

Families of semipermeable curves in differential games with the homicidal chauffeur dynamics[☆]

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Abstract

Families of semipermeable curves in differential games with the homicidal chauffeur dynamics are studied both from theoretical and computational points of view. The knowledge of such families is very useful because semipermeable curves bound capture sets in games of kind. They can also appear as barrier lines on which value functions of time-optimal differential games are discontinuous. Two examples of differential games with the homicidal chauffeur dynamics are considered. Capture sets are constructed using semipermeable curves. The results are verified through computation of level sets of the value functions in the related time-optimal games.

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1. Introduction

Semipermeable curves were introduced in Isaacs (1965). For problems in the plane, a smooth semipermeable curve is a line with the following preventing property: one of the players forbids crossing the curve from the positive side to the negative one, the other player forbids crossing the curve from the negative side to the positive one. The families of semipermeable curves are determined from the dynamics of the system only (including constraints on the controls) and do not depend on the objectives of the players in the game. The knowledge of the structure of these families can be useful for the study of differential games

(Isaacs, 1965; Krasovskii & Subbotin, 1988; Bardi & Capuzzo-Dolcetta, 1997) with some performance indices and for the computation of viability sets (Aubin, 1991). For example, in time-optimal differential games, barrier lines on which the value function is discontinuous are semipermeable curves. In games of kind (Isaacs, 1965), the boundary of the capture set consisting of all points with finite capture time is formed either entirely or partially by semipermeable curves. For many problems, it is very difficult to construct capture sets precisely without any preliminary analysis of families of semipermeable curves. In particular, this is the case for problems where the boundary of the capture set cannot be obtained by the Isaacs approach that uses solely semipermeable curves emanating from endpoints of the usable part of the terminal set.

This paper studies families of semipermeable curves arising in differential games with the “homicidal chauffeur” dynamics. Such a sample variant of the dynamics introduced by Isaacs appears in some important practical problems where the first of two controlled objects (player P) has a bounded radius of turn whereas the second object (player E) is inertialess. Bernhard proposed a more complicated variant of

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this dynamics: the constraint on the velocity of player E depends on the distance between E and P .

When dealing with semipermeable curves in the plane, we distinct smooth semipermeable curves of the first and second type. This classification is based on the roots “–” to “+” and “+” to “–” of the Hamiltonian. We show that there are two families of smooth semipermeable curves of the first type and two families of the second type for problems with the homicidal chauffeur dynamics. The sets where these families are defined overlap. Although differential games with the homicidal chauffeur dynamics were studied in many works (see, for example, Isaacs, 1965; Merz, 1971; Lewin & Breakwell, 1975; Lewin & Olsder, 1979; Cardaliaguet, Quincampoix, & Saint–Pierre, 1999), the families of smooth semipermeable curves are not described precisely for such dynamics.

The possibility to distinct smooth semipermeable curves of the first and second type is a specific feature of differential games in the plane. The advantage of such a distinction was demonstrated by the authors in Patsko (1975) and Patsko and Turova (1995) both for the construction of nonsmooth composite semipermeable curves and for finding capture sets of linear and some nonlinear differential games of kind in the plane.

We conclude this paper with two examples illustrating the application of the families of semipermeable curves. The statement of the first example (acoustic game) is taken from the paper Cardaliaguet et al. (1999), the formulation of the second one (conic surveillance-evasion game) is borrowed from Lewin and Olsder (1979). These examples demonstrate the construction of the capture set using the families of semipermeable curves. The structure of the capture set is verified through the computation of level sets of the value function in the related time-optimal game.

2. Homicidal chauffeur dynamics

Player P has a fixed velocity magnitude $w^{(1)}$, his radius of turn is bounded by a given value R . Player E is inertialess. The kinematic equations are:

$$\begin{aligned}
 P : \quad \dot{x}_P &= w^{(1)} \sin \psi, & E : \quad \dot{x}_E &= v_1, \\
 \dot{y}_P &= w^{(1)} \cos \psi, & \dot{y}_E &= v_2, \\
 \dot{\psi} &= w^{(1)} \varphi / R.
 \end{aligned}$$

Here, φ is the control of player P ; $v = (v_1, v_2)'$ is the control of player E .

The number of equations can be reduced to two (see Isaacs, 1965) if a coordinate system with the origin at P and the axis x_2 directed along the velocity vector of P is used. The axis x_1 is orthogonal to the axis x_2 .

In the reduced coordinates, the dynamics read:

$$\begin{aligned}
 \dot{x}_1 &= -w^{(1)}x_2\varphi/R + v_1, \\
 \dot{x}_2 &= w^{(1)}x_1\varphi/R + v_2 - w^{(1)}.
 \end{aligned} \tag{1}$$

The state vector $x = (x_1, x_2)'$ gives the relative position of E with respect to P .

We suppose that the control φ belongs to the segment $\mathcal{P} = \{\varphi : |\varphi| \leq 1\}$. Two variants of the constraint of player E will be considered. The first variant (classical homicidal chauffeur dynamics) assumes that $v \in Q$, where Q is a circle of radius $w^{(2)}$ centered at the origin. In the second variant (acoustic game), the constraint depends on x ; the dependence is given by the formula

$$\mathcal{Q}(x) = k(x)Q, \quad k(x) = \min\{|x|, s\}/s, \quad s > 0.$$

Here, s is a parameter. We have $\mathcal{Q}(x) = Q$ if $|x| \geq s$. This variant is proposed by Bernhard and Larroturou (1989) and described in Cardaliaguet et al. (1999). One can interpret it as follows: player E must reduce his velocity whenever he comes close to player P in order not to be heard.

In the following, for uniformity of notation, let us agree that $\mathcal{Q}(x) = Q$ in the case of the classical dynamics.

3. Two types of semipermeable curves

3.1. Roots of the Hamiltonian

Consider the Hamiltonian

$$H(\ell, x) = \min_{\varphi \in \mathcal{P}} \max_{v \in \mathcal{Q}(x)} \ell' f(x, \varphi, v), \quad x, \ell \in R^2. \tag{2}$$

Here $f(x, \varphi, v) = p(x)\varphi + v + g$, $g = (0, -w^{(1)})'$ and $p(x) = (-x_2, x_1)' \cdot w^{(1)}/R$.

We study nonzero roots of the equation $H(\ell, x) = 0$, where $x \in R^2$ is fixed. Since the function $\ell \rightarrow H(\ell, x)$ is positively homogeneous, it is convenient to assume that $\ell \in \mathcal{E}$, where \mathcal{E} is the circumference of unit radius centered at the origin.

Let $\ell, \ell_* \in \mathcal{E}$, $\ell \neq \ell_*$. The notation $\ell < \ell_*$ ($\ell > \ell_*$) means that the vector ℓ can be obtained from the vector ℓ_* using a counterclockwise (clockwise) rotation through an angle smaller than π . In fact, this order relation will be used only for vectors that are sufficiently close to each other.

