0} = \xi(0), \quad \xi_t|_{t=0} = \xi_t(0). \tag{3}$$

The model (1) is well known and was in details investigated in [1]. It is convenient to derive (1) using the technique described in [2] and the theory of electro-elasticity developed in [3, 4, 5]. Note that the system (1) should be supplied with some interface conditions that hold on the boundary between S_P and S_B because of the integration by parts when deriving (1) from a weak formulation. We do not write down these conditions because we shall go back to the weak formulation.

2. Differentiability

Assume we apply the method of least squares to estimate the vector of parameters

$$\pi^0 = (\gamma_P^0, \ \gamma_B^0, \ \mu^0, \ K^0) \in \mathcal{P}$$

on the basis of observations

$$y_n(t_m) = g_n(\xi^0(t_m, x_1^1, x_2^1), \xi^0(t_m, x_1^2, x_2^2), ..., \xi^0(t_m, x_1^L, x_2^L)) + \varepsilon_{nm}, \ n = 1, N, \ m = 1, M.$$

Here ξ^0 is a solution of (1) with the set $\pi^0 = (\gamma_P^0, \ \gamma_B^0, \ \mu^0, \ K^0)$ of parameters, $g_n(\sigma_1, ..., \sigma_L)$, $n = 1, N$, are known functions, $t_m \in (0, T)$, $m = 1, M$, and $(x_1^i, x_2^i) \in$ int S, $i = 1, L$, are known times and points, $|\varepsilon_{nm}| \leq \eta$ is an observation error. According to the method of least squares, we form the functional

$$J(\pi) = \sum_{m=1}^{M} \sum_{n=1}^{N} \Big(g_n(\xi(t_m, x_1^1, x_2^1), \xi(t_m, x_1^2, x_2^2), ..., \xi(t_m, x_1^L, x_2^L)) - y_n(t_m) \Big)^2$$

where ξ is a solution of (1) corresponding to a set $\pi = (\gamma_P,\ \gamma_B,\ \mu,\ K) \in \mathcal{P}$ of parameters. An approximation $\tilde{\pi}$ of the system's parameters (an estimation of π^0) is being found as a minimizing point of $J(\pi)$, i.e.

$$\tilde{\pi} = \underset{\pi\in\mathcal{P}}{\arg\min}\, J(\pi).$$

The minimization problem can be solved using a gradient descent method, if one can compute derivatives of J. The formal differentiation of J w.r.t. the components π_α of the vector $\pi \in \mathcal{P}$ yields:

$$\frac{\partial J}{\partial \pi_\alpha} = \sum_{m=1}^{M}\sum_{n=1}^{N}(g_n(\vec{\xi}(t_m)) - y_n(t_m))\sum_{j=1}^{L}\frac{\partial g_n}{\partial \sigma_j}(\vec{\xi}(t_m))\frac{\partial \xi}{\partial \pi_\alpha}(t_m, x_1^j, x_2^j),\ \ \alpha = 1,2,3,4,$$

where $\vec{\xi}(t_m) = (\xi(t_m, x_1^1, x_2^1), \xi(t_m, x_1^2, x_2^2), ..., \xi(t_m, x_1^L, x_2^L)$ and $\frac{\partial\xi}{\partial\pi_\alpha}(t_m, x_1^j, x_2^j)$ is the partial derivative of the function $\pi \to \xi(t_m, x_1^j, x_2^j)$.

PROPOSITION 1. Assume that $\xi \in C([0,T]; H_0^2(S))$ and the map

$$\pi \to \xi : R^4 \to C([0,T]; H_0^2(S)) \tag{4}$$

is differentiable. Then all of the functions $\pi \to \xi(t_m, x_1^j, x_2^j)$ are differentiable. ■

The proof of this proposition immediately follows from the continuous embedding $C([0,T]; H_0^2(S)) \subset C([0,T] \times \bar{S})$ which implies that $\|\cdot\|_{C([0,T]\times\bar{S})} \le \text{const}\cdot\|\cdot\|_{C([0,T];H_0^2(S))}$. Thus, the differentiability with respect to the norm of $C([0,T]; H_0^2(S))$ implies the differentiability with respect to the norm of $C([0,T] \times \bar{S})$ and, therefore, the differentiability of the functions $\pi \to \xi(t_m, x_1^j, x_2^j)$.

