FAMILIES OF SEMIPERMEABLE CURVES IN DIFFERENTIAL GAMES WITH THE HOMICIDAL CHAUFFEUR DYNAMICS

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Abstract: Families of semipermeable curves in differential games with the homicidal chauffeur dynamics are studied both from the theoretical and computational points of view. The knowledge of such families is very useful for the investigation of time-optimal problems because semipermeable curves bound the solvability set and can also appear as barrier lines on which the value function is discontinuous. Three variants of differential games with the homicidal chauffeur dynamics are considered. Results of the computation of level sets of the value function for these games are discussed. Copyright © 2001 IFAC

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1. INTRODUCTION

Semipermeable curves were introduced in (Isaacs, 1965). For problems in the plane, a smooth semipermeable curve is a line with a preventing property: one of the players forbids crossing the curve from the positive side to the negative one, another player forbids crossing the curve from the negative side to the positive one. The families of semipermeable curves are determined from only the dynamics of the system (including constraints on the controls) and do not depend on the form of the terminal set and on the objectives of the players in the game. The knowledge of the structure of these families can be very useful when studying different properties of solutions of time-optimal games. In particular, lines which bound the solvability set are composed from arcs of smooth semipermeable curves.

This paper studies families of semipermeable curves arising in differential games with the homicidal chauffeur dynamics. At the end of the paper, examples of the computation of level sets of the value function for three variants of problems with the homicidal chauffeur dynamics are presented. Both semipermeable curves which are barrier lines and semipermeable curves which form the boundary of the solvability set are given.

2. GAMES WITH THE HOMICIDAL CHAUFFEUR DYNAMICS

The pursuer P has a fixed speed \( w_{1} \) but his radius of turn is bounded by a given quantity \( R \). The evader E is inertialess. He steers by choosing...
his velocity vector \( v = (v_1, v_2)' \) from some set. The kinematic equations are:

\[
P: \begin{align*}
    \dot{x}_p &= w^{(1)} \sin \psi \\
    y_p &= w^{(1)} \cos \psi \\
    \dot{\psi} &= w^{(1)} \varphi / R, \quad |\varphi| \leq 1
\end{align*}
\]

\[
E: \begin{align*}
    \dot{x}_e &= v_1 \\
    \dot{y}_e &= v_2.
\end{align*}
\]

The number of equations can be reduced to two (see Isaacs, 1965) if a coordinate system with the origin at \( P \) and the axis \( x_2 \) in the direction of \( P \)'s velocity vector is used. The axis \( x_1 \) is orthogonal to the axis \( x_2 \).

The dynamics in the reduced coordinates is

\[
\begin{align*}
    \dot{x}_1 &= -w^{(1)} x_2 \varphi / R + v_1 \\
    \dot{x}_2 &= w^{(1)} x_1 \varphi / R + v_2 - w^{(1)}, \quad |\varphi| \leq 1. \tag{1}
\end{align*}
\]

The state vector \( x = (x_1, x_2)' \) gives the relative position of \( E \) with respect to \( P \).

2.1 Classical homicidal chauffeur game

This game is described in (Isaacs, 1965). The control \( v \) is chosen from a circle of radius \( w^{(2)} > 0 \) with the center at the origin. The objective of the control \( \varphi \) of the pursuer is to minimize the time of attainment of a given terminal set \( M \) by the state vector of system (1). The objective of the control \( v \) of the evader is to maximize this time. The payoff of the game is the time of attaining the set \( M \).

2.2 Acoustic game

This variant is proposed by P. Bernhard and described in (Cardaliaguet et al., 1999). The evader must reduce his speed when he comes close to the pursuer in order not to be heard. So, the constraint on the control of player \( E \) depends on \( x \).

It is given by the formula

\[
Q(x) = k(x) Q, \quad k(x) = \min \{|x| \cdot s\} / s, \quad s > 0.
\]

Here \( s \) is a parameter. We have \( Q(x) = Q \) if \( |x| \geq s \). The objective of the pursuer is to minimize the time of attaining a terminal set \( M \). The objective of the evader is to maximize this time.