Fix $x \in R^2$ and consider roots of the equation $H(\ell, x) = 0$, $\ell \in \mathcal{E}$. A vector ℓ_* is called the strict root “–” to “+” if there exist a vector $\kappa \in R^2$ and a neighborhood $S \subset \mathcal{E}$ of the vector ℓ_* such that $H(\ell_*, x) = \ell_*' \kappa = 0$ and $H(\ell, x) \leq \ell' \kappa < 0$ ($H(\ell, x) \geq \ell' \kappa > 0$) for vectors $\ell \in S$ satisfying the relation $\ell < \ell_*$ ($\ell > \ell_*$). Similarly, the strict root “+” to “–” is defined through replacing $\ell < \ell_*$ ($\ell > \ell_*$) by $\ell > \ell_*$ ($\ell < \ell_*$). The roots “–” to “+” and “+” to “–” are called roots of the first and second type, respectively. In the following, when utilizing the notation $\ell < \ell_*$ ($\ell > \ell_*$) we will keep in mind that ℓ is from a neighborhood S like that mentioned in the definition of the roots.

Denote by $\mathcal{P}(\ell, x)$ the collection of all $\varphi \in \mathcal{P}$ that provide the minimum in (2). Since \mathcal{P} is a segment, $\mathcal{P}(\ell, x)$ is either one of the endpoints of \mathcal{P} or $\mathcal{P}(\ell, x) = \mathcal{P}$. If ℓ_* is a strict root of the first (second) type, take $\varphi^{(1)}(\ell_*, x)$ ($\varphi^{(2)}(\ell_*, x)$) equal to $\operatorname{argmin}\{\ell' p(x)\varphi : \varphi \in \mathcal{P}(\ell_*, x)\}$, where $\ell < \ell_*$

($l > l_*$). Note that the result does not depend on the choice of l . Let

$$v^{(1)}(l_*, x) = v^{(2)}(l_*, x) = \operatorname{argmax}\{l'_* v : v \in \mathcal{Q}(x)\}.$$

Since $\mathcal{Q}(x)$ is a circle, $v^{(i)}(l_*, x)$, $i = 1, 2$, is a singleton.

If l_* is a root of the first type, consider the vectograms $f(x, \varphi^{(1)}(l_*, x), \mathcal{Q}(x))$ and $f(x, \mathcal{P}, v^{(1)}(l_*, x))$. We have

$$\begin{aligned} l' f(x, \varphi^{(1)}(l_*, x), v) &\leq \max_{v \in \mathcal{Q}(x)} l' f(x, \varphi^{(1)}(l_*, x), v) \\ &= H(l, x) \leq l' \kappa < 0 \end{aligned} \quad (3)$$

for $v \in \mathcal{Q}(x)$ and $l < l_*$. For $l > l_*$, it holds

$$\max_{v \in \mathcal{Q}(x)} l' v \leq l' v^{(1)}(l_*, x) + l' \kappa / 2$$

because κ is orthogonal to l_* (and hence to $v^{(1)}(l_*, x)$), and $l' \kappa > 0$. Thus,

$$H(l, x) \leq \min_{\varphi \in \mathcal{P}} l' f(x, \varphi, v^{(1)}(l_*, x)) + l' \kappa / 2.$$

The last inequality yields

$$\begin{aligned} l' f(x, \varphi, v^{(1)}(l_*, x)) &\geq \min_{\varphi \in \mathcal{P}} l' f(x, \varphi, v^{(1)}(l_*, x)) \\ &\geq H(l, x) - l' \kappa / 2 \geq l' \kappa / 2 > 0 \end{aligned} \quad (4)$$

for $\varphi \in \mathcal{P}$ and $l > l_*$.

Relations (3) and (4) ensure that the vectograms $f(x, \varphi^{(1)}(l_*, x), \mathcal{Q}(x))$ and $f(x, \mathcal{P}, v^{(1)}(l_*, x))$ do not contain zero and are located with respect to the direction of the vector $f^{(1)} = f(x, \varphi^{(1)}(l_*, x), v^{(1)}(l_*, x))$ as it is shown in Fig. 1(a).

Therefore, the existence of a strict root l_* of the first type at a point x ensures together with taking the control $\varphi^{(1)}(l_*, x)$ ($v^{(1)}(l_*, x)$) by player P (E) that the velocity vector $f(x, \varphi^{(1)}(l_*, x), v)$ ($f(x, \varphi, v^{(1)}(l_*, x))$) is directed to the right (to the left) with respect to the direction of the vector $f^{(1)}$ for any control $v(\varphi)$ of player E (P). Such a disposition of the vectograms means that player P (E) guarantees the trajectories do not go to the left (to the right) with respect to the direction of $f^{(1)}$. The direction of the vector $f^{(1)}$ is called the semipermeable direction of the first type. The vector $f^{(1)}$ is orthogonal to the vector l_* , its direction can be obtained from l_* by a clockwise rotation through the angle $\pi/2$.

Arguing similarly, we obtain that the existence of a strict root l_* of the second type at a point x ensures together with taking the control $\varphi^{(2)}(l_*, x)$ ($v^{(2)}(l_*, x)$) by player P (E) that the velocity vector $f(x, \varphi^{(2)}(l_*, x), v)$ ($f(x, \varphi, v^{(2)}(l_*, x))$) is directed to the left (to the right) with respect to the direction of the vector $f^{(2)} = f(x, \varphi^{(2)}(l_*, x), v^{(2)}(l_*, x))$ for any control of player E (P). This means that player P (E) guarantees the trajectories do not go to the right (to the left) with respect to the direction of $f^{(2)}$. The direction of the vector $f^{(2)}$ is called the semipermeable direction of the second type. The vector $f^{(2)}$ is orthogonal to the vector l_* , its direction can be obtained from l_*

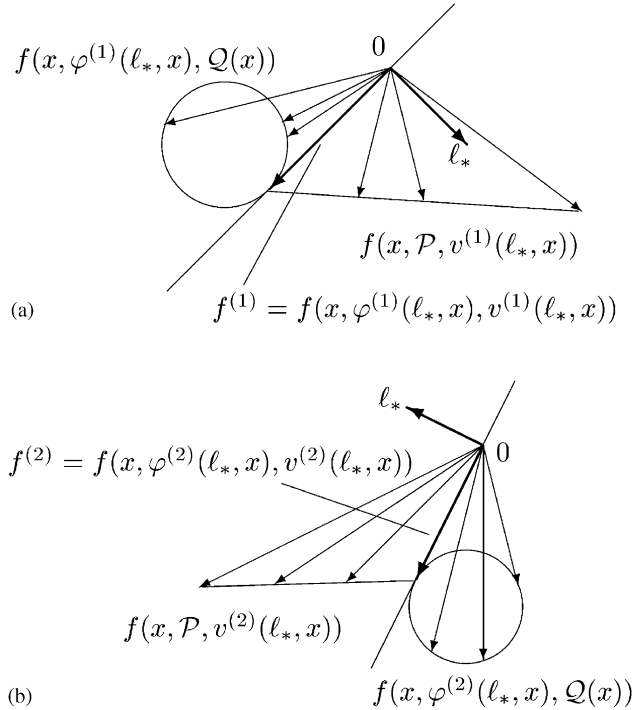


Fig. 1. (a) Semipermeable direction of the first type. (b) Semipermeable direction of the second type.

by a counterclockwise rotation through the angle $\pi/2$ (see Fig. 1(b)).

Thus, there is a significant difference in the location of the vectograms for strict roots of the first and second type. The type of roots will be utilized in the procedure for computing semipermeable curves.