Now we prove the differentiability of the map (4) and show that the partial derivatives can be computed via the corresponding variational equations. The proof is based on the possibility to differentiate the equation (1) w.r.t. t under some compatibility conditions (see [6]). In such a way, one can improve the usual smoothness of solutions of (1) and establish necessary estimates on the difference between the increment of the map (4) and its differential formed by solutions of variational equations.

The weak formulation of (1) is given by the equality

$$\iint\limits_{S_P} \rho_P\xi_{tt}\varphi + \gamma_P\Delta\xi\Delta\varphi + \iint\limits_{S_B} \rho_B\xi_{tt}\varphi + \gamma_B\Delta\xi\Delta\varphi + \tag{5}$$
$$+\iint\limits_{S} \mu\nabla\xi_t\nabla\varphi = \iint\limits_{S_P} v(t)K\Delta\varphi$$

for all $\varphi \in H_0^2(S)$. Consider the following generalization of the last equation:

$$\iint\limits_{S_P} \rho_P\xi_{tt}\varphi + \gamma_P\Delta\xi\Delta\varphi + \iint\limits_{S_B} \rho_B\xi_{tt}\varphi + \gamma_B\Delta\xi\Delta\varphi + \tag{6}$$
$$+\iint\limits_{S} \mu\nabla\xi_t\nabla\varphi = \iint\limits_{S} g\Delta\varphi,$$

which is obtained by generalizing the right-hand-side.

Further, an arbitrary point of $\mathcal{P}$ will be chosen and the differentiability at this point will be investigated. Therefore, we may restrict our considerations to the system's parameters lying in a neighborhood $\Omega \subset \mathcal{P}$ such that

$$\bar{\Omega} \subset \mathcal{P}. \tag{7}$$

PROPOSITION 2. If $\xi(0) \in H_0^2(S)$, $\xi_t(0) \in L_2(S)$, $g \in H^1(0,T;L_2(S))$, then the equation (6) has a unique solution ξ such that $(\xi, \xi_t) \in C([0,T];H_0^2(S)) \times C([0,T];L_2(S))$ and

$$\|\xi\|^2_{C([0,T];H_0^2(S))} + \|\xi_t\|^2_{C([0,T];L_2(S))} \le$$

$$C(\Omega)\left(\|\xi(0)\|^2_{H_0^2(S)} + \|\xi_t(0)\|^2_{L_2(S)} + \|g\|^2_{C([0,T];L_2(S))} + \|g_t\|^2_{L_2(0,T;L_2(S))}\right),$$

with $C(\Omega)$ bounded for any Ω satisfying (7). ■

PROOF OF PROPOSITION 2. The proof of the proposition is similar to that given in Chapter 8 of [7]. Let ω_i, $i = 1, 2...$, be a basis of $H_0^2(S)$. We define approximations $\xi^m(t)$ as follows

$$\xi^m(t) = \sum_{i=1}^{m} r_i^m(t)\omega_i.$$

For each m, the unknown functions $r_i^m(t)$, $i = 1, ..., m$ satisfy the corresponding linear system of ordinary differential equations with initial conditions chosen such that: $\sum_{i=1}^m r_i^m(0)\omega_i = a^m$, $\sum_{i=1}^m d/dt\, r_i^m(0)\omega_i = b^m$, where $a^m \to \xi(0)$ and $b^m \to \xi_t(0)$ in $H_0^2(S)$ and $L_2(S)$, respectively. One can easily obtain the following energy estimate:

$$\|\xi^m\|^2_{L_\infty(0,T;H_0^2(S))} + \|\xi_t^m\|^2_{L_\infty(0,T;L_2(S))} + \mu\|\nabla\xi_t^m\|^2_{L_2(0,T;L_2(S))} \le$$

$$C(\Omega)\left(\|a^m\|^2_{H_0^2(S)} + \|b^m\|^2_{L_2(S)} + \|g\|^2_{C([0,T];L_2(S))} + \|g_t\|^2_{L_2(0,T;L_2(S))}\right)$$

with $C(\Omega)$ bounded for any Ω satisfying (7). Note that $g \in C([0,T];L_2(S))$ (see e.g. [8]). There exists a subsequence ξ^{m_k} and a function ξ such that $\xi^{m_k} \to \xi$ and $\xi_t^{m_k} \to \xi_t$, $*$weakly in $L_\infty(0,T;H_0^2(S))$ and $L_\infty(0,T;L_2(S))$, respectively. Besides, $\nabla\xi_t^{m_k} \to \nabla\xi_t$ weakly in $L_2(0,T;L_2(S))$, if $\mu \ne 0$. One proves that ξ is a unique solution of (6) which, obviously, satisfies

$$\|\xi\|^2_{L_\infty(0,T;H_0^2(S))} + \|\xi_t\|^2_{L_\infty(0,T;L_2(S))} + \mu\|\nabla\xi_t\|^2_{L_2(0,T;L_2(S))} \le$$

$$C(\Omega)\left(\|\xi(0)\|^2_{H_0^2(S)} + \|\xi_t(0)\|^2_{L_2(S)} + \|g\|^2_{C([0,T];L_2(S))} + \|g_t\|^2_{L_2(0,T;L_2(S))}\right).$$