2.3 Conic surveillance-evasion game

The statement of the problem is given in (Lewin and Olsher, 1979). The control \( v \) satisfies the restriction \( |v| \leq w^{(2)} \). The terminal set \( M \) is the complement of the open detection cone about the axis \( x_2 \) with the apex at the origin (Figure 1). The objective of the evader is to minimize the time of attaining \( M \). The objective of the pursuer is to maximize this time. Therefore, in contrast to the Isaacs’ homicidal chauffeur game, the roles of the players change: the evader is the “minimizing” player and the pursuer is the “maximizing” one.

3. FAMILIES OF SEMIPERMEABLE CURVES

The families of smooth semipermeable curves are determined from only the dynamics of the system and the bounds on the controls of the players.

In the following, for the uniformity of notation of the constraint of player \( E \), let us agree that \( Q(x) = Q \) for problems 2.1 and 2.3.

![Fig. 1. Detection cone.](image-url)

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ℓ < ℓ∗ means that the direction of the vector ℓ can be obtained from the direction of the vector ℓ∗ using a counterclockwise rotation through an angle not exceeding π. The roots − to + and the roots + to − are called roots of the first and second type, respectively.

We denote roots of the first type by ℓ(1),r(x) and roots of the second type by ℓ(2),r(x). The right index takes the value 1 or 2, and indicates whether the vector belongs to the half-plane \{ℓ ∈ ℝ2 : ℓ'p(x) < 0\} or \{ℓ ∈ ℝ2 : ℓ'p(x) ≥ 0\}. Due to the above property of the piecewise convexity of the function \(H(\ell, x)\), the equation \(H(\ell, x) = 0\) can have at most two roots of each type for any given x.

We now describe how the families of smooth semipermeable curves can be constructed.

3.1 Constraint Q on the control of player E does not depend on x

Assume that the constraint Q does not depend on x that is Q(x) = Q. Denote

\[
A_\ast = \{x : x_1 = \frac{v_2 R}{w(1)} - R, x_2 = -\frac{v_1 R}{w(1)}, v \in Q\},
\]

(3)

\[
B_\ast = \{x : x_1 = -\frac{v_2 R}{w(1)} + R, x_2 = \frac{v_1 R}{w(1)}, v \in Q\}.
\]

(4)

The set \(B_\ast\) is symmetric to the set \(A_\ast\) with respect to the origin. Let \(C_\ast = A_\ast \cap B_\ast\).

3.1.1. Roots of equation \(H(\ell, x) = 0\). For all \(x \notin C_\ast\), the equation \(H(\ell, x) = 0\) has at least one root of the first type and one root of the second type.

Let \(x \in \text{int} C_\ast\). Roots of the first and second type do not exist for \(x \in \text{int} C_\ast\). Due to continuity of \(H\), strict roots do not exist for \(x \in \partial C_\ast\) too.

The proofs of these assertions are given in (Patsko and Turova, 2000).

3.1.2. Case \(C_\ast = \emptyset\). We consider cones spanned onto the sets \(A_\ast\) and \(B_\ast\) with the apex at the origin. Denote these cones by cone\(A_\ast\) and cone\(B_\ast\), respectively. The part of cone\(A_\ast\) after deleting the set

\[
\{x : x_1 = \frac{v_2 R}{w(1)}\phi - R, x_2 = -\frac{v_1 R}{w(1)}\phi, 1 < \phi < \infty, v \in Q\}
\]

is denoted by \(A\). Similarly, the set \(B\) as the part of cone\(B_\ast\) is introduced.

One can find the domains of the functions \(\ell^{(j),i}(\cdot)\), \(j = 1, 2, i = 1, 2\). Figure 2 presents the

sets \(A\) and \(B\) and the domains of the functions \(\ell^{(j),i}(\cdot)\) for the case where the set \(Q\) is a polygonal approximation of a circle of some radius \(w(2)\). The boundaries of \(A\) and \(B\) are drawn with the thick lines. There exist two roots of the first type and two roots of the second type at each internal point of the sets \(A\) and \(B\). For any point in the exterior of \(A\) and \(B\), there exist one root of the first type and one root of the second type.