3.2. Smooth semipermeable curves

A smooth curve is called semipermeable of the first (second) type if the direction of the tangent vector at any point along this curve is the semipermeable direction of the first (second) type. The side of a semipermeable curve that player P (E) can keep is called positive (negative). The positive (negative) side of a semipermeable curve of the first type is on the right (on the left) when looking along the semipermeable direction. The opposite is valid for semipermeable curves of the second type.

Fig. 2 illustrates the role of semipermeable curves of the first and second type in solving a game of kind with the classical homicidal chauffeur dynamics, the restriction Q is of a rather large radius. The objective of player P is to bring trajectories to the terminal set M which is a circle centered at the origin, the objective of player E is opposite. Denote by a and b the endpoints of the usable part (Isaacs, 1965) on the boundary of M . The semipermeable curves of the first and second type that are tangent to the set M and pass through the points a and b , respectively, define completely the capture set \mathcal{B} of all points for which guaranteed time

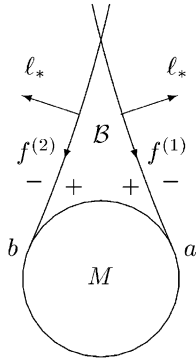


Fig. 2. Semipermeable curves of the first and second type and capture set.

of attaining M is finite. The curves are faced towards each other with the positive sides.

The following property is important for the construction of smooth semipermeable curves of the first and second type. Let a strict root $\tilde{\ell} \in \mathcal{E}$ of the first (second) type exists at a point \tilde{x} . Then there exists a neighborhood $O(\tilde{x})$ and a unique Lipschitz function $x \rightarrow \ell(x)$ defined on $O(\tilde{x})$ with values in \mathcal{E} such that $\ell(\tilde{x}) = \tilde{\ell}$ and $\ell(x)$ is the strict root of the first (second) type for any $x \in O(\tilde{x})$. This property can be derived from a nonsmooth version of the implicit function theorem (Dem'yanov, 1995) by applying it to the relations $H(\ell, x) = 0$, $\ell \in \mathcal{E}$ with the Lipschitz function H .

4. Families of semipermeable curves

The function $\ell \rightarrow H(\ell, x)$, $\ell \in R^2$, is composed of two convex functions:

$$H(\ell, x) = \begin{cases} \max_{v \in \mathcal{Q}(x)} \ell'v + \ell'p(x) + \ell'g, & \ell'p(x) < 0, \\ \max_{v \in \mathcal{Q}(x)} \ell'v - \ell'p(x) + \ell'g, & \ell'p(x) \geq 0. \end{cases}$$

Due to piecewise convexity of H , the equation $H(\ell, x) = 0$, $\ell \in \mathcal{E}$, can have at most two roots of each type for any given x . We denote the roots of the first type by $\ell^{(1),i}(x)$ and the roots of the second type by $\ell^{(2),i}(x)$. The index i assumes the value 1 or 2 and indicates whether the vector fulfils the inequality $\ell'p(x) < 0$ or $\ell'p(x) \geq 0$. Equivalently, the index i determines whether $\ell'p(x)\varphi$ attains the minimum over $\varphi \in \mathcal{P}$ at $\varphi = 1$ or at $\varphi = -1$.

Consider the two-dimensional differential equation

$$dx/dt = \Pi \ell^{(j),i}(x), \tag{5}$$

where Π is the matrix of rotation through the angle $\pi/2$, the rotation being clockwise or counterclockwise if $j=1$ or $j=2$, respectively. The function $\ell^{(j),i}(\cdot)$ is locally Lipschitz in the interior of its domain due to the local Lipschitz property mentioned in Section 3.2. Hence, solutions to (5) are unique and can be extended up to the boundary of the domain of the function $\ell^{(j),i}(\cdot)$. Since the tangent vectors at points of the trajectories defined by this equation are semipermeable directions, the trajectories are semipermeable curves.

The computation of semipermeable curves using Eq. (5) is being done as follows. Let $j = 1$, $i = 1$ for definiteness.

Step 1: Fix an integration stepwidth δ and a positive integer n_{\max} that determines the maximal number of integration steps. Set $n := 0$, $m := 0$.

Step 2: Check whether a strict root “-” to “+” of the function $\ell \rightarrow H(\ell, x_n)$ exists in the set $\{\ell \in \mathcal{E} : \ell'p(x_n) < 0\}$ or not. If it exists, go to step 3. Note that the function $\ell \rightarrow H(\ell, x_n)$ is convex on the set $\{\ell \in R^2 : \ell'p(x_n) < 0\}$, therefore, the root is unique if exists. If it does not exist, go to step 6.

Step 3: Compute $\ell^{(1),1}(x_n)$ and $\Pi \ell^{(1),1}(x_n)$.

Step 4: Compute $x_{n+1} = x_n + \Pi \ell^{(1),1}(x_n)\delta$.

Step 5: Set $n := n + 1$. If $n > n_{\max}$ then go to the next step else go to step 2.

Step 6: If $m = 0$, then set $\delta := -\delta$, $m := m + 1$, $n := 0$ and go to step 2. If $m \neq 0$, stop.

When computing the root in step 3, the set \mathcal{Q} is approximated by a polygon, which makes the function $H(\ell, x_n)$ piecewise linear. The root can be computed with any desired accuracy if the number of the polygon vertices is sufficiently large.

Using this algorithm, one obtains a semipermeable curve $p^{(1),1}$ passing through the point x_0 . If the root $\ell^{(1),1}(x_0)$ does not exist, then no semipermeable curve $p^{(1),1}$ goes through the point x_0 . The construction of the curves $p^{(1),2}$, $p^{(2),1}$, and $p^{(2),2}$ passing through the point x_0 can be done in a similar way.

The presence of two roots at some point means that two semipermeable curves (one of the first and one of the second type) pass through this point. Respectively, the presence of four roots at some point means that four semipermeable curves (two of the first and two of the second type) pass through this point.

Below, the families of smooth semipermeable curves will be described. To find regions where the families are defined, some geometrical considerations will be used.

4.1. Constraint Q does not depend on x

The peculiarity of dynamics (1) is that the phase trajectories are circular for any fixed controls $\varphi \neq 0$ and v . The circles are centered at the point

$$X(\varphi, v) = \left(-\frac{v_2 R}{w^{(1)}\varphi} + \frac{R}{\varphi}, \frac{v_1 R}{w^{(1)}\varphi} \right)'$$

at which the right-hand side of (1) becomes zero. If $\varphi > 0$, then a counterclockwise rotation occurs; if $\varphi < 0$, a clockwise rotation takes place.