Then one establishes the additional regularity of ξ with respect to t. To this end, one uses the energy equality:

$$\iint_{S_P} \rho_P(\xi_t)^2 + \iint_{S_B} \rho_B(\xi_t)^2 + \iint_{S_P} \gamma_P(\Delta\xi)^2 + \iint_{S_B} \gamma_B(\Delta\xi)^2 =$$

$$\iint_{S_P} \rho_P(\xi_t(0))^2 + \iint_{S_B} \rho_B(\xi_t(0))^2 + \iint_{S_P} \gamma_P(\Delta\xi(0))^2 + \iint_{S_B} \gamma_B(\Delta\xi(0))^2 - 2\iint_{S} g(0)\Delta\xi(0) +$$

$$2\iint\limits_S g\Delta\xi - 2\int\limits_0^t\iint\limits_S g_t\Delta\xi - 2\int\limits_0^t\iint\limits_S \mu(\nabla\xi_t)^2.$$

Note that the function $t \to \iint\limits_S g(t)\Delta\xi(t)$ is continuous due to Lemma 8.2 of [7]. Since $\xi(0) \in H_0^2(S)$ and $\xi_t(0) \in L_2(S)$, all terms on the right-hand-side are well defined and the function

$$t \to \iint\limits_{S_P} \rho_P(\xi_t)^2 + \iint\limits_{S_B} \rho_B(\xi_t)^2 + \iint\limits_{S_P} \gamma_P(\Delta\xi)^2 + \iint\limits_{S_B} \gamma_B(\Delta\xi)^2$$

is continuous on $[0,T]$. Then, using the same arguments as in the proof of Theorem 8.2 of [7], one shows that $\xi \in C([0,T];H_0^2(S))$ and $\xi_t \in C([0,T];L_2(S))$. ■

PROPOSITION 3. Let $v \in H^3(0,T)$, $v(0) = v'(0) = 0$, and $\xi(0) = \xi_t(0) = 0$, then the solution of (5) has the form

$$\xi(t,\cdot) = \int\limits_0^t d\tau \int\limits_0^\tau \ddot{\xi}(\theta,\cdot)d\theta,$$

where $\ddot{\xi}$ satisfies the equation

$$\iint\limits_{S_P} \rho_P\ddot{\xi}_{tt}\varphi + \gamma_P\Delta\ddot{\xi}\Delta\varphi + \iint\limits_{S_B} \rho_B\ddot{\xi}_{tt}\varphi + \gamma_B\Delta\ddot{\xi}\Delta\varphi + \tag{8}$$

$$+\iint\limits_S \mu\nabla\ddot{\xi}_t\nabla\varphi = \iint\limits_{S_P} v''(t)K\Delta\varphi$$

with the initial conditions $\ddot{\xi}(0) = \ddot{\xi}_t(0) = 0$.■

PROOF OF PROPOSITION 3. We have

$$\iint\limits_{S_P} \rho_P\xi_{tt}\varphi + \iint\limits_{S_B} \rho_B\xi_{tt}\varphi = \iint\limits_{S_P} \rho_P\ddot{\xi}\varphi + \iint\limits_{S_B} \rho_B\ddot{\xi}\varphi =$$

$$\iint\limits_{S_P} \rho_P \left(\int\limits_0^t d\tau \int\limits_0^\tau \ddot{\xi}_{tt}d\theta\right)\varphi + \iint\limits_{S_B} \rho_B \left(\int\limits_0^t d\tau \int\limits_0^\tau \ddot{\xi}_{tt}d\theta\right)\varphi =$$

$$\int\limits_0^t d\tau \int\limits_0^\tau d\theta \left(\iint\limits_{S_P} \rho_P\ddot{\xi}_{tt}\varphi + \iint\limits_{S_B} \rho_B\ddot{\xi}_{tt}\varphi\right) =$$