The function \(\ell^{(j),i}(\cdot)\) is Lipschitz continuous on any closed bounded subset of the interior of its domain. Consider the two-dimensional differential equation

\[
dx/dt = \Pi \ell^{(j),i}(x),
\]

(5)

where \(\Pi\) is the matrix of rotation through the angle \(\pi/2\), the rotation being clockwise or counterclockwise if \(j = 1\) or \(j = 2\), respectively. Since the tangent vector at each point of the trajectory defined by this equation is a semipermeable direction, the trajectories are semipermeable curves. Therefore player P can keep the state vector \(x\) on one side of the curve, and player E can keep \(x\) on the other side. Equation (5) specifies a family \(\Lambda^{(j),i}\) of smooth semipermeable curves \(\mu^{(j),i}\). The family \(\Lambda^{(1),1}\) for the games from sections 2.1 and 2.3 in the case \(C_\ast = \emptyset\) is depicted in Figure 3. Each smooth semipermeable curve is a trajectory of system (1) for controls of the players that deliver minimum

\[
\text{Fig. 3. Family } \Lambda^{(1),1} \text{ of semipermeable curves. Set } Q \text{ does not depend on } x; C_\ast = \emptyset.
\]
and maximum in (2). The arrows show the direction of motion in reverse time. The pictures of three other families of smooth semipermeable curves can be obtained from Figure 3 by reflections in the $x_1$- and $x_2$-axes. Thereby, the reflection in the $x_1$-axis changes the direction of the motion.

3.1.3. Case $C_* \neq \emptyset$. There are no roots in the set $C_*$, there are four roots in the set $\mathbb{R}^2 \setminus (A_* \cup B_*)$, and there are two roots (one root of the first type and one root of the second type) in the rest part of the plane. Figure 4 shows the domains of the functions $\ell^{(j)}(\cdot)$ for this case. The set $Q$ is a circle of a radius $w^{(2)} > w^{(1)}$. The digits 4, 2 and 0 state the number of roots. Using (5), one can produce the families $\Lambda^{(j)}$ for this case.

The following important property holds true for any point $x \in C_*$ for any $\varphi \in [-1, 1]$ there exists $v \in Q$ such that $f(x, \varphi, v) = 0$. Therefore, in the region $C_*$, player $E$ can control any control of player $P$, so the state remains immovable all the time. We call regions of such points the superiority sets of player $E$.

3.2 Constraint $Q$ on the control of player $E$ depends on $x$

Using the form of the domains of $\ell^{(j)}(\cdot)$ from section 3.1, one can construct the domains for the case $Q(x) = k(x)Q$. Let us describe how it can be done.

First note that $k(x) = \text{const}$ for the points $x$ of any circumference of some fixed radius with the center at $(0, 0, 0)$. It holds $k(x) = 1$ outside the circle of radius $s$. Let $\Omega(r)$ be a circumference of radius $r$ with the center at $(0, 0, 0)$; $k(r) = \min \{r, s\}/s$ and $Q(r) = k(r)Q$. We have $Q(x) = Q(|x|)$.

Form the sets $A_*(r)$ and $B_*(r)$ substituting the set $Q(r)$ instead of $Q$ in formulae (3) and (4) for $A_*$ and $B_*$. Let $C_*(r) = A_*(r) \cap B_*(r)$. Using $A_*(r)$ and $B_*(r)$, construct domains of $\ell^{(j)}(\cdot)$, the cases $C_*(r) = \emptyset$ and $C_*(r) \neq \emptyset$ being distinguished. Put the circumference $\Omega(r)$ onto the constructed domains. As a result, a division of $\Omega(r)$ onto arcs is obtained. The number and the type of roots are the same for all points of each arc. This technique is applied for every $r$ in $[0, s]$, and identically named division points are connected. Thus the circle of radius $s$ is divided into parts according to the kinds of roots. Outside this circle, the dividing lines coincide with the lines constructed for the case when $Q$ does not depend on $x$.