For any $\varphi \neq 0$, the mapping $v \rightarrow X(\varphi, v)$ defined on \mathcal{Q} is one-to-one. The set $X(\varphi, \mathcal{Q})$ is a circle of radius $w^{(2)}R/(w^{(1)}|\varphi|)$ with the center at $x_1 = R/\varphi$, $x_2 = 0$. The mapping $\varphi \rightarrow X(\varphi, v)$ defined on $\mathcal{P} \setminus \{0\}$ is also one-to-one for any $v \neq (0, w^{(1)})'$. The set $X(\mathcal{P} \setminus \{0\}, v)$ is the collection of rotation centers for every given v . It is composed of

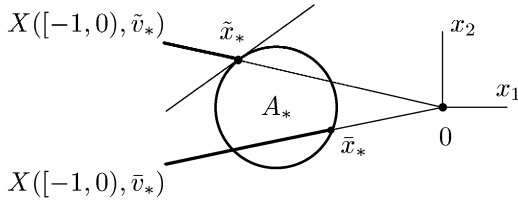


Fig. 3. Explanation to the separation property.

two rays $X([-1, 0], v)$ and $X((0, 1], v)$ which are located on the line $x_2(w^{(1)} - v_2) = v_1x_1$.

Consider the sets

$$A_* = X(-1, Q), \quad B_* = X(1, Q). \quad (6)$$

Let $C_* = A_* \cap B_*$.

The sets A_* and B_* are collections of the rotation centers for $\varphi = -1$ and 1 , respectively.

Let us explain the set C_* . It follows from the definition of the set A_* (B_*) that for any $x \in A_*$ ($x \in B_*$), there exists a unique $v \in Q$ such that $f(x, -1, v) = 0$ ($f(x, 1, v) = 0$). Therefore, for any $x \in C_* = A_* \cap B_*$ and any $\varphi \in \mathcal{P}$, there exists $v \in Q$ such that $f(x, \varphi, v) = 0$. The ensuring value is $v = ((1 - \varphi)\bar{v} + (1 + \varphi)\tilde{v})/2$, where \bar{v} (\tilde{v}) is the control of player E that provides zero velocity for $\varphi = -1$ ($\varphi = 1$). Thus, in the region C_* , player E can counter any control of player P , so the state remains immovable all the time. Regions of such points are called the superiority sets of player E .

4.1.1. Case $C_* = \emptyset$

We have the following simple property: the direction of the velocity vector $f(x, \varphi, v)$ of system (1) can be obtained by a rotation through the angle $\pi/2$ of the vector directed from x to the rotation center $X(\varphi, v)$; the rotation being counterclockwise or clockwise, if $\varphi = -1$ or 1 , respectively. This enables to replace the analysis of the separation of the vectograms by an analysis of the separation of the convex sets A_* and $X([-1, 0], v)$ with the line passing through the points x and $X(-1, v)$ and separation of the sets B_* and $X([1, 0], v)$ with the line passing through the points x and $X(1, v)$. We will utilize that when constructing domains of the functions $\ell^{(j),i}(\cdot)$.

Find those points x_* on the boundary of A_* for which a support line to A_* separates the set A_* and the half-line $X([-1, 0], v_*)$ with v_* such that $X(-1, v_*) = x_*$. In Fig. 3, two half-lines $X([-1, 0], \tilde{v}_*)$ and $X([-1, 0], \bar{v}_*)$ are shown for two points \tilde{x}_* and \bar{x}_* . The property of separation is valid for \tilde{x}_* and is not valid for \bar{x}_* . The separation property is fulfilled for the points of the closed arc $a_1a_2a_3$ (Fig. 4) and is not fulfilled for the points of the open arc $a_1a_4a_3$. Closed arcs include their endpoints, open arcs do not include them. The points a_1 and a_3 are located on the lines passing tangent to the set A_* through the origin.

Similarly, one can find a closed arc $b_1b_2b_3$ (Fig. 4) on the boundary of B_* consisting of points x_* for which a support line to the set B_* separates B_* and the half-line $X((0, 1], v_*)$

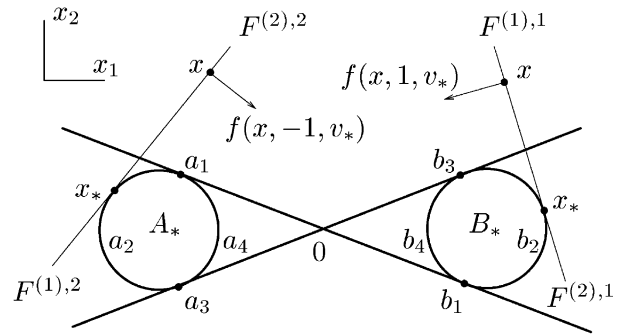


Fig. 4. Explanation to the construction of the domains of roots $\ell^{(j),i}(x)$.

with v_* such that $X(1, v_*) = x_*$. The arc $b_1b_2b_3$ is symmetric to the arc $a_1a_2a_3$ with respect to the origin.

Take a point x_* from the open arc $a_1a_2a_3$ ($b_1b_2b_3$). Passing through x_* , the support line to the set A_* (B_*) is divided into two half-lines by this point. Denote the half-lines by $F^{(2),2}(x_*)$ and $F^{(1),2}(x_*)$ ($F^{(1),1}(x_*)$ and $F^{(2),1}(x_*)$). In Fig. 4, a reduced notation for the half-lines is used.

Consider a point $x \in F^{(2),2}(x_*)$, $x \neq x_*$. The vector $f(x, -1, v_*)$ is the velocity of system (1) for $\varphi = -1$ and $v = v_*$. Therefore, it is the vector of a clockwise rotation around the point x_* . When looking at x_* from the point x , the set A_* remains to the left, and the half-line $X([-1, 0], v_*)$ remains to the right. The half-line $X((0, 1], v_*)$ is on the left but the direction of the rotation for $\varphi \in (0, 1]$ is opposite to the direction of the rotation for $\varphi \in [-1, 0]$. Such a location of the rotation centers yields the separation of the vectograms like that in Fig. 1(b) (with $f^{(2)} = f(x, -1, v_*)$), which means that $f(x, -1, v_*)$ is the semipermeable direction of the second type. In addition, the root $\ell^{(2),2}(x)$ of “+” to “-” is directed along the half-line $F^{(2),2}(x_*)$ towards the point x_* .

With similar arguments, one can obtain that the vector $f(x, 1, v_*)$ is the semipermeable direction of the first type for any point $x \in F^{(1),1}(x_*)$, $x \neq x_*$. The root $\ell^{(1),1}(x)$ of “-” to “+” is directed along the half-line $F^{(1),1}(x_*)$ towards the point x_* .

For any point $x \neq x_*$ on $F^{(1),2}(x_*)$ ($F^{(2),1}(x_*)$), the direction of the vector $f(x, \varphi, v)$ for $\varphi = -1$ ($\varphi = 1$) and $v = v_*$ is a semipermeable direction of the first (second) type. The root $\ell^{(1),2}(x)$ ($\ell^{(2),1}(x)$) of “-” to “+” (“+” to “-”) is directed along the half-line $F^{(1),2}(x_*)$ ($F^{(2),1}(x_*)$) backward from the point x_* .

The interior of the domain of the root $\ell^{(2),2}$ ($\ell^{(1),2}$) is the set covered by the half-line $F^{(2),2}(x_*)$ ($F^{(1),2}(x_*)$) when the point x_* runs over the open arc $a_1a_2a_3$. Similarly, considering the half-lines $F^{(1),1}(x_*)$ ($F^{(2),1}(x_*)$) for $x_* \in b_1b_2b_3$, the domain of $\ell^{(1),1}$ ($\ell^{(2),1}$) is obtained.