$$\int\limits_0^t d\tau \int\limits_0^\tau d\theta \left(\iint\limits_{S_P} v''(\theta)K\Delta\varphi - \iint\limits_{S_P} \gamma_P\Delta\ddot{\xi}\Delta\varphi - \iint\limits_{S_B} \gamma_B\Delta\ddot{\xi}\Delta\varphi - \iint\limits_S \mu\nabla\ddot{\xi}_t\nabla\varphi\right) =$$

$$\iint\limits_{S_P} v(t)K\Delta\varphi - \iint\limits_{S_P} \gamma_P\Delta\xi\Delta\varphi - \iint\limits_{S_B} \gamma_B\Delta\xi\Delta\varphi - \iint\limits_S \mu\nabla\xi_t\nabla\varphi,$$

that is ξ is the solution of (5).

COROLLARY 1. If conditions of Proposition 3 hold, then (5) has a unique solution ξ satisfying: $(\xi, \xi_t) \in C^2([0,T]; H_0^2(S)) \times C^2([0,T]; L_2(S))$.■

This follows from Proposition 2 applied to (8) and Proposition 3.

Let $u = \xi - \xi^0$, where ξ and ξ^0 are solutions of (5) corresponding to parameter sets π and π^0, respectively. It is easy to see that u satisfies the equation:

$$\begin{aligned}
&\iint\limits_{S_P} \rho_P^0 u_{tt}\varphi + \gamma_P^0 \Delta u \Delta\varphi + (\gamma_P - \gamma_P^0)\Delta\xi\Delta\varphi + \\
&\iint\limits_{S_B} \rho_B^0 u_{tt}\varphi + \gamma_B^0 \Delta u \Delta\varphi + (\gamma_B - \gamma_B^0)\Delta\xi\Delta\varphi + \\
&\iint\limits_{S} \mu^0 \nabla u_t \nabla\varphi + (\mu - \mu^0)\nabla\xi_t\nabla\varphi = \iint\limits_{S_P} v(t)(K - K^0)\Delta\varphi,
\end{aligned} \tag{9}$$

with the initial conditions $u(0) = 0, \ u_t(0) = 0$.

PROPOSITION 4. Let conditions of Proposition 3 hold. Then the solution of (9) has the form

$$u(t, \cdot) = \int_0^t \dot{u}(\theta, \cdot)d\theta,$$

where $\dot{u}$ satisfies the equation

$$\begin{aligned}
&\iint\limits_{S_P} \rho_P^0 \dot{u}_{tt}\varphi + \gamma_P^0 \Delta \dot{u} \Delta\varphi + (\gamma_P - \gamma_P^0)\Delta\xi_t\Delta\varphi + \\
&\iint\limits_{S_B} \rho_B^0 \dot{u}_{tt}\varphi + \gamma_B^0 \Delta \dot{u} \Delta\varphi + (\gamma_B - \gamma_B^0)\Delta\xi_t\Delta\varphi + \\
&\iint\limits_{S} \mu^0 \nabla \dot{u}_t \nabla\varphi + (\mu - \mu^0)\nabla\xi_{tt}\nabla\varphi = \iint\limits_{S_P} v'(t)(K - K^0)\Delta\varphi.
\end{aligned} \tag{10}$$

PROOF OF PROPOSITION 4. The proof of this proposition is similar to that of Proposition 3, if one takes into account that

$$\iint\limits_{S} \nabla\xi_{tt}\nabla\varphi = \iint\limits_{S} \xi_{tt}\Delta\varphi \tag{11}$$

for all $\varphi \in H_0^2(S)$.

COROLLARY 2. If conditions of Proposition 3 hold, then (9) has a unique solution u satisfying: $(u, u_t) \in C^1([0,T]; H_0^2(S)) \times C^1([0,T]; L_2(S))$ and the following estimate holds

$$\|u\|^2_{C^1([0,T];H_0^2(S))} + \|u_t\|^2_{C^1([0,T];L_2(S))} \leq$$

$$C(\Omega)\left[(\gamma_P-\gamma_P^0)^2+(\gamma_B-\gamma_B^0)^2+(\mu-\mu^0)^2+(K-K^0)^2\right]$$

with $C(\Omega)$ bounded for any Ω satisfying (7). ■

PROOF OF COROLLARY 2. Note that $\Delta\xi_t,\ \xi_{tt}\in C^1(0,T;L_2(S))$ due to Corollary 1. Thus, taking into account (11) , one can apply Proposition 2 to (10) and use Proposition 4. ■