Since $Q$ is a circle of radius $w^{(2)}$, then $Q(r)$ is a circle of radius $w^{(2)}(r) = \min \{r, s\}w^{(2)}/s$, and the condition $C_*(r) \neq \emptyset$ is equivalent to the relation $w^{(2)}(r) \geq w^{(1)}$. If $w^{(2)}(r) \leq w^{(1)}$, we put the points $x \in \Omega(r)$ onto the domains of Figure 5 constructed for $w^{(2)} = w^{(2)}(r)$. Otherwise, if $w^{(2)}(r) > w^{(1)}$, we put these points onto the domains of Figure 4.

Figures 5, 7 and 9 were constructed in this way for the parameters $w^{(1)} = 1$, $R = 0.8$, $s = 0.75$ and

Fig. 5. Domains of $\ell^{(j)}(\cdot)$. Set $Q$ depends on $x$; $w^{(2)} = 0.8$.

Fig. 6. Family $\Lambda^{(1,1)}$ of semipermeable curves. Set $Q$ depends on $x$; $w^{(2)} = 0.8$. 

![Diagram](image-url)
4. LEVEL SETS OF THE VALUE FUNCTION
AND SEMIPERMEABLE CURVES

The level set $W_c$ of the value function $V$ is the set of all points $x$ such that $V(x) \leq c$. The front is the collection of all points on the boundary of $W_c$.

where $V(x) = c$.

We give three examples of the backward computation of level sets of the value function for problems introduced in sections 2.1, 2.2 and 2.3.

In Figure 11, level sets for the classical homicidal chauffeur game are presented. The following values of parameters are used: $w^{(1)} = 2, w^{(2)} = 0.6, R = 0.2$. The set $M$ is a polygon inscribed into the circle of radius 0.015 with the center at $(0.2, 0.3)$. The step $\Delta$ of the backward procedure is 0.001. Every 8th front is depicted.

The first self-intersection of the front occurs at $\tau = 0.725$. Here $\tau$ is the reverse time of the backward procedure. For $\tau > 0.725$, only fronts that propagate into the “region of turn” are drawn. The greatest value of $\tau$ in the region of turn is 0.95. This corresponds to the time when the fronts complete filling the gap on the left hand side of the axis $x_2$.

The value function is discontinuous on two barrier lines which emanate from the boundary of the terminal set and end on the boundary of the
sets $A$ and $B$. The barrier lines coincide with the semipermeable curves $p^{(1),1}$ and $p^{(2),2}$.

Figure 12 is done for the acoustic game with $w^{(1)} = 1.7$, $R = 0.8$, $w^{(2)} = 1.9$ and $s = 0.75$. The set $M$ is the rectangle $\{x \in R^2 : -3.5 \leq x_1 \leq 3.5, -0.2 \leq x_2 \leq 0\}$. The step $\Delta$ is 0.01. The upper and lower fronts are computed until $\tau = 8.42$ and $\tau = 1.6$, respectively. The solution is symmetric with respect to the axis $x_2$. Every 5th front is drawn. The semipermeable curves bounded the solvability set are marked. It is interesting that the boundary of the solvability set in this example is a connected curve which can not be obtained using the boundary of the terminal set and semipermeable curves emitted from this boundary. One should additionally use semipermeable curves emitted from the boundary of the superiority set $C_U$ of player $E$.

Level sets of the value function of the surveillance-evasion game are presented in Figure 13. The parameter $w^{(1)} = 1.7$. The set $Q$ is a regular hexagon inscribed into the unit circle with the center at the origin. The detection cone is non-symmetric with respect to the axis $x_2$. This gives a non-symmetric set $M$. The step $\Delta$ is 0.01. The computation is done until $\tau = 5.8$. Every 10th front is drawn. The solvability set of the problem (escape zone of player $E$) is infinite. The boundary is formed by the semipermeable curves $p^{(2),1}$ and $p^{(2),2}$. The value function is discontinuous on the barrier line which is a semipermeable curve $p^{(1),2}$.

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