Let D be the preimage of the closed arc $a_1a_2a_3$ under the mapping $v \rightarrow X(-1, v)$, $v \in Q$. The set D is an arc on the boundary of the set Q . Furthermore, due to the symmetry of the arcs $a_1a_2a_3$ and $b_1b_2b_3$ with respect to the origin, D

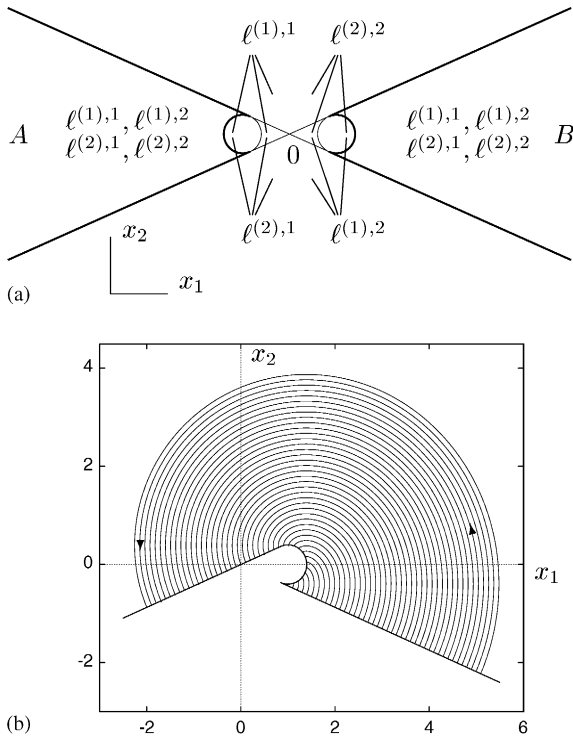


Fig. 5. Set \mathcal{Q} does not depend on x ; $C_* = \emptyset$. (a) Domains of $\ell^{(j),i}(\cdot)$. (b) Family $A^{(1),1}$ of semipermeable curves.

is the preimage of the closed arc $b_1b_2b_3$ under the mapping $v \rightarrow X(1, v)$, $v \in Q$. Let

$$A = X([-1, 0), D), \quad B = X((0, 1], D). \tag{7}$$

Fig. 5(a) presents the sets A , B , and the domains of the functions $\ell^{(j),i}(\cdot)$, $j = 1, 2$, $i = 1, 2$. The boundaries of A and B are drawn with thick lines. There exist two roots of the first type and two roots of the second type at each internal point of the sets A and B . For any point in the exterior of A and B , there exist one root of the first type and one root of the second type.

The family $A^{(1),1}$ of smooth semipermeable curves computed using Eq. (5) in the domain of the root $\ell^{(1),1}$ for $w^{(1)}=2$, $w^{(2)}=0.8$ is depicted in Fig. 5(b). The arrows show the motion in direct time. The initial points for the curves are located on the straight line going through the fourth quadrant and bounding the domain of $\ell^{(1),1}$ from below.

The picture of the family $A^{(2),2}$ can be obtained from Fig. 5(b) by the reflection in the x_2 -axis. The family $A^{(2),1}$ ($A^{(1),2}$) is obtained from the family $A^{(1),1}$ ($A^{(2),2}$) by the reflection in the x_1 -axis and reversion of the direction of motion.

If the radius $w^{(2)}$ of the circle Q decreases, the cones, which have the apex in the origin and contain the sets A and B , become smaller. If $w^{(2)} = 0$, the sets A and B degenerate into half-lines. In this case, the points where more than one strict root of the first (second) type exist, disappear.

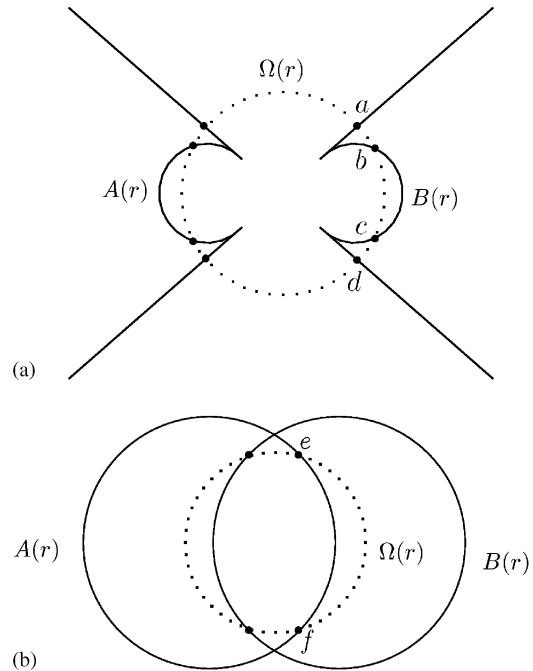


Fig. 6. Explanation to the construction of domains of functions $\ell^{(j),i}(\cdot)$ when \mathcal{Q} depends on x : (a) $w^{(2)}(r) \leq w^{(1)}$; (b) $w^{(2)}(r) > w^{(1)}$.

4.1.2. Case $C_* \neq \emptyset$

As an immediate consequence of the superiority property, we have $H(\ell, x) \geq 0$ for any $\ell \in \mathcal{E}$ and $x \in C_*$. Hence, the function $\ell \rightarrow H(\ell, x)$ does not possess any roots “-” to “+” and “+” to “-” for $x \in C_*$. There exists a root $\ell^{(2),2}(x)$ ($\ell^{(1),1}(x)$) and a root $\ell^{(1),2}(x)$ ($\ell^{(2),1}(x)$) for any $x \notin A_*$ ($x \notin B_*$). Therefore, there exist two roots of the first type and two roots of the second type for any $x \in R^2 \setminus (A_* \cup B_*)$. In the case $C_* \neq \emptyset$, the arc D is the whole boundary of the circle Q . The sets A and B defined by formula (7) are the closures of the sets $R^2 \setminus A_*$ and $R^2 \setminus B_*$, respectively.

4.2. Constraint \mathcal{Q} depends on x

Let us describe how one can construct the domains of $\ell^{(j),i}(\cdot)$ for the case $\mathcal{Q}(x) = k(x)Q$ using the form of the domains from Section 4.1. The idea of this construction is explained in Fig. 6.

First note that $k(x) = \text{const}$ for the points x of any circumference $\Omega(r)$ of radius $r \leq s$ with the center at $(0, 0)$. We have $k(x) = 1$ outside the circle of radius s . Let $h(r) = \min\{r, s\}/s$, $s > 0$, and $G(r) = h(r)Q$. It holds $\mathcal{Q}(x) = G(|x|)$.

Form sets $A_*(r)$ and $B_*(r)$ substituting the set $G(r)$ instead of Q in formula (6). Let $C_*(r) = A_*(r) \cap B_*(r)$. Using $A_*(r)$ and $B_*(r)$, construct the sets $A(r)$ and $B(r)$, then the domains of $\ell^{(j),i}(\cdot)$ as it is described in 4.1.1 and 4.1.2. Put the circumference $\Omega(r)$ onto the constructed domains. As a result, a division of $\Omega(r)$ into arcs is obtained. The division points separate the arcs whose points have the same number and the same type of roots. This technique is applied for

every $r \in [0, s]$, identical division points are connected. Thus, the circle of radius s is divided into parts according to the kind of roots. Outside this circle, the dividing lines coincide with the lines related to the case when \mathcal{Q} does not depend on x .