The variational equations that formally define the derivatives

$$\delta^1=\frac{\partial\xi}{\partial\gamma_P},\quad \delta^2=\frac{\partial\xi}{\partial\gamma_B},\quad \delta^3=\frac{\partial\xi}{\partial\mu},\quad \delta^3=\frac{\partial\xi}{\partial K}$$

at the point $\pi^0=(\gamma_P^0,\ \gamma_B^0,\ \mu^0,\ K^0)$ have the form:

$$\iint\limits_{S_P}\rho_P^0\delta_{tt}^1\varphi+\gamma_P^0\Delta\delta^1\Delta\varphi+\Delta\xi^0\Delta\varphi+\iint\limits_{S_B}\rho_B^0\delta_{tt}^1\varphi+\gamma_B^0\Delta\delta^1\Delta\varphi+ \\ +\iint\limits_{S}\mu^0\nabla\delta_t^1\nabla\varphi=0, \tag{12}$$

$$\iint\limits_{S_P}\rho_P^0\delta_{tt}^2\varphi+\gamma_P^0\Delta\delta^2\Delta\varphi+\iint\limits_{S_B}\rho_B^0\delta_{tt}^2\varphi+\gamma_B^0\Delta\delta^2\Delta\varphi+\Delta\xi^0\Delta\varphi+ \\ +\iint\limits_{S}\mu^0\nabla\delta_t^2\nabla\varphi=0, \tag{13}$$

$$\iint\limits_{S_P}\rho_P^0\delta_{tt}^3\varphi+\gamma_P^0\Delta\delta^3\Delta\varphi+\iint\limits_{S_B}\rho_B^0\delta_{tt}^3\varphi+\gamma_B^0\Delta\delta^3\Delta\varphi+ \\ +\iint\limits_{S}\mu^0\nabla\delta_t^3\nabla\varphi+\nabla\xi_t^0\nabla\varphi=0, \tag{14}$$

$$\iint\limits_{S_P}\rho_P^0\delta_{tt}^4\varphi+\gamma_P^0\Delta\delta^4\Delta\varphi+\iint\limits_{S_B}\rho_B^0\delta_{tt}^4\varphi+\gamma_B^0\Delta\delta^4\Delta\varphi+ \\ +\iint\limits_{S}\mu^0\nabla\delta_t^4\nabla\varphi=\iint\limits_{S_P}v(t)\Delta\varphi. \tag{15}$$

PROPOSITION 5. If conditions of Proposition 3 hold, then each of the equations (12) – (15) has a unique solution satisfying $(\delta^i,\delta_t^i)\in C^1([0,T];H_0^2(S))\times C^1([0,T];L_2(S)),i=1,4$. All $(\delta^i,\delta_t^i),i=1,4$, are Lipschitzian with respect to $\gamma_P^0,\gamma_B^0,\mu^0,K^0$ in $C^1([0,T];H_0^2(S))\times C^1([0,T];L_2(S))$. ■

PROOF OF PROPOSITION 5. Let us prove the proposition for (12). The other equations can be handled in a similar way. First, apply Proposition 2 to (12) to establish the existence and uniqueness of a solution. Then prove that the solution is of the form

$$\delta^1(t,\cdot) = \int\limits_0^t \dot{\delta}^1(\theta,\cdot)d\theta,$$

where $\dot{\delta}^1$ satisfies the equation

$$\iint\limits_{S_P} \rho_P^0\dot{\delta}^1_{tt}\varphi + \gamma_P^0\Delta\dot{\delta}^1\Delta\varphi + \Delta\xi^0_t\Delta\varphi + \iint\limits_{S_B} \rho_B^0\dot{\delta}^1_{tt}\varphi + \gamma_B^0\Delta\dot{\delta}^1\Delta\varphi + \tag{16}$$

$$+\iint\limits_{S} \mu^0\nabla\dot{\delta}^1_t\nabla\varphi = 0.$$

Then apply Proposition 2 to (16) to prove the first claim of Proposition 5.
To prove the second claim, note that the difference $z = \bar{\delta}^1_t - \delta^1_t$, where δ^1 and $\bar{\delta}^1$ are solutions of (12) with parameter sets $\pi^0 = (\gamma_P^0,\ \gamma_B^0,\ \mu^0,\ K^0)$ and $\pi = (\gamma_P,\ \gamma_B,\ \mu,\ K)$, respectively, satisfies the equation:

$$\iint\limits_{S_P} \rho_P^0 z_{tt}\varphi + \gamma_P\Delta z\Delta\varphi + (\gamma_P - \gamma_P^0)\Delta\delta^1_t\Delta\varphi + \Delta u_t\Delta\varphi + \tag{17}$$

$$\iint\limits_{S_B} \rho_B^0 z_{tt}\varphi + \gamma_B\Delta z\Delta\varphi + (\gamma_B - \gamma_B^0)\Delta\delta^1_t\Delta\varphi +$$

$$\iint\limits_{S} \mu\nabla z_t\nabla\varphi + (\mu - \mu^0)\delta^1_t\Delta\varphi = 0.$$

Applying Proposition 2 to (17) and taking into account Corollary 2 yields

$$\|z\|^2_{C([0,T];H_0^2(S))} + \|z_t\|^2_{C([0,T];L_2(S))} \leq$$

$$C(\Omega)\left[(\gamma_P - \gamma_P^0)^2 + (\gamma_B - \gamma_B^0)^2 + (\mu - \mu^0)^2\right],$$

where $C(\Omega)$ is bounded for any Ω satisfying (7). This proves the second part of the proposition. ■

Keep the notation $u = \xi - \xi^0$ and estimate the function

$$w = u - \delta^1(\gamma_P - \gamma_P^0) - \delta^2(\gamma_B - \gamma_B^0) - \delta^3(\mu - \mu^0) - \delta^4(K - K^0).$$

One can check that the function w satisfies the following equation:

$$\iint\limits_{S_P} \rho_P^0 w_{tt}\varphi + \gamma_P^0\Delta w\Delta\varphi + (\gamma_P - \gamma_P^0)\Delta u\Delta\varphi +$$

$$\iint\limits_{S_B} \rho_B^0 w_{tt}\varphi + \gamma_B^0\Delta w\Delta\varphi + (\gamma_B - \gamma_B^0)\Delta u\Delta\varphi + \tag{18}$$

$$\iint\limits_{S} \mu^0\nabla w_t\nabla\varphi + (\mu - \mu^0)\nabla u_t\nabla\varphi = 0.$$

PROPOSITION 6. Let conditions of Proposition 3 hold. Then $(w,w_t)\in C([0,T];H_0^2(S))\times C([0,T];L_2(S))$ and

$$\|w\|_{C([0,T];H_0^2(S))}+\|w_t\|_{C([0,T];L_2(S))}\le C(\Omega)\left[(\gamma_P-\gamma_P^0)^2+(\gamma_B-\gamma_B^0)^2+(\mu-\mu^0)^2+(K-K^0)^2\right],$$

where $C(\Omega)$ is bounded for any Ω satisfying (7). ■

PROOF OF PROPOSITION 6. Note that

$$\int\!\!\int_S \nabla u_t\nabla\varphi=\int\!\!\int_S u_t\Delta\varphi$$

for all $\varphi\in H_0^2(S)$. Let $g=[(\gamma_P-\gamma_P^0)I_{S_P}+(\gamma_B-\gamma_B^0)I_{S_B})]\Delta u+(\mu-\mu^0)u_t$. Using Corollary 2, one can verify that g satisfies conditions of Proposition 2. Applying Proposition 2 to (18) yields

$$\|w\|^2_{C([0,T];H_0^2(S))}+\|w_t\|^2_{C([0,T];L_2(S))}\le$$
$$C_1(\Omega)\left[(\gamma_P-\gamma_P^0)^2+(\gamma_B-\gamma_B^0)^2\right]\|u\|^2_{C^1([0,T];H_0^2(S))}+C_2(\Omega)(\mu-\mu^0)^2\|u_t\|^2_{C^1([0,T];L_2(S))}\le$$
$$C(\Omega)\left[(\gamma_P-\gamma_P^0)^2+(\gamma_B-\gamma_B^0)^2+(\mu-\mu^0)^2+(K-K^0)^2\right]^2$$

due to Corollary 2. ■

THEOREM 1. Let $v\in H^3(0,T)$, $v(0)=v'(0)=0$, and $\xi(0)=\xi_t(0)=0$. Then the map $(\gamma_P,\gamma_B,\mu,K)\to(\xi,\xi_t):\mathcal{P}\to C([0,T];H_0^2(S))\times C([0,T];L_2(S))$ is continuously differentiable at each point of $\mathcal{P}$. Partial derivatives of this map satisfy equations (12) – (15). ■
The proof immediately follows from Propositions 5 and 6.