Since \mathcal{Q} is a circle of radius $w^{(2)}$, the set $G(r)$ is a circle of radius $w^{(2)}(r) = h(r)w^{(2)}$. The condition $C_*(r) = \emptyset$ means $w^{(2)}(r) < w^{(1)}$, whereas the condition $C_*(r) \neq \emptyset$ is equivalent to the relation $w^{(2)}(r) \geq w^{(1)}$. If $w^{(2)}(r) < w^{(1)}$, we put the circumference $\Omega(r)$ onto the domains of the functions $\ell^{(j),i}(\cdot)$ (see Fig. 5(a)) constructed for $w^{(2)} = w^{(2)}(r)$. In Fig. 6(a), the division points a, b, c, d , and the symmetric ones located in the left half-plane, are shown. The set $\Omega(r)$ is drawn with the dotted line. Otherwise, if $w^{(2)}(r) \geq w^{(1)}$, we put the circumference $\Omega(r)$ onto the domains of the functions $\ell^{(j),i}(\cdot)$ from Section 4.1.2. In Fig. 6(b), the division points e, f and those symmetric to them are depicted.

Using this technique, domains of $\ell^{(j),i}(\cdot)$ and families of semipermeable curves can be computed for any set of parameters of the problem. Fig. 7(a) (Fig. 8(a)) is drawn in this way for $w^{(1)} = 1$, $R = 0.8$, $s = 0.75$, and $w^{(2)} = 0.8$ ($w^{(2)} = 1.9$). In Fig. 7(a), the domains of the functions $\ell^{(j),i}(\cdot)$ are shown; the sets that are analogous to A and B in Fig. 5(a) are marked. In Fig. 8(a), two symmetric superiority sets of player E arise, the upper set being denoted by C_U , the lower set by C_L . The digits 0, 2, 4 mean the number of roots of the equation $H(\ell, x) = 0$. The arcs which separate the domains of the functions $\ell^{(j),i}$ (they would be similar to the arcs in the central part of Fig. 7(a)) are not included. If we increase $w^{(2)}$, the sets C_U and C_L expand and form a doubly connected region.

In Figs. 7(b) and 8(b), the family $A^{(1),1}$ of semipermeable curves for the values of parameters of Figs. 7(a) and 8(a) is shown. The computation is done using Eq. (5). The initial points for the semipermeable curves in Fig. 7(b) are located on the boundary of the set B in the fourth quadrant. The initial points for emitting the semipermeable curves in Fig. 8(b) are uniformly distributed over the circumference of radius 4 with the center at the origin.

5. Two examples

The construction of capture sets using semipermeable curves is an important question when solving differential games of kind. The authors do not possess a receipt how to do that in general case. However, such a construction might be not very difficult for some particular problems with dynamics (1). We will demonstrate this using two examples.

The first example deals with the acoustic game. The objective of player P is to bring the state vector to a terminal set M . Player E strives to avoid this. The values of the parameters are: $w^{(1)} = 1$, $R = 0.8$, $w^{(2)} = 1.9$ and $s = 0.75$. The set M is the rectangle $\{x \in R^2 : -3.5 \leq x_1 \leq 3.5, -0.2 \leq x_2 \leq 0\}$.

The boundary of the capture set \mathcal{B} can be obtained (Fig. 9) using Eq. (5) by the computation in reverse time of

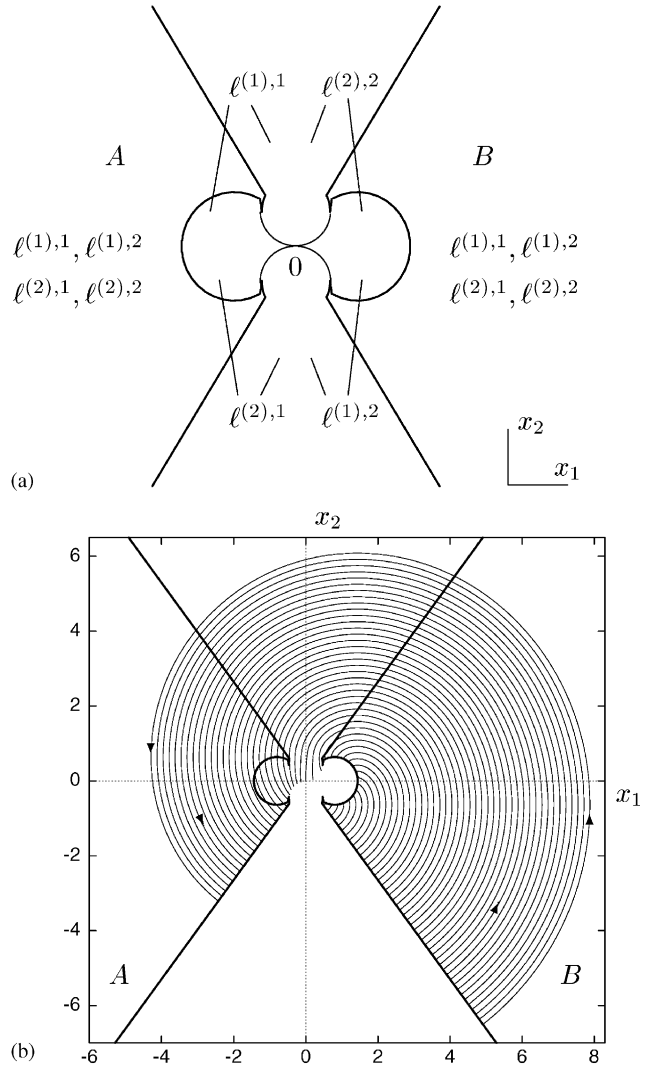


Fig. 7. Set \mathcal{Q} depends on x ; $w^{(2)} = 0.8$. (a) Domains of $\ell^{(j),i}(\cdot)$. (b) Family $A^{(1),1}$ of semipermeable curves.

the semipermeable curves $p^{(1),1}$ and $p^{(2),1}$ from the points a and b , the curves $p^{(1),2}$ and $p^{(2),2}$ from the points c and d , the curve $p^{(1),2}$ from the point d , and symmetric to them with respect to x_2 -axis semipermeable curves emanating from the points a', b', c', d' . Remember that $p^{(j),i}$ denotes a curve of the family $A^{(j),i}$.