The next theorem state the existence of derivatives of the high order.

THEOREM 2. Let $v\in H^{k+2}(0,T)$, $v(0)=v^{(1)}(0)=...=v^{(k)}(0)=0$, and $\xi(0)=\xi_t(0)=0$. Then the map $(\gamma_P,\gamma_B,\mu,K)\to(\xi,\xi_t):\mathcal{P}\to C([0,T];H_0^2(S))\times C([0,T];L_2(S))$ is k times continuously differentiable at each point of $\mathcal{P}$. Partial derivatives of this map satisfy equations that are obtain by the formal differentiation of (5) w.r.t. the parameters. ■
This theorem can be proved using the induction technique. We give a sketch of the proof. Let

$$\begin{aligned}&w^0=\xi(\pi),\\&w^1=w^0-\xi(\pi_0),\\&w^2=w^1-\Phi^{(1)}(\pi_0)(\pi-\pi_0),\\&...\\&w^{k+1}=w^k-\tfrac{1}{k!}\Phi^{(k)}(\pi_0)(\pi-\pi_0),\end{aligned}$$

where $\Phi^{(k)}(\pi_0)(\pi-\pi_0)$ is the kth formal derivative of the map $\pi\to\xi(\pi)$ (k-linear form of $\pi-\pi_0$) which is obtained via the formal differentiation of (5) with respect to π. One can verify that w^{k+1} satisfies the equation:

$$\int\!\!\int_{S_P}\rho_P^0 w_{tt}^{k+1}\varphi+\gamma_P^0\Delta w^{k+1}\Delta\varphi+(\gamma_P-\gamma_P^0)\Delta w^k\Delta\varphi+$$

$$\iint\limits_{S_B} \rho_B^0 w_{tt}^{k+1}\varphi + \gamma_B^0 \Delta w^{k+1}\Delta\varphi + (\gamma_B - \gamma_B^0)\Delta w^k \Delta\varphi + \tag{19}$$

$$\iint\limits_{S} \mu^0 \nabla w_t^{k+1}\nabla\varphi + (\mu - \mu^0)\nabla w_t^k \nabla\varphi = 0,$$

if $k \geq 1$ (w_0 and w_1 satisfy equations identical with (5) and (9), respectively). If $v \in H^{k+2}(0,T)$, one can successive "differentiate" the equations (19) with respect to t and apply Proposition 2. Finally one obtains that

$$\|w^{k+1}\|^2_{C([0,T];H_0^2(S))} + \|w_t^{k+1}\|^2_{C([0,T];L_2(S))} \leq$$

$$C(\Omega)\left[(\gamma_P - \gamma_P^0)^2 + (\gamma_B - \gamma_B^0)^2 + (\mu - \mu^0)^2 + (K - K^0)^2\right]^{k+1},$$

which proves the theorem.

3. Numerical simulation

The exact values of parameters to be estimated were chosen as follows:

$$\gamma_p = 0.1, \quad \gamma_B = 0.01, \quad \mu = 0, \quad K = 5.$$

The other parameters are:

$$S = [0,1]\times[0,1], \quad S_P = [0.4,0.6]\times[0.2,0.5], \quad \rho_P = 1.5, \quad \rho_B = 1, \quad v(t) = 2\sin^2 5t.$$

Solution values at five points $(x_1^n, x_2^n) = (n\cdot 0.1, 0.1),\ n = 1,..,5$, are observed for the times $t_m = m\cdot 0.01,\ m = 1,..,10$. Thus, $y_n(t_m) = \xi^0(t_m, x_1^n, x_2^n),\ n = 1,..,5,\ m = 1,..,10$, and

$$J(\pi) = \sum_{m=1}^{10}\sum_{n=1}^{5}\Big(\xi(t_m, x_1^n, x_2^n) - y_n(t_m)\Big)^2.$$

The gradient descent method is applied to the minimization of J, where the gradients are computed using the variational equations (12)-(15): The variation of the functional is

$$J(\pi + \delta\pi) \approx \sum_{m=1}^{10}\sum_{n=1}^{5}\left(\xi(t_m, x_1^n, x_2^n) + \sum_{l=1}^{4}\delta^l(t_m, x_1^n, x_2^n)\delta\pi_l - y_n(t_m)\right)^2. \tag{20}$$

Here $\delta^l,\ l = 1,..,4$ are solutions of (12)-(15) with the parameter set π. The descent direction $\delta\pi$ is found as a minimizing point of the right-hand-side of (20) which is a quadratic form. Then we put $\pi := \pi + \lambda\delta\pi$, where λ is a step length, and repeat the procedure.

Figure 2 demonstrate the convergence of the method. The integers under horizontal axis indicate the number of steps of the gradient descent method.

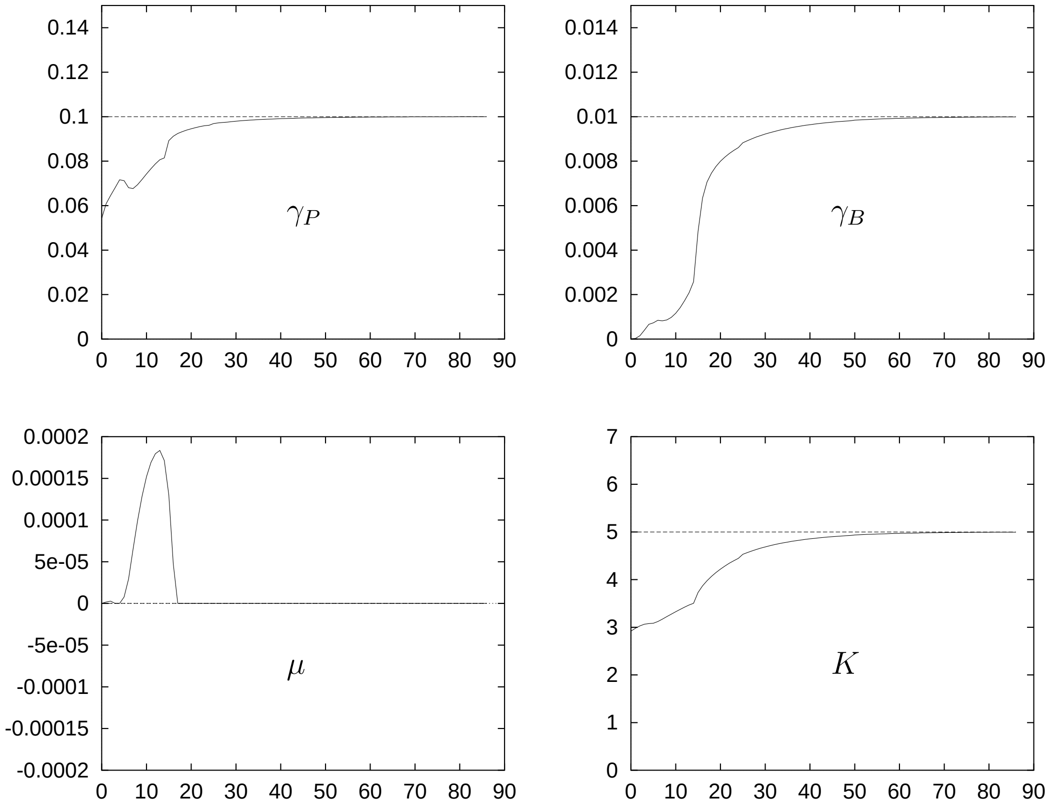

Figure 2.

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References

1. Banks, H.T., Wang, Y., Inman. D.J., and Slater, J.C.: Approximation and parameter identification for damped second order systems with unbounded input operators. In: Report of the Center for Research in Scientific Computation. North Carolina State University: CRSC-TR93-9, May, 1993.

2. Landau, L.D. and Lifschitz, E.M.: Elastizitätstheorie. Berlin: Akademie–Verlag 1975.

3. Maugin, G.A.: Continuum mechanics of electromagnetic solids. In: North-Holland series in: Applied Mathematics and mechanics. **33**, North-Holland 1987.

4. Tiersten, H.F.: On the nonlinear equations of thermoelectroelasticity. Int. J. Engng Sci., **9**, 587-604 (1971).

5. Zelenka, J.: Piezoelectric Resonators and their Applications. In: Studies in Electrical and Electronic Engineering, **24**, Elsevier 1986.

6. WLOKA, J: Partielle Differentialgleichungen. Stuttgart: B.G. Teubner 1982.

7. LIONS, J.L. and MAGENES, E.: Non-homogeneous Boundary Value Problems and applications, **I**, Berlin - Heidelberg-New York: Springer Verlag 1972.

8. SIMON, J. Compact sets in the space $L^p(0, T; b)$. Ann. Mat. Pura Appl. **IV**, 146, 65-96, (1987).

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