The usable part of M through which the penetration in M is possible consists of three segments: the first one is cc' , the second and third segments are symmetric to each other and belong to bb' . Let us consider the one of these two segments that lies on the right from the origin. The right endpoint of this segment is b , the left endpoint is g (it does not plotted in Fig. 9) with $g_1 = w^{(2)}R/w^{(1)} + R = 2.32$. The point g is on the right from the point a that belongs to the intersection of the segment bb' with the boundary of the domain of the family $A^{(1),1}$. The point a' symmetric to a belongs to the intersection of the segment bb' with the boundary of the domain of the family $A^{(2),2}$. Therefore, the semipermeable

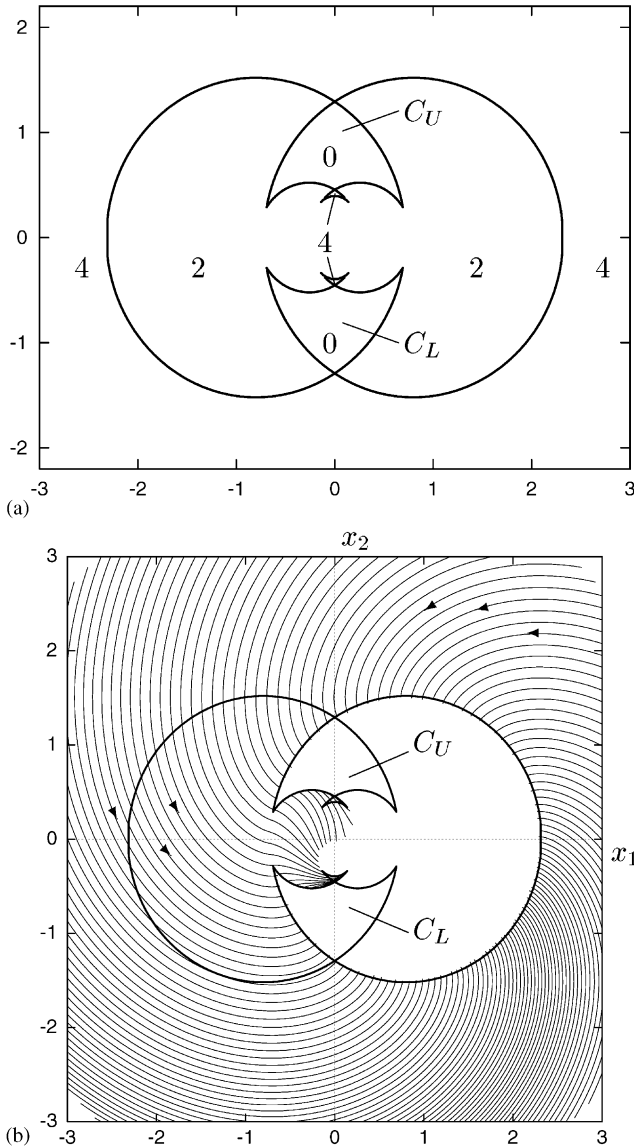


Fig. 8. Set \mathcal{L} depends on x ; $w^{(2)} = 1.9$. (a) Superiority sets C_U and C_L of player E . (b) Family $A^{(1),1}$ of semipermeable curves.

curves $p^{(1),1}$ and $p^{(2),2}$ located in the lower part of Fig. 9 emanate from the points on the boundary of M which are not endpoints of the usable part.

It might seem that, according to the Isaacs approach, one should continue the curves $p^{(1),2}$ and $p^{(2),1}$ emanated from the points c and c' up to their intersection. However, the set bounded by such curves and by the segment cc' would contain the superiority set C_U for which the capture time is infinite. Hence, one can assume that some additional points of emanation for semipermeable curves should exist and try to find them based on an analysis of families of semipermeable curves. In the example considered, these points are d and d' lying on the boundary of C_U .

Removing the parts of mentioned semipermeable curves of the first and second type beyond the intersection points

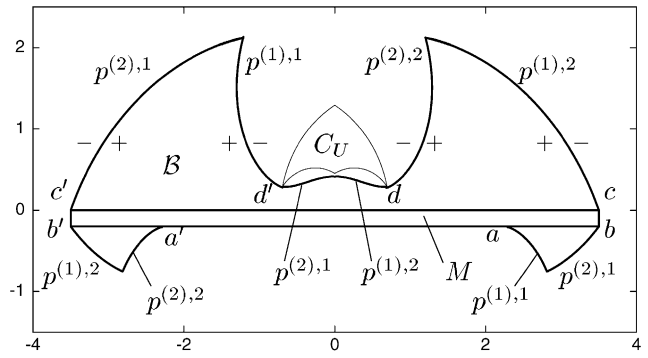


Fig. 9. Capture set of the acoustic game.

yields the boundary of \mathcal{B} . The positive sides of semipermeable curves forming the boundary are oriented to the interior of \mathcal{B} .

Note that, for $x_1 \geq 0, x_2 \geq 0$ ($x_1 \leq 0, x_2 \geq 0$), all of the three semipermeable curves on the boundary of \mathcal{B} correspond to the control $\varphi = -1$ ($\varphi = 1$). Both of the two semipermeable curves in the right (left) lower part correspond to $\varphi = 1$ ($\varphi = -1$). This enables to define the following simple control law $\varphi(x)$ of player P that guarantees the attainment of the set M from the inside of the set \mathcal{B} : if $x_1 \geq 0, x_2 > 0$ or $x_1 < 0, x_2 < 0$, put $\varphi(x) = -1$; if $x_1 \leq 0, x_2 > 0$ or $x_1 > 0, x_2 < 0$, put $\varphi(x) = 1$. The control $v(x)$ of player E that ensures the evasion from the set M for initial points outside the closure of the set \mathcal{B} can be defined using semipermeable curves that are located in a small neighborhood of the set \mathcal{B} and are similar to those lying on the boundary of \mathcal{B} . The initial points from the boundary of \mathcal{B} should be considered specifically. Thus, we have outlined an idea of the control construction, the precise proof is not trivial.

To verify the correctness of the constructed capture set, level sets of the value function $x \rightarrow V(x)$ of the related time-optimal differential game are presented in Fig. 10. The level set W_c of the value function V is the set of all points x such that $V(x) \leq c$. The front is the collection of all points on the boundary of W_c where $V(x) = c$. The capture set \mathcal{B} of the game of kind can be equivalently defined as the set which is filled out with fronts of the time-optimal game. Numerical computation of level sets of the value function is done using an algorithm from Patsko and Turova (2001) for the backward construction in time. The step Δ of the backward procedure is 0.01. The upper and lower fronts are computed until $\tau = 8.42$ and $\tau = 1.6$, respectively. Here τ is the reverse time of the backward procedure. Every 5th front is drawn. The semipermeable curves from Fig. 9 are also plotted. The lines from Fig. 8(b) that define the domains of the families $A^{(j),i}$ of smooth semipermeable curves are depicted. The part of the plane filled out with the fronts forms the capture set \mathcal{B} as $\tau \rightarrow \infty$.

The second example is the conic surveillance–evasion game with the classical homicidal chauffeur dynamics. We take $w^{(1)} = 1.7$. The terminal set M is the complement of an

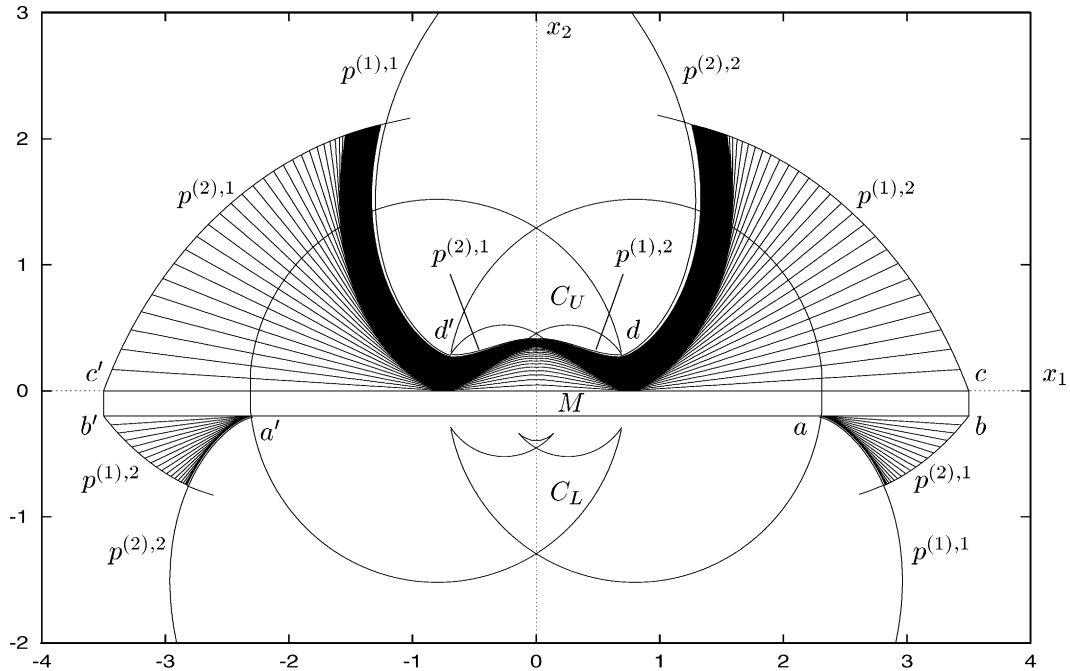


Fig. 10. Level sets in the acoustic game.

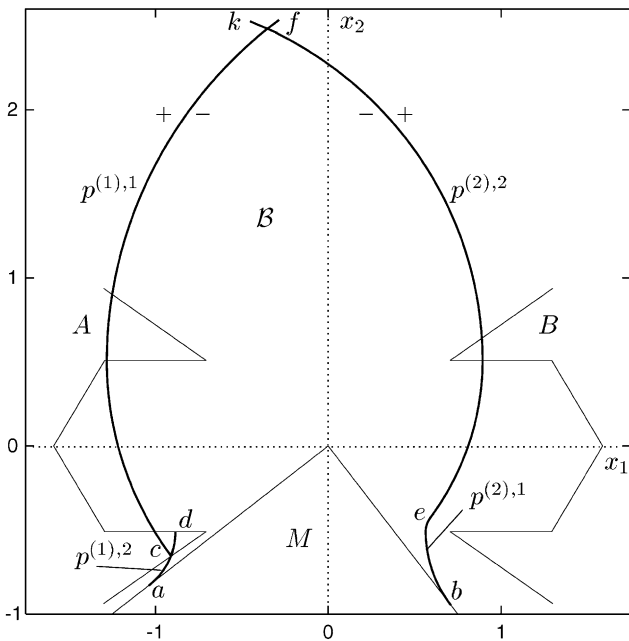


Fig. 11. Capture set of the surveillance–evasion game.

open detection cone around the axis x_2 with the apex at the origin (see Fig. 11). The objective of player E is to bring the state vector to the set M . Player P has the opposite objective. The set Q is a regular hexagon inscribed into the unit circle with the center at the origin. The detection cone is nonsymmetric with respect to the axis x_2 so that the set M is nonsymmetric too.

First find endpoints a and b of the usable part of M . Compute the curve $p^{(1),2}$ backward in time from the point a up to the boundary of the domain $A^{(1),2}$ (d is the endpoint of the curve). Moving along the curve $p^{(1),2}$ from d toward a , look for a sewing point from which a semipermeable curve $p^{(1),1}$ of the family $A^{(1),1}$ emanates such that the composite curve formed by the initial part of $p^{(1),2}$ and the curve $p^{(1),1}$ would possess the semipermeability property at the sewing point. One can establish that such a sewing point can only lie on the boundary of the domain of $A^{(1),1}$. Denote this point by c . We obtain piecewise-smooth composite semipermeable curve acf of the first type. Then compute the semipermeable curve $p^{(2),1}$ from the point b up to the boundary of the domain of the family $A^{(2),1}$ (until the point e). The curve $p^{(2),1}$ is joined smoothly with the curve $p^{(2),2}$ of the family $A^{(2),2}$. As a result, smooth composite semipermeable curve bek of the second type is obtained. The intersection of the curves acf and bek defines the capture set \mathcal{B} . The negative sides of these curves are oriented to the inside of \mathcal{B} .

Level sets of the value function in the time-optimal problem are presented in Fig. 12. The step Δ is 0.01. The computation is done until $\tau = 3.8$. Every 10th front is drawn. The value function is discontinuous on the barrier line cd . The fronts approach the curve $p^{(1),1}$ as $\tau \rightarrow \infty$.

6. Conclusions

The paper describes the construction of families of smooth semipermeable curves for differential games with the homicidal chauffeur dynamics. The construction is based on the

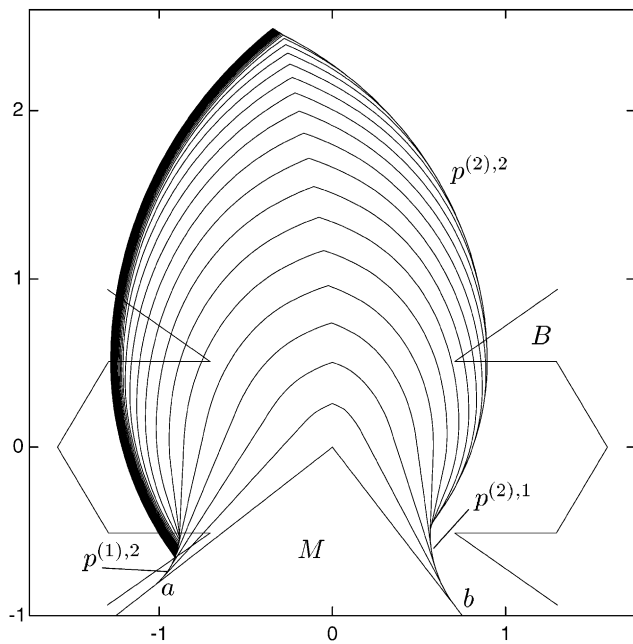


Fig. 12. Level sets in the surveillance-evasion game.

computation of roots “–” to “+” and “+” to “–” of the associated Hamiltonians. The Hamiltonian for the “homicidal chauffeur” dynamics is a positively homogeneous function composed of two convex pieces. This implies the presence of two families of semipermeable curves of the first type related to roots “–” to “+” and two families of semipermeable curves of the second type related to roots “+” to “–”. The paper describes the domains of these families, gives an algorithm for numerical construction of semipermeable curves, and presents results of computation of the families.

The families of smooth semipermeable curves are completely determined by the dynamics and can be useful for game problems with various performance indices. In particular, they form the boundary of the capture set in differential games of kind. In time-optimal game problems, the value function can be discontinuous on semipermeable curves only.

The existence of smooth semipermeable curves of the first and second type is the speciality of two-dimensional problems, which can be used for constructing piecewise smooth semipermeable curves and finding solutions to differential games of kind.

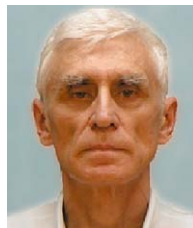
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